BAYESIAN RELIABILITY ANALYSIS FOR THE GUMBEL FAILURE MODEL

CHRIS P. TSOKOS AND BRANKO MILADINOVIC

Department of Mathematics and Statistics, University of South Florida, Tampa, FL 33620

ABSTRACT. The Gumbel or double exponential probability distribution is used to characterize the failure times of a given system. The ordinary and Bayesian settings of the reliability functions are being studied. In the Bayesian reliability analysis, we utilize Jeffrey's non-informative prior to obtain estimates of the target time of a system to achieve a desired reliability. Lindley's approximation procedure is used to obtain numerical estimates of the target time.

1. Introduction

Extreme value probability distributions have been used effectively to model various problems in engineering, environment, business, etc. Some key references are (Burton and Markopoulos, 1985; Naess, 1998; Osella et al., 1992; Ramachandran, 1982; Rao et al., 1997; Sastry and Pi, 1991; Silbergleit, 1996; Suzuki and Ozaha, 1994; and Tsokos, 1999). The object of the present study is to use the classical Gumbel or double exponential probability distribution to characterize the failure times of a given system, both in the ordinary and Bayesian settings. In the Bayesian setting, we assume that the prior probability density function is the Jeffrey's non-informative prior under the mean square error loss function. We are interested in obtaining ordinary and Bayesian estimates of a target time t_α, subject to a desired and specified reliability. That is, for a given system what is the time to failure, t_α, with at least (1 - α)% assurance. For example, we want to be at least 95% certain that the system will be operable to time $t_{0.05}$. We develop both ordinary and Bayesian estimates of t_α and introduce Lindley's approximation procedure that is used to obtain numerical results that illustrate the usefulness of the study.

2. The Gumbel Model

For the Gumbel model, the probability distribution function (p.d.f) and the cumulative distribution function (c.d.f) of the failure time T are given, respectively, by

$$f(t) = \frac{1}{\sigma} e^{-\frac{t-\mu}{\sigma} - e^{-\frac{t-\mu}{\sigma}}}, \tag{2.1}$$

$$-\infty < t < \infty, \ -\infty < \mu < \infty, \ \sigma > 0$$

and

$$F(t) = exp\{-exp\{(-\frac{(t-\mu)}{\sigma}\}\} \tag{2.2}$$

where μ and σ are the location and scale parameters, respectively. This model has been used in fire protection, insurance problems, prediction of earthquake magnitudes, carbon dioxide levels in the atmosphere, high return levels of wind speeds in the design of structures among others. In the present study, we shall apply the subject model in reliability analysis and, more specifically, Bayesian reliability modeling.

3. **Reliability modeling**

Let t_1, t_2, t_3,..., t_n be the failure times that follow the Gumbel p.d.f. given by (2.1). The likelihood function L(μ, σ), is given by

$$L(\mu, \ \sigma) = \sigma^{-n} exp\{-\sum_{i=1}^{n} \frac{t_i - \mu}{\sigma} - \sum_{i=1}^{n} exp(-\frac{t_i - \mu}{\sigma})\}$$

and its logarithmic form is

$$LogL = -nln\sigma - \sum_{i=1}^{n}(\frac{t_i - \mu}{\sigma}) - \sum_{i=1}^{n} e^{-(\frac{t_i-\mu}{\sigma})} \tag{3.1}$$

The maximum likelihood estimates (MLE) for μ and σ can be obtained from the likelihood functions by solving the following equations

$$\hat{\sigma} + \frac{\Sigma t_i e^{-t_i/\hat{\sigma}}}{\Sigma e^{-t_i/\hat{\sigma}}} = \bar{t} \tag{3.2}$$

and

$$\hat{\mu} = -\hat{\sigma} ln\{\frac{1}{n}\Sigma e^{-t_i/\hat{\sigma}}\} \tag{3.3}$$

Equations (3.2) and (3.3) are not analytically tractable and must be solved numerically to obtain approximate MLE's of μ and σ, that is $\hat{\mu}$ and $\hat{\sigma}$. By taking the natural logarithm of both sides of equation (2.2) and solving for t we obtain the expression for the target time t_α under the desired reliability 1 - α given by

$$t_\alpha = \mu - \sigma(ln(-ln(\alpha)) \tag{3.4}$$

Thus, by the invariance property of the MLE's we can obtain the MLE of the target time t_α

$$\hat{t}_\alpha = \hat{\mu} - \hat{\sigma}(ln(-ln(\alpha)) \tag{3.5}$$

Classical estimates and confidence intervals for t_α can be obtained using the method of maximum likelihood and the normal approximation for different extreme value models. In the present study, we shall examine the estimation of t_α for an extreme value model in a Bayesian setting under a specified prior and mean square error loss function.

4. **Bayesian approach to the Gumbel Model**

In the Bayesian approach we regard μ and σ behaving as random variables with a joint p.d.f. $\pi(\mu,\ \sigma)$. We shall investigate the point estimator of t_α for Jeffrey's non-informative prior.

4.1. **Jeffrey's non-informative prior.** Jeffrey's non-informative prior chooses the prior $\pi(\mu,\ \sigma)$ to be proportional to $\sqrt{detI(\theta)}$, where I(θ) is the expected Fisher information matrix. That is,

$$I(\hat{\mu},\hat{\sigma}) = -E\begin{vmatrix} \frac{\partial^2}{\partial\mu^2}\ln L & \frac{\partial^2 \ln L}{\partial\mu\partial\sigma^2} \\ \frac{\partial^2 \ln L}{\partial\mu\partial\sigma^2} & \frac{\partial^2}{\partial\sigma^2}\ln L \end{vmatrix}.$$

Using logL as given in (3.1), we obtain I(θ) as

$$I(\theta) = \begin{vmatrix} \frac{n}{\sigma^2} & -\frac{n}{\sigma^2}(1-\gamma) \\ \frac{-n}{\sigma^2}(1-\gamma) & \frac{n}{\sigma^2}\{\gamma^2 - 2\gamma + 2 + \eta(2,2)\} \end{vmatrix}. \tag{4.1}$$

where γ is the Euler's constant and

$$\eta(p,q) = \frac{1}{\Gamma(p)}\int_0^\infty \frac{t^{p-1}e^{-qt}}{1-e^{-t}}\,dt.$$

Hence,

$$det(I(\theta)) = \frac{K}{\sigma^4}$$

implying that the Jeffrey's non-informative prior is given by

$$\pi(\mu,\ \sigma) = \frac{1}{\sigma^2} \tag{4.2}$$

We remark that π is an improper prior p.d.f. Consistent with the aim of the present study in identifying the target time t_α, we proceed to obtain its analytical form.

4.2. **Posterior distribution.** The posterior probability density function of (μ,σ) given the failure times $t_1,...,t_n$ is given by

$$\pi(\mu,\sigma \mid t_1,t_2,...,t_n) = \frac{L(\mu,\sigma \mid t_1,...,t_n)\pi(\mu,\sigma)}{\int_0^\infty\int_{-\infty}^\infty L(\mu,\sigma \mid t_1,t_2,...,t_n)\pi(\mu,\sigma)d\mu d\sigma}$$

where $L(\mu,\sigma \mid t_1,...,t_n)$ is given by (3.1). We shall first compute the marginal probability density function, that is,

$$\int_0^\infty\int_{-\infty}^\infty L(\mu,\sigma \mid t_1,...,t_n)\pi(\mu,\sigma)\ d\mu\ d\sigma.$$

Using the prior $\pi(\mu,\sigma) = \frac{1}{\sigma^2}$, $\sigma > 0$ and letting $x = \sum_{i=1}^n t_i$, we obtain

$$\int_0^\infty \int_{-\infty}^\infty L(\mu,\sigma \mid t_1,...,t_n)\pi(\mu,\sigma)d\mu d\sigma$$

$$= \int_0^\infty \sigma^{-n-2} \int_{-\infty}^{+\infty} e^{-x/\sigma} e^{(n\mu/\sigma)} e^{-e^{(\mu/\theta)}\sum_{i=1}^n e^{-t_i/\sigma}} d\mu d\sigma. \tag{4.3}$$

Let $u = e$, and $a = \sum_{l=1}^n e^{-(t_1/\sigma)}$, the expression (4.3) can be written as

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty & L(\mu,\sigma \mid t_1,...,t_n)\pi(\mu,\sigma)d\mu\, d\sigma \\ &= \int_0^\infty \sigma^{-n-1} e^{-x/\sigma} \int_0^\infty u^{n-1} e^{-au} du d\sigma \\ &= \Gamma(n) \int_0^\infty \sigma^{-n-1} e^{-x/\sigma} a^{-n} d\sigma \\ &= \Gamma(n) \int_0^\infty v^{n-1} e^{-xv} (\sum_{l=1}^n e^{-t_i v})^n dv \\ &= \Gamma(n) \int_0^\infty v^{n-1} (\sum_{l=1}^n e^{-v(t_i+\bar{t})})^n dv \end{aligned} \tag{4.4}$$

where $x = \sum_{l=1}^n t_i = n\bar{t}$.

4.3. **Bayesian estimation of t_α for Jeffrey's prior.** The Bayes estimate of

$$t_\alpha = \mu - \sigma \ln(-\ln(\alpha))$$

for squared error loss is given by

$$\hat{t}_B = E(t_\alpha \mid t_1, t_2, ..., t_n)$$

$$\int_0^\infty \int_{-\infty}^\infty [\mu - \sigma \ln(-\ln(\alpha)] L(\mu,\sigma \mid t_1,t_2,...,t_n)\pi(\mu,\sigma)d\mu d\sigma$$

or

$$\hat{t}_B = \frac{\int_0^\infty \int_{-\infty}^\infty [\mu - \sigma \ln(-\ln \alpha)] L(\mu,\sigma \mid t_1,t_2,...,t_n)\pi(\mu,\sigma)\, d\mu d\sigma}{\Gamma(n) \int_0^\infty v^{n-1} (\sum e^{-(t_i+\bar{t})v})^n dv} \tag{4.5}$$

Proceeding as we did before for obtaining the marginal probability distribution we can write

$$E(\mu \mid \vec{t}) = \int_0^\infty v^{n-2} e^{-xv} \int_0^\infty (\ln u) u^{n-1} e^{au} du dv \tag{4.6}$$

where

$$a = \sum_{l=1}^n e^{-t_i v},$$

and

$$E(\sigma \mid \vec{t}) = \Gamma(n) \int_0^\infty v^{n-2} (\sum_{l=1}^{n} e^{-v(t_i+\bar{t})})^n dv. \tag{4.7}$$

Hence,

$$E(\hat{t}_\alpha \mid t_1, ..., t_n) = \frac{\int_0^\infty v^{n-2} e^{-xv} \int_0^\infty (\ln u) u^{n-1} e^{-au} du dv}{\Gamma(n) \int_0^\infty v^{n-1} [\sum_{l=1}^{n} e^{-v(t_i+\bar{t})}]^n dv}$$
$$+(-\sigma \ln(-\ln \alpha)) \frac{\int_0^\infty v^{n-2} [\sum e^{-v(t_i+\bar{t})}]^n dv}{\int_0^\infty v^{n-1} [\sum_{l=1}^{n} e^{-v(t_i+\bar{t})}]^n dv}$$

where

$$a = \sum_{l=1}^{n} e^{-t_i v}$$

and

$$\bar{t} = \frac{\sum_{l=1}^{n} t_i}{n}.$$

To evaluate the above expression to obtain approximate Bayesian estimates of t_α, we shall use Lindley's approximation method.

4.4. **The Lindley Approximation.** Similar to our work in the previous chapter, let

$$I = \frac{\int u(\theta) v(\theta) e^{L(\theta)} d\theta}{\int v(\theta) e^{L(\theta)} d\theta}$$

where $\theta = (\theta_1, \theta_2, ..., \theta_k)$, a vector of parameters. Also, let L=log(likelihood function) Note that I is the posterior expectation of $u(\vec{\theta})$ given the failure data, for a prior $v(\theta)$. Denote by

$$u_1 = \frac{\partial u}{\partial \theta_1} \quad u_2 = \frac{\partial u}{\partial \theta_2}$$
$$u_{11} = \frac{\partial^2 u}{\partial \theta_1^2} \quad u_{22} = \frac{\partial^2 u}{\partial \theta_2^2}$$
$$p = \pi(\theta_1, \theta_2)$$
$$p_1 = \frac{\partial p}{\partial \theta_1}; \quad p_2 = \frac{\partial p}{\partial \theta_2}$$
$$L_{20} = \frac{\partial^2 L}{\partial \theta_1^2}; \quad L_{02} = \frac{\partial^2 L}{\partial \theta_2^2}$$
$$L_{30} = \frac{\partial^3 L}{\partial \theta_1^3}; \quad L_{03} = \frac{\partial^3 L}{\partial \theta_2^3}$$

and

$$\sigma_{11} = (-L_{20})^{-1} \; and \; \sigma_{22} = (-L_{02})^{-1}$$

Furthermore,

$$E(u(\theta) \mid \vec{t}) = u(\hat{\theta}_1, \hat{\theta}_2) + \frac{1}{2}(u_{11}\sigma_{11} + u_{22}\sigma_{22}) + P_1 u_1 \sigma_{11} + P_2 u_2 \sigma_{22}$$
$$+\frac{1}{2}(L_{30} u_1 \sigma_{11}^2 + L_{03} u_2 \sigma_{22}^2 + L_{21} u_2 \sigma_{11} \sigma_{22} + L_{12} u_1 \sigma_{22} \sigma_{11})$$

evaluated at $(\hat{\theta}_1, \hat{\theta}_2)$, where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the MLEs of θ_1 and θ_2. The target time for the Gumbel model given by

$$t_B = \mu - b\sigma = u(\mu, \sigma),$$

where $\theta_1 = \mu$ and $\theta_2 = \sigma$. Also, $u_1 = 1$ and $u_2 = -b$ where $b = \ln(-\ln \alpha)$ where $u_{11} = 0$ and $u_{22} = 0$. Thus, we can write

$$P(\theta_1, \theta_2) = \pi(\mu, \sigma) = \frac{1}{\sigma^2}$$

and

$$P_1 = 0 \ \ and \ P_2 = -\frac{2}{\sigma^3}.$$

Let $\hat{\mu}$ and $\hat{\sigma}$ be the classical MLEs for μ and σ, respectively. Furthermore, we have

$$L = \sigma^{-n} \exp\left\{ -\sum_{i=1}^{n} (\frac{t_i - \mu}{\sigma}) - \sum_{i=1}^{n} \exp(-\frac{t_i - \mu}{\sigma}) \right\}$$

or

$$\ln L = -n \ln \sigma - \sum_{i=1}^{n} (\frac{t_i - \mu}{\sigma}) - \sum_{i=1}^{n} e^{-\frac{t_i - \mu}{\sigma}}.$$

Thus,

$$\frac{\partial \ln L}{\partial \mu} = \frac{n}{\sigma} - \frac{1}{\sigma} \sum_{i=1}^{n} e^{-\frac{t_i - \mu}{\sigma}}$$

and

$$L_{2,0} = \frac{\partial^2 \ln L}{\partial \mu^2} = -\frac{1}{\sigma^2} \sum_{i=1}^{n} e^{-\frac{t_i - \mu}{\sigma}}.$$

Also,

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{(t_i - \mu)}{\sigma^2} - \frac{1}{\sigma^2} \sum_{i=1}^{n} e^{-(\frac{t_i - \mu}{\sigma)}} (t_i - \mu)$$

and

$$L_{0,2} = \frac{\partial^2 \ln L}{\partial \sigma^2} = \frac{n}{\sigma^2} - 2[\sum_{i=1}^{n} (t_i - \mu)] \frac{1}{\sigma^3} + \frac{2}{\sigma^3} \sum_{i=1}^{n} e^{\frac{t_i - \mu}{\sigma}} (t_i - \mu) - \frac{1}{\sigma^4} \sum_{i=1}^{n} e^{-(\frac{t_i - \mu}{\sigma})} (t_i - \mu)^2$$

which can be expressed as

$$\frac{n}{\sigma^2} - (\sum_{i=1}^{n} (t_i - \mu)) \frac{1}{\sigma^3} + (\sum_{l=1}^{n} e^{\frac{t_i - \mu}{\sigma}}) \frac{(t_i - \mu)}{\sigma^3} - \frac{1}{\sigma^4} \sum_{i=1}^{n} e^{-\frac{t_i - \mu}{\sigma}} (t_i - \mu)^2$$

or

$$\frac{n}{\sigma^2} - [\sum_{l=1}^{n} 2(t_i - \mu)[1 - e^{-\frac{(t_i - \mu)}{\sigma}}]] \frac{1}{\sigma^3} + (\sum_{i=1}^{n} e^{-(\frac{t_i - \mu}{\sigma})} (t_i - \mu)^2) \frac{1}{\sigma^4}.$$

We proceed to find $L_{3,0}$ and $L_{0,3}$, that is

$$L_{3,0} = \frac{\partial^3 \ln L}{\partial \mu^3} = -\frac{1}{\sigma^3} \sum_{i=1}^{n} e^{-\frac{t_i - \mu}{\sigma}}$$

and

$$L_{0,3} = \frac{\partial^3 \ln L}{\partial \sigma^3} = \frac{-2n}{\sigma^3} + 6[\sum_{i=1}^{n}(t_i - \mu)[1 - e^{-\frac{(t_i-\mu)}{\sigma}}]]\frac{1}{\sigma^4}$$

$$+6[\sum_{i=1}^{n} e^{-\frac{(t_i-\mu)}{\sigma}}(t_i-\mu)^2]\frac{1}{\sigma^5} - 1[\sum_{i=1}^{n} e^{-\frac{(t_i-\mu)}{\sigma}}(t_i-\mu)^3]\frac{1}{\sigma^6}.$$

Also,

$$L_{21} = \frac{\partial}{\partial \sigma}(\frac{\partial^2 \ln L}{\partial \mu^2}) = [\sum_{i=1}^{n} e^{-(\frac{t_i-\mu}{\sigma})}]\frac{2}{\sigma^3} - [\sum_{i=1}^{n} e^{-(\frac{t_i-\mu}{\sigma})(t_i-\mu)}]\frac{1}{\sigma^4}$$

and

$$L_{12} = \frac{\partial}{\partial \mu}(\frac{\partial^2 \ln L}{\partial \sigma^2})$$

or

$$L_{12} = \frac{2}{\sigma^3}(n - \sum_{i=1}^{n} e^{\frac{t_i-\mu}{\sigma}}) + \frac{4}{\sigma^4}\sum(t_i-\mu)e^{-\frac{t_i-\mu}{\sigma}} - \frac{1}{\sigma^5}\sum_{i=1}^{n}(t_i-\mu)^2 e^{-\frac{t_i-\mu}{\sigma}}$$

Thus, a Bayesian approximate estimate for t_α is given by

$$\hat{t}_B = \hat{t}_\alpha(MLE) + P_2 u_2 \sigma_{22} + \frac{1}{2}(L_{30}\sigma_{11}^2 + L_{03}u_2\sigma_{22}^2 + L_{21}u_2\sigma_{11}\sigma_{22} + L_{12}\sigma_{22}\sigma_{11}) \tag{4.8}$$

evaluated at the MLE of μ and σ, $\hat{\mu}$ and $\hat{\sigma}$.

5. **Numerical Analysis**

In this section we present a numerical study in order to compare the maximum likelihood and Bayes estimates for determining the target time of the Gumbel failure model subject to specified reliability. Our numerical simulation was conducted in the following manner:

1. Under the assumption that the location parameter μ and the scale parameter σ behave randomly and independently, we simulated m (m = 50, 100, 200) location parameters from the normal distribution. In order to study the effects of the prior variance on our estimates, we simulated location parameters from the normal distribution with mean 25 and variances equal to 1, 4, and 9 respectively.
2. We assumed the scale parameter follows the uniform distribution. However, in order to see what effects the increase of variance has on our estimates, we let σ equal to 1, 2 and 4 respectively.
3. Using the obtained m pairs of μ and σ, we generated n (n = 50, 100, 200) observations from the Gumbel p.d.f. and calculated both the maximum likelihood and Bayes estimates of the target time.
4. For comparison purposes, we calculated the absolute value of the difference between the true target time and the corresponding ML and Bayes estimates for 99% reliability.

A schematic diagram of the complete step-by-step process of the numerical analysis is presented in Figure 1.

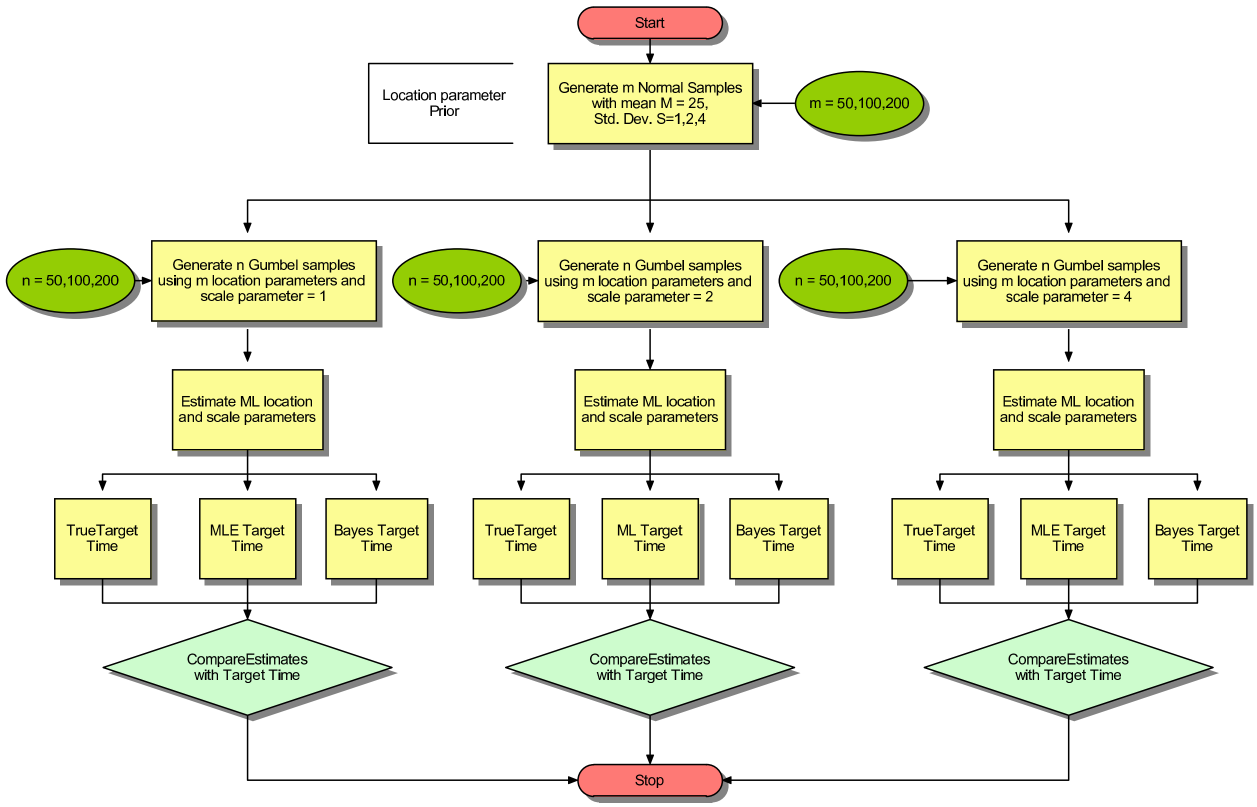

FIGURE 1. Numerical Study of the Gumbel Failure Time

m	n	$\mu_B, \hat{\mu}$	$\sigma_B, \hat{\sigma}$	$\mid t_\alpha - \hat{t}_\alpha \mid$	$\mid t_\alpha - \hat{t}_B \mid$
50	50	25.0526, 25.2053	1, 0.9683	0.2011	0.0938
50	100	25.0526, 25.1542	1, 0.8594	0.3165	0.2548
50	200	25.0526, 24.999	1, 0.9608	0.008	0.01
50	50	25.0526, 25.2433	2, 1.9397	0.2827	0.1485
50	100	25.0526,25.1688	2, 1.9276	0.2267	0.1607
50	200	25.0526, 25.1454	2, 1.9591	0.1552	0.1214
50	50	25.0526, 25.455	4, 4.047	0.3312	0.0362
50	100	25.0526, 25.0701	4, 4.043	0.2167	0.0805
50	200	25.0526, 25.0342	4, 3,951	0.1543	0.0726

TABLE 1. Comparison between ML and Bayesian Estimates of Reliability Time: $\mu \sim$ N(25, 1), σ = 1,2,4, $\alpha = 0.01$

Due to the size of our simulation some of the numerical results are given in Tables 1-3 under 99% reliability. In each table we present the size of the prior sample m used to calculate the Bayes estimate μ_B, while $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of the location and scale parameters. $\mid t_\alpha - \hat{t}_\alpha \mid$ and $\mid t_\alpha - \hat{t}_B \mid$represent the absolute value of the difference between the true target time, and maximum likelihood and Bayes target time estimates respectively. As we can see from Table 1, by keeping the prior sample

m	n	$\mu_B, \hat{\mu}$	$\sigma_B, \hat{\sigma}$	$\mid t_\alpha - \hat{t}_\alpha \mid$	$\mid t_\alpha - \hat{t}_B \mid$
100	50	25.0701, 25.1942	1, 0.9796	0.1552	0.0611
100	100	25.0701, 25.1968	1, 0.9696	0.1416	0.1226
100	200	25.0701, 25.1252	1, 0.9584	0.1185	0.0921
100	50	25.0701, 25.145	2, 1.771	0.4792	0.3587
100	100	25.0701, 25.145	2, 1.8397	0.3195	0.2566
100	200	25.0701, 25.06	2, 1.774	0.2195	0.1956
100	50	25.0701, 25.42	4, 4.05	0.2167	0.1805
100	100	25.0701, 24.92	4, 3.904	0.07	0.069
100	200	25.0701, 24.77	4, 3.85	0.01	0.01

Table 2. Comparison between ML and Bayesian Estimates of Reliability Time: $\mu \sim$ N(25, 2), σ = 1,2,4, $\alpha = 0.01$

size m = 50 and prior variance fixed and varying the sample size of the failure model from n = 50 to n = 200, the absolute value of the difference $\mid t_\alpha - \hat{t}_\alpha \mid$ and $\mid t_\alpha - \hat{t}_B \mid$ decreases. This behavior is consistent as we increase σ and n, except that we notice

m	n	$\mu_B, \hat{\mu}$	$\sigma_B, \hat{\sigma}$	$\mid t_\alpha - \hat{t}_\alpha \mid$	$\mid t_\alpha - \hat{t}_B \mid$
200	50	21.982, 21.964	1, 0.8696	0.1805	0.0921
200	100	21.982, 22.01	1, 0.9562	0.0904	0.0372
200	200	21.982, 21.03	1, 0.999	0.0472	0.0214
200	50	21.982,21.991	2, 1.8243	0.278	0.148
200	100	21.982, 22.011	2, 2.151	0.203	0.21
200	200	21.982, 21.845	2, 2.1965	0.1303	0.130
200	50	21.982, 22.1561	4,3.7306	0.2834	0.291
200	100	21.982, 22.201	4, 3.8402	0.1625	0.159
200	200	21.982, 21.756	4, 3.6615	0.091	0.101

Table 3. Comparison between ML and Bayesian Estimates of Reliability Time: $\mu \sim$ N(25, 3), σ = 1,2,4, $\alpha = 0.01$

a significant improvement in the ML estimate. In Table 2 and Table 3 we increase the prior sample size m to 100 and 200 and prior variance to 4 and 9 respectively and also observe that the absolute value of the difference $\mid t_\alpha - \hat{t}_\alpha \mid$ and $\mid t_\alpha - \hat{t}_B \mid$ decreases. The increase in the prior variance has no effect on the behavior of our estimates. This is consistent as we increase σ and n, and we again notice a significant improvement in the ML estimate. In almost every case the Bayes estimate is closer to the true target time than its maximum likelihood counterpart.

6. **Conclusion**

As expected, the Monte Carlo simulation indicates that the Bayes estimate under the non-informative prior is closer to the true reliability time than its maximum likelihood counterpart. However, the following findings are in order:

1. An increase in the prior sample size for the location parameter has no effect on the behavior of the estimates.
2. An increase in the sample size of the simulated Gumbel data results in the improvement of both the maximum likelihood and Bayesian estimates.
3. When we increase the variance of the prior distribution from 1 to 4 to 9 and the variance of the simulated Gumbel data from 1 to 2 to 4, we notice a significant improvement in the maximum likelihood estimate. We therefore conclude that for large sample size and high variance there is very little difference between the maximum likelihood and the Bayes estimates of the target time subject to specified reliability.

REFERENCES

[1] Burton, P. W. and Markopoulos, K. C. (1985). Seismic risk of circum-Pacific earthquakes, II. Extreme values using Gumbel's third distribution and the relationship with strain energy release. *Pure and Applied Geophysics*.123:849-866.

[2] Naess, A. (1998). Estimations of long return period design values for wind speeds. *J. Eng. Mech. A. Soc. Civil Eng.*. 124: 252:259.

[3] Osella, A. M., Sabbione, N. C., & Cernados, D. C. (1992). Statistical analysis of seismic data from north- western and western Argentina. *Pure and Applied Geophysics*. 139:277-292.

[4] Ramachandran, G. (1982), Properties of extreme order statistics and their application to fire protection and insurance problems. *Fire Saf. J.*. 5:59-76.

[5] Rao, N. M., Rao, P. P., & Kaila, K. L. (1997). The first and third asymptotic distributions of extremes as applied to the seismic source regions of India and adjacent areas. *Geophys. J. Int.*. 128:639-646.

[6] Sastry, S. & Pi, J. I. (1991). Estimating the minimum of partitioning and floor planning problems. *IEEETrans. Comput. Aided Design Integr. Circuits Syst.*. 10:273-282.

[7] Silbergleit, V. M. (1996). On the occurrence of geomagnetic storms with sudden commencements. *J. Geomagn. Geoelect.*. 48:1011-1016.

[8] Suzuki, M & Ozaha, Y (1994). Seismic risk analysis based on strain-energy accumulations in focal region. *J. Res. Natl. Inst. Stand. Technol.*. 99:421-434.

[9] Tsokos, C. P. & Nadarajah S. (2003). Extreme value models for software reliability. *Stoch. Anal. & Applic.*. 21: 719-735.

[10] Tsokos, C. P. (1999). Ordinary and Bayesian approach to life testing using the extreme value distributions. *Frontiers in Reliability Analysis*. IAPOQ: 379-395.

Acknowledgment The authors wish to acknowledge our late colleague Dr. A.N.V. Rao for our fruitful discussions and assistance concerning the present study.