Neural, Parallel and Scientific Computations 16 (2008) 201–207

Positive solutions of singular IVPs for integro-differential equations

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Abstract: *We study singular initial value problems for integro-differential equations of the form*

$$u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))ds, \quad t \in (0, 1],$$
$$u(0) = 0,$$

where $f(t, x)$ is singular at $x = 0$. We prove the existence of a positive solution by means of the lower and upper solutions method and the Brouwer fixed point theorem in conjunction with perturbation methods to approximate regular problems.

Key words: Singular boundary value problem, integro-differential equation, lower and upper solutions, Brouwer fixed point theorem, approximate regular problems.
AMS Subject Classification: 45J05, 47G20

1 Preliminaries

This paper deals with an upper and lower solution method used in finding positive solutions to a given type of integro-differential equation. For a few works in the area of integro-differential equations that deal with upper and lower solutions, one may look to [5, 7]. In addition, if the reader has an interest in upper and lower solution methods in general, they may wish to see [1–4], which give a wide variety of different applications.

The methods of this paper rely heavily on upper and lower solutions methods in conjunction with an application of the Brouwer fixed point theorem [6]. We will provide definitions of appropriate upper and lower solutions. The upper and lower solutions will be applied to nonsingular perturbations of our nonlinear problem, ultimately giving rise to our boundary value problem by passing to the limit.

In this section we will state the definitions that are used in the remainder of the paper.

Consider the integro-differential equation,

$$\begin{aligned} u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))ds, \quad t \in (0, 1], \\ u(0) = 0. \end{aligned} \tag{1}$$

Our goal is to prove the existence of a positive solution of problem (1).

Received January 15, 2008

Definition 1.1 By a solution of problem (1), we mean a function $u : [0, 1] \to \mathbb{R}$ such that u satisfies equation (1) in the normal sense. If $u(t) > 0$ for $t \in (0, 1]$, we say u is a positive solution of problem (1).

Definition 1.2 Let $f : [0, 1] \times \mathcal{D} \to \mathbb{R}$, where $\mathcal{D} \subseteq \mathbb{R}$. If $\mathcal{D} = \mathbb{R}$, problem (1) is called regular. If $\mathcal{D} \neq \mathbb{R}$ and f has singularities on $\partial\mathcal{D}$, then problem (1) is called singular.

We will assume throughout this paper that the following hold:

(A): $\mathcal{D} = (0, \infty)$.

(B): f is continuous on $[0, 1] \times \mathcal{D}$, and K is continuous on $[0, 1] \times [0, 1] \times \mathcal{D}$.

(C): $f(t, x)$ has a singularity at $x = 0$, i.e. $\limsup\limits_{x\to 0^+} |f(t, x)| = \infty$ for each $t \in (0, 1]$.

2 Lower and upper solutions method for regular problems

Let us first consider the regular integro-differential equation,

$$\begin{gathered} u'(t) = h(t, u(t)) + \int_0^t H(t, s, u(s))ds, \quad t \in (0, 1], \\ u(0) = 0, \end{gathered} \tag{2}$$

where h is continuous on $[0, 1] \times \mathbb{R}$, and H is continuous on $[0, 1] \times [0, 1] \times \mathbb{R}$. We establish a lower and upper solutions method for regular problem (2).

Definition 2.1 $\alpha : [0, 1] \to \mathbb{R}$ is called a lower solution of (2), if

$$\begin{gathered} \alpha'(t) \le h(t, \alpha(t)) + \int_0^t H(t, s, \alpha(s))ds, \quad t \in (0, 1], \\ \alpha(0) \le 0. \end{gathered} \tag{3}$$

Definition 2.2 $\beta : [0, 1] \to \mathbb{R}$ is called an upper solution of (2), if

$$\begin{gathered} \beta'(t) \ge h(t, \beta(t)) + \int_0^t H(t, s, \beta(s))ds, \quad t \in (0, 1], \\ \beta(0) \ge 0. \end{gathered} \tag{4}$$

Theorem 2.1 (Lower and Upper Solutions Method) *Let α and β be lower and upper solutions of (2), respectively, and $\alpha \le \beta$ on $[0, 1]$. Let $h(t, x)$ be continuous on $[0, 1] \times \mathbb{R}$ and let $H(t, s, x)$ be continuous on $[0, 1] \times [0, 1] \times \mathbb{R}$. Then (2) has a solution u satisfying*

$$\alpha(t) \le u(t) \le \beta(t), \quad t \in [0, 1].$$

Proof: We proceed through a sequence of steps involving modifications of the functions h and H.

Step 1. For $s, t \in [0, 1]$ and $x \in \mathbb{R}$, define

$$\widetilde{h}(t, x) = \begin{cases} h(t, \beta(t)) - \frac{x-\beta(t)}{x-\beta(t)+1}, & x > \beta(t), \\ h(t, x), & \alpha(t) \le x \le \beta(t), \\ h(t, \alpha(t)) + \frac{\alpha(t)-x}{\alpha(t)-x+1}, & x < \alpha(t), \end{cases}$$

and

$$\widetilde{H}(t,s,x)=\begin{cases} H(t,s,\beta(s))-\frac{x-\beta(t)}{x-\beta(t)+1}, & x>\beta(t),\\ H(t,s,x), & \alpha(t)\le x\le\beta(t),\\ H(t,s,\alpha(s))+\frac{\alpha(t)-x}{\alpha(t)-x+1}, & x<\alpha(t), \end{cases}$$

Thus, $\widetilde{h}$ is continuous on $[0,1]\times\mathbb{R}$ and there exists $M>0$ so that,

$$|\widetilde{h}(t,x)|\le M,\quad t\in[0,1],x\in\mathbb{R}.$$

Also, $\widetilde{H}$ is continuous on $[0,1]\times[0,1]\times\mathbb{R}$ and there exists $N>0$ so that,

$$|\widetilde{H}(t,s,x)|\le N,\quad t,s\in[0,1],x\in\mathbb{R}.$$

We now study the auxiliary equation,

$$\begin{gathered} u'(t)=\widetilde{h}(t,u(t))+\textstyle\int_0^t\widetilde{H}(t,s,u(s))ds,\quad t\in(0,1],\\ u(0)=0. \end{gathered}\tag{5}$$

Our immediate goal is to prove the existence of a solution of (5).

Step 2. We lay the foundation to use the Brouwer fixed point theorem. To this end, define

$$E=\{u:[0,1]\to\mathbb{R}:u\text{ is continuous}\}$$

and also define

$$\|u\|=\sup\{|u(t)|:t\in[0,1]\}.$$

E is a Banach space. Further, we define an operator $\mathcal{T}:E\to E$ by,

$$(\mathcal{T}u)(t)=\int_0^t\left(\widetilde{h}(s,u(s))+\int_s^t\widetilde{H}(\sigma,s,u(s))d\sigma\right)ds\tag{6}$$

$\mathcal{T}$ is a continuous operator. Moreover, from the bounds placed on $\widetilde{h}$ and $\widetilde{H}$ in Step 1 and from (6), if

$$r>M+N,$$

then $\mathcal{T}(\overline{B(r)})\subset\overline{B(r)}$, where $B(r)=\{u\in E:\|u\|<r\}$. Therefore, by the Brouwer fixed point theorem [6], there exists $u\in\overline{B(r)}$ such that $u=\mathcal{T}u$.

Step 3. We now show that u is a fixed point of $\mathcal{T}$ iff u is a solution of (5).

First assume $u=\mathcal{T}u$. Then $u\in E$ and clearly,

$$\begin{aligned} u(0)&=\int_0^0\left(\widetilde{h}(s,u(s))+\int_s^t\widetilde{H}(\sigma,s,u(s))d\sigma\right)ds\\ &=0. \end{aligned}$$

Via the Fundamental Theorem of Calculus and a change in the order of integration,

$$u'(t)=\widetilde{h}(t,u(t))+\int_0^t\widetilde{H}(t,s,u(s))ds,$$

and thus, u solves (5).

On the other hand, if we let $u(t)$ solve (5), then $u \in E$ and

$$\begin{aligned}\int_0^t u'(s)ds &= u(t) - u(0) \\ &= u(t) \\ &= \int_0^t \left(\widetilde{h}(s,u(s)) + \int_0^s \widetilde{H}(s,\sigma,u(\sigma))d\sigma\right) ds \\ &= \int_0^t \left(\widetilde{h}(s,u(s)) + \int_s^t \widetilde{H}(\sigma,s,u(s))d\sigma\right) ds,\end{aligned}$$

via a change in the order of integration. Therefore, $u = \mathcal{T}u$.

Step 4. We now show that solutions $u(t)$ of (5) satisfy

$$\alpha(t) \leq u(t) \leq \beta(t), \quad t \in [0,1].$$

Consider the case of obtaining $u(t) \leq \beta(t)$. Let $v(t) = u(t) - \beta(t)$. For the sake of establishing a contradiction, assume that

$$\sup\{v(t) : t \in [0,1]\} := v(l) > 0.$$

From (5) and (4), we see that $v'(l) \leq 0$. Therefore,

$$u'(l) \geq \beta'(l). \tag{7}$$

On the other hand,

$$\begin{aligned}v'(l) &= u'(l) - \beta'(l) \\ &= \widetilde{h}(l,u(l)) + \int_0^l \widetilde{H}(l,s,u(s))ds - \beta'(l) \\ &= \widetilde{h}(l,\beta(l)) - \frac{u(l)-\beta(l)}{u(l)-\beta(l)+1} + \int_0^l \left(\widetilde{H}(l,s,\beta(s)) - \frac{u(l)-\beta(l)}{u(l)-\beta(l)+1}\right) ds - \beta'(l) \\ &= \widetilde{h}(l,\beta(l)) + \int_0^l \widetilde{H}(l,s,\beta(s))ds - \beta'(l) - \int_0^l \frac{u(l)-\beta(l)}{u(l)-\beta(l)+1} ds - \frac{u(l)-\beta(l)}{u(l)-\beta(l)+1} \\ &\leq -\frac{u(l)-\beta(l)}{u(l)-\beta(l)+1}(l-0) - \frac{u(l)-\beta(l)}{u(l)-\beta(l)+1} \\ &= -\frac{u(l)-\beta(l)}{u(l)-\beta(l)+1}(l+1) \\ &< 0.\end{aligned}$$

Hence, $u'(l) < \beta'(l)$, but this contradicts (7). Therefore, $v(l) \leq 0$, for all $l \in [0,1]$. This implies that

$$u(t) \leq \beta(t), \quad t \in [0,1].$$

A similar argument shows that $\alpha(t) \leq u(t), t \in [0,1]$.

Thus, the conclusion of the theorem holds and our proof is complete. □

3 Existence Result

In this section, we make use of Theorem 2.1 to obtain positive solutions of the singular problem (1). In particular, in applying Theorem 2.1, we deal with a sequence of regular perturbations of (1). Ultimately, we obtain a desired solution of (1) by passing to the limit on a sequence of solutions for the perturbations.

Theorem 3.1 *Assume conditions (A), (B), and (C) hold, along with the following:*

(D): $\lim\limits_{x \to 0^+} f(t,x) = \infty$ *for* $t \in [0,1]$.

(E): There exists $c \in (0, \infty)$ *such that*

$$\lim_{x \to c} \left(f(t,x) + \int_0^t K(t,s,x)ds \right) = 0, \text{ for all } t \in [0,1],$$

and

$$f(t,x) + \int_0^t K(t,s,x)ds > 0, \text{ for all } t \in (0,1].$$

Then (1) has a solution u *satisfying,*

$$0 < u(t) \leq c, \quad t \in (0,1].$$

Proof: Again for the proof, we proceed through a sequence of steps.

Step 1. For $k \in \mathbb{N}, t \in [0,1], x \in \mathbb{R}$, define

$$f_k(t,x) = \begin{cases} f(t,|x|), & |x| \geq \frac{1}{k}, \\ f\left(t, \frac{1}{k}\right), & |x| < \frac{1}{k}. \end{cases}$$

Then f_k is continuous on $[0,1] \times \mathbb{R}$.

Assumption (D) implies that there exists k_0, such that, for all $k \geq k_0$,

$$f_k(t,0) = f\left(t, \frac{1}{k}\right) > 0, \quad t \in [0,1].$$

Consider, for each $k \geq k_0$,

$$\begin{gathered} u'(t) = f_k(t,u(t)) + \int_0^t K(t,s,u(s))ds, \quad t \in (0,1], \\ u(0) = 0 \end{gathered} \tag{8k}$$

Define $\alpha(t) = 0$ and $\beta(t) = c$. Then α and β are lower and upper solutions for (8k) and $\alpha(t) \leq \beta(t)$ on $[0, 1]$. Thus, by Theorem 2.1, there exists u_k a solution of (8k) satisfying

$$0 \leq u_k(t) \leq c, \quad t \in [0, 1], k \geq k_0.$$

Step 2. Let $k \in \mathbb{N}, k \geq k_0$. Since $u_k(t)$ solves (8k), we get from work similar to that exhibited in Theorem 2.1,

$$u_k(t) = \int\limits_0^t \left(f(s, u_k(s)) + \int_s^t K(\sigma, s, u_k(s)) d\sigma \right) ds.$$

Clearly, $u_k(0) = 0$ and $u_k'(0) > 0$. Also, from assumption (E), we know that

$$u_k'(t) = f(t, u_k(t)) + \int_0^t K(t, s, u_k(s)) ds > 0, \quad t \in (0, 1].$$

Now, there exists $R_1 > 0$ and $R_2 > 0$ such that $|f(t, x)| \leq R_1$ for $(t, x) \in [0, 1] \times [0, c]$ and $|K(t, x, y)| \leq R_2$ for $(t, s, x) \in [0, 1] \times [0, 1] \times [0, c]$. This implies that for $k \geq k_0$, $|f(t, u_k(t))| \leq R_1$, and $\left| \int_0^t K(t, x, u_k(s)) ds \right| \leq R_2$. Thus, we have that for $k \geq k_0$,

$$\begin{aligned} |u_k'(t)| &\leq |f_k(t, u_k(t))| + \left| \int_0^t K(t, s, u_k(s)) ds \right| \\ &\leq R_1 + R_2 \\ &:= \overline{R}. \end{aligned}$$

Hence, we have a uniformly equicontinuous set of functions $\{u_k\}$. Define $u_k'(0) := m$. Then, there exists $\varepsilon_k(t) : [0, 1] \to \mathbb{R}$, defined by

$$\varepsilon_k(t) = \begin{cases} \frac{mt}{2}, & 0 \leq t \leq \delta, \\ \frac{m\delta}{2}, & \delta \leq t \leq 1, \end{cases}$$

for some $\delta > 0$, such that $\varepsilon_k(t) \leq u_k(t), 0 \leq t \leq 1$. By construction,

$$0 < \varepsilon_k(t) \leq u_k(t) \leq c, \quad 0 < t \leq 1.$$

Hence, from the continuous and bounded nature of $\{u_k(t)\}$, we can now choose a subsequence $\{u_{k_n}(t)\} \subseteq \{u_k(t)\}$ such that $\lim\limits_{n\to\infty} u_{k_n}(t) = u(t)$ uniformly on $[0, 1], u \in E$, where E is defined as in Step 2 of Theorem 2.1. Moreover, for sufficiently large n,

$$u_{k_n}(t) = \int\limits_0^t \left(f(s, u_{k_n}(s)) + \int_s^t K(\sigma, s, u_{k_n}(s)) d\sigma \right) ds.$$

From the continuity of f and K, as we let $n \to \infty$,

$$u(t) = \int\limits_0^t \left(f(s, u(s)) + \int_s^t K(\sigma, s, u(s)) d\sigma \right) ds.$$

Hence,

$$u'(t) = f(t, u(t)) + \int_0^t K(t, s, u(s))ds,$$

and

$$0 < u(t) \leq c, \quad 0 < t \leq 1.$$

□

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