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THE hp FINITE ELEMENT METHOD FOR SINGULARLY PERTURBED SYSTEMS OF REACTION-DIFFUSION EQUATIONS

CHRISTOS XENOPHONTOS

Department of Mathematics and Statistics, University of Cyprus
Nicosia 1678, Cyprus

ABSTRACT. We consider the approximation of a coupled system of two singularly perturbed reaction-diffusion equations, with the finite element method. The solution to such problems contains boundary layers which overlap and interact, and the numerical approximation must take this into account in order for the resulting scheme to converge uniformly with respect to the singular perturbation parameters. We present results on a high order hp finite element scheme which includes elements of size $O(\varepsilon p)$ and $O(\mu p)$ near the boundary, where ε, μ are the singular perturbation parameters and p is the degree of the approximating polynomials. Under the assumption of analytic input data, the method yields *exponential* rates of convergence as $p \to \infty$, independently of ε and μ. Numerical computations supporting the theory are also presented.

Key Words. singularly perturbed system, boundary layers, hp finite element method

1. INTRODUCTION

The numerical solution of (scalar) singularly perturbed boundary value problems has received a lot of attention during the last two decades. It is well known that the main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), has been the main focus of numerous research endeavors (see, e.g., the books by Miller et al., Morton, Roos et al. and the references therein). Problems of this type arise in numerous applications from science and engineering, such as fluid flow at high Reynolds number, heat transfer with small diffusion parameters and bending of thin structures, just to name a few. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin or Bakhvalov mesh), or the use of the high order p and hp versions on specially designed (variable) meshes such as the one presented by Schwab and Suri. In both cases, the a-priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established.

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In recent years researchers have turned their attention to *systems* of singularly perturbed problems, which have two (or more) overlapping boundary layers, such as the one considered below: Find $\overrightarrow{u}$ such that

$$L\overrightarrow{u} := \begin{bmatrix} -\varepsilon^2 \frac{d^2}{dx^2} & 0 \\ 0 & -\mu^2 \frac{d^2}{dx^2} \end{bmatrix} \overrightarrow{u} + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1) \tag{1}$$

where $0 < \varepsilon \leq \mu \leq 1$,

$$A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}, \quad \overrightarrow{f}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \tag{2}$$

along with the boundary conditions on $\partial\Omega$

$$\overrightarrow{u}(0) = \overrightarrow{\gamma}_0, \quad \overrightarrow{u}(1) = \overrightarrow{\gamma}_1. \tag{3}$$

The data ε, μ, A, $\overrightarrow{f}$, $\overrightarrow{\gamma}_0$ and $\overrightarrow{\gamma}_1$ are given and the unknown solution is $\overrightarrow{u}(x) = [u_1(x), u_2(x)]^T$. Without loss of generality we will take $\overrightarrow{\gamma}_0 = \overrightarrow{\gamma}_1 = \overrightarrow{0}$, and in addition we will assume that

$$a_{12}(x) \leq 0, \quad a_{21}(x) \leq 0 \quad \forall\, x \in \overline{\Omega}, \tag{4}$$

$$\min_{\overline{\Omega}} \{a_{11}(x) + a_{12}(x),\ a_{21}(x) + a_{22}(x)\} \geq \alpha^2 > 1, \tag{5}$$

for some $\alpha \in \mathbb{R}$. The assumption that α is positive guarantees that A is invertible with $\|A^{-1}\|$ bounded, and the fact that A is an M-matrix allows us to find a diagonal matrix D, such that DA is positive definite. Hence, if A is not itself positive definite then we scale the problem (1) by D which results in a system with a positive definite coefficient matrix – this then contributes to the existence and uniqueness of a solution. The assumption that α is greater than 1 is made for convenience, as it is needed for technical reasons in one of the proofs. In fact, one could always scale the problem (1) by a constant such that (5) would hold with $\alpha > 1$.

The presence of ε and μ in (1) causes the solution $\overrightarrow{u}$ to have boundary layers near the endpoints of Ω, which, in general, overlap and interact. Problems of this type arise in the modelling of turbulance in water waves (see the article by Thomas), as well as in the finite element approximation of shells, where the singular perturbation parameters are related to the thickness t of the shell; for example, Pitkäranta et al. showed that in Naghdi-type thin shell models in mechanics there is an $O(t)$ layer due to shear deformation and there is a second layer (or length scale) $O(t^\beta)$, with $\beta \in \{1/2, 1/3, 1/4\}$ (depending on the principal curvatures of the shell's midsurface), due to bending and membrane coupling. The 2-scale reaction-diffusion system (1), (3) could be considered as a model problem for this situation, with $\varepsilon = t$ and $\mu = t^\beta$.

Matthews et al. studied the above problem for the cases $0 < \varepsilon = \mu \ll 1$ and $0 < \varepsilon \ll \mu = 1$, obtaining an approximation using finite differences which converged indepedently of ε and μ. The more general case of $0 < \varepsilon \leq \mu \leq 1$ was studied by

Madden and Stynes and by Linß and Madden in the context of finite differences, and by Linß and Madden in the context of the h version of the FEM with piecewise linear basis functions. In all the works mentioned, estimates were obtained showing that the approximation converged (at the expected rate) indepedently of ε and μ. In this article we present results corresponding to the approximation of the system (1), (3) by a high order hp FEM, which yields *exponential* rates of convergence, indepedently of ε and μ.

In what follows, the space of squared integrable functions on an interval $\Omega \subset \mathbb{R}$ will be denoted by $L^2(\Omega)$, with associated inner product

$$(u, v)_\Omega := \int_\Omega u(x)v(x)dx.$$

We will also utilize the usual Sobolev space notation $H^k(\Omega)$ to denote the space of functions on Ω with $0, 1, 2, \ldots, k$ generalized derivatives in $L^2(\Omega)$, equipped with norm and seminorm $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. For vector functions $\overrightarrow{u} = [u_1(x), u_2(x)]^T$, we will write

$$\|\overrightarrow{u}\|^2_{k,\Omega} = \|u_1\|^2_{k,\Omega} + \|u_2\|^2_{k,\Omega}.$$

We will also use the space

$$H^1_0(\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\right\},$$

where $\partial\Omega$ denotes the boundary of Ω. Finally, the letter C will be used to denote a generic positive constant, independent of any discretization or singular perturbation parameters and possibly having different values in each occurrence.

2. FINITE ELEMENT FORMULATION

The variational formulation of (1), (3) reads: Find $\overrightarrow{u} \in [H^1_0(\Omega)]^2$ such that

$$B(\overrightarrow{u}, \overrightarrow{v}) = F(\overrightarrow{v}) \quad \forall\, \overrightarrow{v} \in \left[H^1_0(\Omega)\right]^2, \tag{6}$$

where

$$\begin{aligned} B(\overrightarrow{u}, \overrightarrow{v}) &= \varepsilon^2 (u_1', v_1')_\Omega + \mu^2 (u_2', v_2')_\Omega + \sum_{i=1}^{2}\sum_{j=1}^{2} (a_{ij}u_j, v_i)_\Omega, \\ F(\overrightarrow{v}) &= \sum_{i=1}^{2} (f_i, v_i)_\Omega. \end{aligned}$$

From the discussion following equation (5), we have that for any $x \in \overline{\Omega}$,

$$\overrightarrow{\xi}^T A \overrightarrow{\xi} \geq \alpha^2 \overrightarrow{\xi}^T \overrightarrow{\xi} \quad \forall\, \overrightarrow{\xi} \in \mathbb{R}^2, \tag{7}$$

and it follows that the bilinear form $B(\cdot,\cdot)$ is coercive with respect to the *energy norm*

$$\|\overrightarrow{u}\|^2_{E,\Omega} := \varepsilon^2 |u_1|^2_{1,\Omega} + \mu^2 |u_2|^2_{1,\Omega} + \alpha^2 \left(\|u_1\|^2_{0,\Omega} + \|u_2\|^2_{0,\Omega}\right), \tag{8}$$

i.e.,

$$B(\overrightarrow{u},\overrightarrow{u}) \geq \|\overrightarrow{u}\|_{E,\Omega}^2 \quad \forall\, \overrightarrow{u} \in \left[H_0^1(\Omega)\right]^2. \tag{9}$$

This, along with the continuity of $B(\cdot,\cdot)$ and $F(\cdot)$, imply the unique solvability of (6). We also have the following *a priori* estimate

$$\|\overrightarrow{u}\|_{E,\Omega} \leq \frac{1}{\alpha}\left\|\overrightarrow{f}\right\|_{0,\Omega}. \tag{10}$$

For the discretization, we choose a finite dimensional subspace S_N of $H_0^1(\Omega)$ and solve the problem: Find $\overrightarrow{u}_N \in [S_N]^2$ such that

$$B(\overrightarrow{u}_N,\overrightarrow{v}) = F(\overrightarrow{v}) \quad \forall\, \overrightarrow{v} \in [S_N]^2. \tag{11}$$

The unique solvability of the discrete problem (11) follows from (7) and (9). Moreover, by the well-known orthogonality relation we have

$$\|\overrightarrow{u} - \overrightarrow{u}_N\|_{E,\Omega} \leq \inf_{\overrightarrow{v}\in[S_N]^2} \|\overrightarrow{u} - \overrightarrow{v}\|_E. \tag{12}$$

In order to achieve exponential convergence we assume that the functions $a_{ij}(x)$ and $f_i(x)$ are analytic on $\overline{\Omega}$ and that there exist constants $C_f,\ \gamma_f,\ C_a,\ \gamma_a > 0$ such that

$$\left\|f_i^{(n)}\right\|_{\infty,\Omega} \leq C_f \gamma_f^n n! \quad \forall\, n \in \mathbb{N}_0,\ i = 1,2, \tag{13}$$

$$\left\|a_{ij}^{(n)}\right\|_{\infty,\Omega} \leq C_a \gamma_a^n n! \quad \forall\, n \in \mathbb{N}_0,\ i,j = 1,2. \tag{14}$$

Note that by the analyticity of a_{ij} and f_i, we have that u_i are analytic. Moreover, we have the following theorem.

Theorem: *Let $\overrightarrow{u}$ be the solution to* (1), (3) *with* $0 < \varepsilon \leq \mu \leq 1$. *Then there exist constants C and $K > 0$, independent of ε and μ, such that*

$$\|\overrightarrow{u}^{(n)}\|_{0,\Omega} \leq CK^n \max\{n,\varepsilon^{-1}\}^n \quad \forall\, n \in \mathbb{N}_0. \tag{15}$$

The above theorem does not suffice for the analysis of the method we will be considering, hence we will describe how the solution $\overrightarrow{u}$ can be decomposed into

$$\overrightarrow{u} = \overrightarrow{w} + \overrightarrow{u}^- + \overrightarrow{u}^+ + \overrightarrow{r}, \tag{16}$$

where $\overrightarrow{w}$ is the smooth part, $\overrightarrow{u}^{\pm}$ are the two boundary layer parts and $\overrightarrow{r}$ is the smooth remainder. The nature of $\overrightarrow{w}$ and $\overrightarrow{r}$ depend on the relationship between ε and μ, and will be discussed in Section 3 below. The boundary layer parts are defined, independently of the relationship between ε and μ, by:

$$L\overrightarrow{u}^- = \overrightarrow{0} \text{ in } \Omega\,,\ \overrightarrow{u}^-(0) = -\overrightarrow{w}(0)\,,\ \overrightarrow{u}^-(1) = \overrightarrow{0}, \tag{17}$$

$$L\overrightarrow{u}^+ = \overrightarrow{0} \text{ in } \Omega\,,\ \overrightarrow{u}^+(0) = \overrightarrow{0}\,,\ \overrightarrow{u}^+(1) = -\overrightarrow{w}(1). \tag{18}$$

3. REGULARITY OF THE SOLUTION

In what follows we will present results that describe the regularity of the each part in the decomposition for the solution $\overrightarrow{u}$, as given by (16), and we will restrict ourselves to the cases $0 < \varepsilon = \mu < 1$ and $0 < \varepsilon < \mu < 1$.

3.1. The case $0 < \varepsilon = \mu < 1$

When $0 < \varepsilon = \mu < 1$, the boundary value problem (1) can be written as

$$L\overrightarrow{u} := -\varepsilon^2 \overrightarrow{u}'' + A\overrightarrow{u} = \overrightarrow{f} \quad \text{in} \quad \Omega = (0,1), \tag{19}$$

and the analysis can be extended beyond just two equations – for simplicity, we will continue to only consider two equations. We note that in this case both components of $\overrightarrow{u}$ will have a boundary layer of width $O\left(|\varepsilon \ln \varepsilon|\right)$. To obtain the decomposition (16) we insert the formal ansatz

$$\overrightarrow{u}(x) \sim \sum_{i=0}^{\infty} \varepsilon^i \overrightarrow{u}^{[i]}(x), \tag{20}$$

into the differential equation (19), and equate like powers of ε so that we can define the smooth part $\overrightarrow{w}$ as

$$\overrightarrow{w}(x) := \sum_{i=0}^{M} \varepsilon^{2i} \overrightarrow{u}^{[2i]}(x), \tag{21}$$

where the terms $\overrightarrow{u}^{[2i]}$ are defined recursively by

$$\overrightarrow{u}^{[0]} := A^{-1}\overrightarrow{f}\ ,\ \overrightarrow{u}^{[2i+2]} := A^{-1}\left(\overrightarrow{u}^{[2i]}\right)'',\ i = 0, 2, 4, \ldots \tag{22}$$

A calculation shows that

$$L(\overrightarrow{u} - \overrightarrow{w}) = \varepsilon^{2M+2}\left(\overrightarrow{u}^{[2M]}\right)'', \tag{23}$$

hence, as $\varepsilon \to 0$, $\overrightarrow{w}(x)$ defined by (21) satisfies the differential equation, but not the boundary conditions. The boundary layer functions $\overrightarrow{u}^{\pm}$ defined by (17), (18) correct this, and by construction $\overrightarrow{u}$ given by (16) satsifies the differential equation (1) and the boundary conditions (3). Finally, the remainder $\overrightarrow{r}$ is defined by

$$L\overrightarrow{r} = \varepsilon^{2M+2}\left(\overrightarrow{u}^{[2M]}\right)'',\ \overrightarrow{r}(0) = \overrightarrow{r}(1) = \overrightarrow{0}. \tag{24}$$

The following results analyze the behavior of the terms in the decomposition of $\overrightarrow{u}$ and establish the claim made earlier, namely that the solution can be decomposed into a smooth part, two boundary layer parts and a smooth remainder.

Lemma: *Let $\overrightarrow{u}^{[2i]}$ be defined as in (22). Then there exist positive constants C, K_1, K_2 depending only on A and $\overrightarrow{f}$ such that for any $i, n \in \mathbb{N}_0$*

$$\left\|\left(\overrightarrow{u}^{[2i]}\right)^{(n)}\right\|_{\infty,\Omega} \leq C K_1^{2i} K_2^{n} (2i)! n!$$

Using the above lemma we can prove the following.

Theorem: *There exist constants* C, $\overline{K}_1$, $\overline{K}_2 \in \mathbb{R}^+$ *depending only on* $\overrightarrow{f}$ *and* A *such that if* $0 < 2M\varepsilon\overline{K}_1 \leq 1$, *then* $\overrightarrow{w}(x)$ *given by* (21), *satisfies*

$$\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} \leq C\overline{K}_2^n n! \;\forall\, n \in \mathbb{N}_0.$$

By the previous theorem we have that the boundary layer functions, defined by (17)–(18), are bounded independently of ε for $x \in \partial\Omega$. The following theorem gives bounds on the boundary layer part $\overrightarrow{u}^-$, valid for all $x \in \overline{\Omega}$. Analogous bounds hold for $\overrightarrow{u}^+$.

Theorem: *Let* $\overrightarrow{u}^-$ *be the solution of* (17). *Then there exist constants* C, $K > 0$ *independent of* ε *and* n *such that for any* $x \in \overline{\Omega}$, $n \in \mathbb{N}_0$,

$$\left| \left(\overrightarrow{u}^- \right)^{(n)}(x) \right| \leq CK^n e^{-x\alpha/\varepsilon} \max\{n, \varepsilon^{-1}\}^n.$$

The final result of this section shows that the remainder $\overrightarrow{r}$ is small, provided $2M\varepsilon K_1 < 1$ (i.e. $2M\varepsilon$ is small). In the complementary case the asymptotic expansion (20) looses its meaning.

Theorem: *There are constants* C, K_1, $K_2 > 0$ *depending only on the input data such that the remainder* $\overrightarrow{r}$ *defined by* (24) *satisfies*

$$\|\overrightarrow{r}\|_{E,\Omega} \leq CK_2\varepsilon^2 \left(2M\varepsilon K_1\right)^{2M}.$$

3.2. The case $0 < \varepsilon < \mu < 1$

This is, arguably, the most challenging (and interesting) case, because while both compoments of $\overrightarrow{u}$ will have a boundary layer of width $O\left(|\mu \ln \mu|\right)$, the first component $u_1(x)$ will have an *additional sublayer* of width $O\left(|\varepsilon \ln \varepsilon|\right)$. The decomposition (16) in this case is obtained by inserting the format ansatz

$$\overrightarrow{u}(x) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \varepsilon^i \mu^j \overrightarrow{u}^{[i,j]}(x), \tag{25}$$

into the differential equation (1), and equating like powers of ε and μ, so that we can define the smooth part $\overrightarrow{w}$ as

$$\overrightarrow{w}(x) := \sum_{i=0}^{M}\sum_{j=0}^{M} \varepsilon^{2i} \mu^{2j} \overrightarrow{u}^{[2i,2j]}(x), \tag{26}$$

where the terms $\overrightarrow{u}^{[2i,2j]}$ are defined recursively by

$$\overrightarrow{u}^{[0,0]} = A^{-1}\overrightarrow{f}, \tag{27}$$

$$\overrightarrow{u}^{[2i,0]} = A^{-1}\begin{bmatrix} \left(u_1^{[2i-2,0]}\right)'' \\ 0 \end{bmatrix}, \qquad \overrightarrow{u}^{[0,2j]} = A^{-1}\begin{bmatrix} 0 \\ \left(u_2^{[0,2j-2]}\right)'' \end{bmatrix}, \tag{28}$$

$$\overrightarrow{u}^{[2i,2j]} = A^{-1}\begin{bmatrix} \left(u_1^{[2i-2,2j]}\right)'' \\ \left(u_2^{[2i,2j-2]}\right)'' \end{bmatrix}, \; i,j = 1,2,\ldots \tag{29}$$

A calculation shows that

$$L(\overrightarrow{u}-\overrightarrow{w}) = \varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}\begin{bmatrix}\left(u_1^{[2M,2j]}\right)''\\ 0\end{bmatrix} + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}\begin{bmatrix}0\\ \left(u_2^{[2i,2M]}\right)''\end{bmatrix}, \tag{30}$$

hence, as ε, $\mu \to 0$, $\overrightarrow{w}(x)$ defined by (26) satisfies the differential equation, but not the boundary conditions, something that is corrected by the boundary layer functions $\overrightarrow{u}^{\pm}$ defined by (17), (18). Finally, we define the remainder $\overrightarrow{r}$ by

$$\begin{aligned} L\overrightarrow{r} &= \varepsilon^{2M+2}\sum_{j=0}^{M}\mu^{2j}\begin{bmatrix}\left(u_1^{[2M,2j]}\right)''\\ 0\end{bmatrix} + \mu^{2M+2}\sum_{i=0}^{M}\varepsilon^{2i}\begin{bmatrix}0\\ \left(u_2^{[2i,2M]}\right)''\end{bmatrix}\\ \overrightarrow{r}(0) &= \overrightarrow{r}(1) = \overrightarrow{0}. \end{aligned} \tag{31}$$

The behavior of the terms in the decomposition of $\overrightarrow{u}$ is analogous to the previous case, and is summarized by the results that follow.

Lemma: *Let* $\overrightarrow{u}^{[2i,2j]}$ *be defined as in* (27)–(29). *Then there exist constants* $C, K_1, K_2 > 0$ *depending only on* $\overrightarrow{f}$ *and* A *such that for all* $i, j, n \in \mathbb{N}_0$

$$\left\| \left(\overrightarrow{u}^{[2i,2j]}\right)^{(n)} \right\|_{\infty,\Omega} \leq C K_1^{2i+2j} K_2^n (2i+2j)! n!.$$

The next result, which is obtained with the help of the above lemma, bounds the derivatives of $\overrightarrow{w}$, independently of ε and μ. It also shows that the boundary layer functions defined by (17)–(18) are bounded independently of ε and μ for $x \in \partial\Omega$.

Lemma: *There exist constants* C, $\overline{K}_1$, $\overline{K}_2 \in \mathbb{R}^+$ *depending only on* $\overrightarrow{f}$ *and* A *such that if* $4M\mu\overline{K}_1 < 1$, $\overrightarrow{w}(x)$ *given by* (26), *satisfies*

$$\left\| \overrightarrow{w}^{(n)} \right\|_{\infty,\Omega} \leq C\overline{K}_2^n n! \ \forall\, n \in \mathbb{N}_0.$$

We now present bounds on the boundary layer part $\overrightarrow{u}^-$ valid for all $x \in \overline{\Omega}$. The bounds for $\overrightarrow{u}^+$ can be derived in an analogous way.

Theorem: *Let* $\overrightarrow{u}^-$ *be the solution of* (17). *Then there exist constants* C, $K > 0$ *independent of* ε, μ *and* n *such that for any* $x \in \overline{\Omega}$,

$$\left|\left(u_1^-\right)(x)\right| \leq Ce^{-\alpha x/\mu}\ ,\ \left|\left(u_2^-\right)(x)\right| \leq Ce^{-\alpha x/\mu},$$

$$\left|\left(u_1^-\right)^{(n)}(x)\right| \leq CK^n\left(e^{-x\alpha/\varepsilon}\max\{n,\varepsilon^{-1}\}^n + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right),\ \forall\, n \in \mathbb{N},$$

$$\left|\left(u_2^-\right)'(x)\right| \leq C\mu^{-1}e^{-\alpha x/\mu}\ ,\ \left|\left(u_2^-\right)''(x)\right| \leq C\mu^{-2}e^{-\alpha x/\mu},$$

$$\left|\left(u_2^-\right)^{(n)}(x)\right| \leq CK^n\left(e^{-x\alpha/\varepsilon}\max\{n^n,\mu^{-2}\varepsilon^{-n+2}\} + e^{-x\alpha/\mu}\max\{n,\mu^{-1}\}^n\right)\ \forall\, n = 3, 4, \ldots.$$

The following result establishes that the boundary layer functions can be separated into a part that depends on ε and a part that depends on μ. This was also shown by Madden and Stynes, where derivative growth estimates were established for the first three derivatives of $\overrightarrow{u}^-$. Our result extends that of Madden and Stynes and gives estimates valid for a much higher number of derivatives, something that is needed for our approximation.

Theorem: *For each* $n = 1, 2, \ldots, q \in \mathbb{N}$*, with* $q < \alpha(\mu - \varepsilon)/(\varepsilon\mu \ln(\mu/\varepsilon))$*, there exist functions* $u^-_{1,\varepsilon}$*,* $u^-_{1,\mu}$*,* $u^-_{2,\varepsilon}$ *and* $u^-_{2,\mu}$ *such that*

$$u_1^-(x) = u^-_{1,\varepsilon}(x) + u^-_{1,\mu}(x), \qquad u_2^-(x) = u^-_{2,\varepsilon}(x) + u^-_{2,\mu}(x),$$

with

$$\left|\left(u^-_{1,\varepsilon}\right)^{(j)}(x)\right| \le CK^j \varepsilon^{-j} e^{-x\alpha/\varepsilon}\ ,\ j = 1, \ldots, n,$$

$$\left|\left(u^-_{2,\varepsilon}\right)^{(j)}(x)\right| \le CK^j \mu^{-2} \varepsilon^{-j+2} e^{-x\alpha/\varepsilon}\ ,\ j = 1, \ldots, n,$$

$$\left|\left(u^-_{1,\mu}\right)^{(n)}(x)\right| \le CK^n \mu^{-n} e^{-x\alpha/\mu},$$

$$\left|\left(u^-_{2,\mu}\right)^{(n)}(x)\right| \le CK^n \mu^{-n} e^{-x\alpha/\mu}\ ,\ \textit{provided } n > 2,$$

for all $x \in \overline{\Omega}$*, where the constants* $C, K > 0$ *are independent of* ε *and* μ*.*

The final theorem of this section gives bounds on the remainder $\overrightarrow{r}$ in terms of μ, the order M of the asymptotic expansion (16) and the input data. In particular, it shows that $\overrightarrow{r}$ is small provided $4M\mu$ is small – in the case when $4M\mu$ is large the asymptotic expansion is not meaningful.

Theorem: *There exists constants* $C, K_1 > 0$ *depending only on the input data* A *and* $\overrightarrow{f}$ *such that if* $4M\mu K_1 < 1$*, the remainder* $\overrightarrow{r}$ *defined by* (31) *satisfies*

$$\|\overrightarrow{r}\|_{E,\Omega} \le C\mu^2 \left(4M\mu K_1\right)^{2M}.$$

4. FINITE ELEMENT APPROXIMATION

In this section we describe the specific choice of the subspace S_N, which will allow us to approximate the solution of (11) at an exponential rate.

Let $\Delta = \{0 = x_0 < x_1 < \cdots < x_{\mathcal{M}} = 1\}$ be an arbitrary partition of $\Omega = (0, 1)$ and set

$$I_j = (x_{j-1}, x_j)\,,\ h_j = x_j - x_{j-1},\ j = 1, \ldots, \mathcal{M}.$$

Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the j^{th} element I_j by the linear mapping

$$x = Q_j(t) = \frac{1}{2}(1 - t)\, x_{j-1} + \frac{1}{2}(1 + t)\, x_j.$$

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\le p$ on I_{ST}, we define our finite dimensional subspaces as

$$S_N \equiv S^{\overrightarrow{p}}(\Delta) = \left\{u \in H_0^1(\Omega) : u\left(Q_j(t)\right) \in \Pi_{p_j}(I_{ST})\,, j = 1, \ldots, \mathcal{M}\right\}$$

and

$$\overrightarrow{S}{}_0^p(\Delta) := [S^{\overrightarrow{p}}(\Delta) \cap H_0^1(\Omega)]^2, \tag{32}$$

where $\overrightarrow{p} = (p_1, \ldots, p_{\mathcal{M}})$ is the vector of polynomial degrees assigned to the elements.

The definition below describes the mesh used for the method: If we are in the asymptotic range of p, i.e. $p \geq 1/\varepsilon \geq 1/\mu$, then a single element suffices since p will be sufficiently large to give us exponential convergence without any refinement. If we are in the pre-asymptotic range of p then the mesh consists of either five or three elements as described below. We should point out that this is the *minimal* mesh-degree combination for attaining exponential convergence; obviously, refining within each element will retain the convergence rate but would require more degrees of freedom – one such example is the so-called *geometrically graded* mesh discussed by Melenk for the corresponding scalar problem.

Definition: *For $\kappa > 0$, $p \in \mathbb{N}$ and $0 < \varepsilon \leq \mu < 1$, define the spaces $\overrightarrow{S}(\kappa, p)$ of piecewise polynomials by*

$$\overrightarrow{S}(\kappa, p) := \begin{cases} \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, 1\} & \text{if } \kappa p\varepsilon \geq \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, \kappa p\mu, 1-\kappa p\mu, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\mu < \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \kappa p\varepsilon < \frac{1}{2}\ \&\ \kappa p\mu \geq \frac{1}{2}, \\ \overrightarrow{S}_0^p(\Delta);\ \Delta = \{0, \kappa p\varepsilon, 1-\kappa p\varepsilon, 1\} & \text{if } \varepsilon = \mu\ \&\ \kappa p\varepsilon < \frac{1}{2}, \end{cases}$$

In all cases above the polynomial degree is uniformly p on all elements.

We now state our main result.

Theorem: *Let $\overrightarrow{f}$ and A be composed of functions that are analytic on $\overline{\Omega}$ and satisfy the conditions in* (4), (5), (13), (14). *Let $\overrightarrow{u} = [u_1, u_2]^T$ be the solution to* (1), (3), *with $0 < \varepsilon \leq \mu < 1$. Then there exist constants $\kappa, C, \beta > 0$ depending only on $\overrightarrow{f}$ and A such that there exists $\mathcal{I}_p\overrightarrow{u} = [\mathcal{I}_p u_1, \mathcal{I}_p u_2]^T \in \overrightarrow{S}(\kappa, p)$ with $\mathcal{I}_p\overrightarrow{u} = \overrightarrow{u}$ on $\partial\Omega$ and*

$$\|\overrightarrow{u} - \mathcal{I}_p\overrightarrow{u}\|_{E,\Omega}^2 \leq Cp^3 e^{-\beta p},$$

as $p \to \infty$.

The proof of the above theorem uses the regularity results discussed in the previous section. In particular, we decompose $\overrightarrow{u}$ as in (16) and we approximate each component separetely. Since $\overrightarrow{w}$ is smooth, its approximation by (piecewise) polynomials of degree p will converge at an exponential rate as $p \to \infty$, independently of the mesh used. For the approximation of the two boundary layer parts we must use the three or five element mesh (depending on the relationship between ε and μ) as follows: over the (thin) elements near the boundary we approximate the boundary layers with polynomials of degree p and over the (long) element in the middle of the domain we approximate them by piecewise linears. This yields an approximation that captures the layer effects for both scales, and converges at an exponential rate $p \to \infty$. Finally, the remainder $\overrightarrow{r}$ is not approximated at all, since its energy norm is sufficiently small. The triangle inequality then allows us to obtain the desired result.

Now, using the above theorem and the quasioptimality result (12) we have the following.

Corollary: *Let* $\overrightarrow{u}$ *be the solution to* (1), (3), *with* $0 < \varepsilon \leq \mu < 1$, *and let* $\overrightarrow{u}_{FE} \in \overrightarrow{S}_0^p(\Delta)$ *be the solution to* (11). *Then exist constants* $\kappa, C, \sigma > 0$ *depending only on the input data* $\overrightarrow{f}$ *and* A *such that as* $p \to \infty$

$$\|\overrightarrow{u} - \overrightarrow{u}_{FE}\|_{E,\Omega} \leq Cp^{3/2}e^{-\sigma p}.$$

The above result shows that as $p \to \infty$ the method converges at an exponential rate, independently of the singular perturbation parameters ε and μ, when the error is measured in the energy norm.

6. NUMERICAL RESULTS

In this section we present the results of numerical computations for two model problems, having as our goals the illustration of the theoretical findings and the comparison of the proposed method with other finite element schemes found in the literature.

6.1. The constant coefficient case. First we consider a constant coefficient problem, in which

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \overrightarrow{f}(x) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \overrightarrow{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is available, hence the computations we report are reliable. We will comparing the following methods:

- The h version on a uniform mesh, with $p = 1, 2$ and 3.
- The p version on a single element.
- The hp version we are proposing.
- The h version on the piecewise uniform Shishkin mesh, with $p = 1, 2$ and 3.
- The h version on a non-uniform (exponentially graded) mesh which includes refinement near the endpoints of the domain, with $p = 1, 2$ and 3.

We expect that the first two methods will not be robust, while the last three will – the Shishkin mesh is quite popular for this type of problem mainly for its simple nature. We will be plotting the percentage relative error in the energy norm, given by

$$100 \times \frac{\|\overrightarrow{u}_{EXACT} - \overrightarrow{u}_{FEM}\|_{E,\Omega}}{\|\overrightarrow{u}_{EXACT}\|_{E,\Omega}}, \tag{33}$$

versus the number of degrees of freedom N, on a log-log scale.

Figure 1 shows the error when $\varepsilon = 0.1$ and $\mu = 1$. Since these values are "large," we see that the h version on a uniform mesh performs sufficiently well (with

$O(N^{-p})$ convergence rate), and the p version on a single element yields exponential convergence.

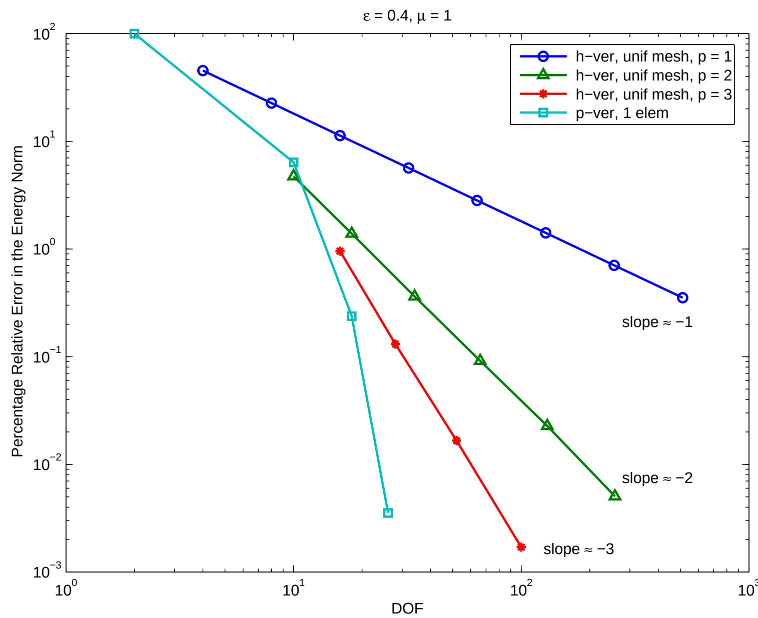

FIGURE 1. Energy norm convergence for $\varepsilon = 0.4$ and $\mu = 1$.

In figure 2, which corresponds to $\varepsilon = 0.01$ and $\mu = 0.1$, we see the performance of the aforementioned methods beginning to deteriorate, even though the p version still converges exponentially due to the fact that we continue to have $p \approx \varepsilon^{-1}$.

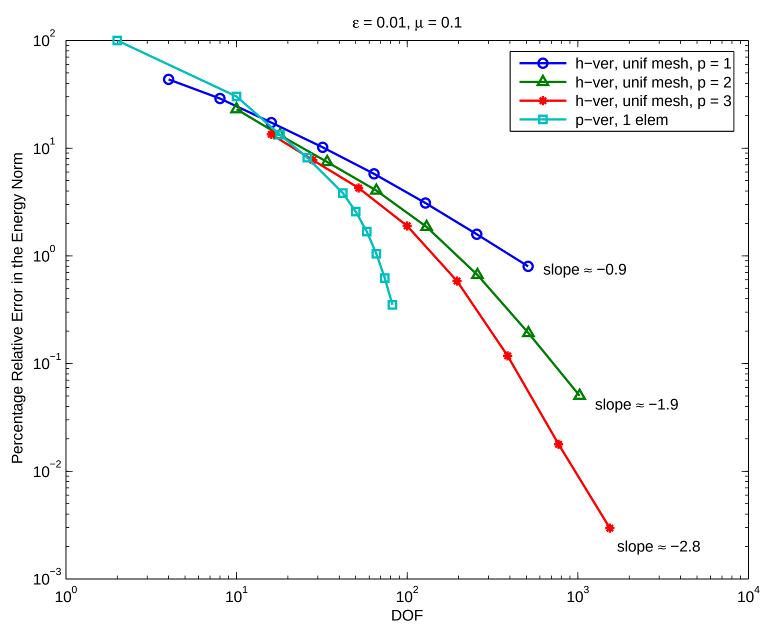

FIGURE 2. Energy norm convergence for $\varepsilon = 0.01$ and $\mu = 0.1$.

The deterioration of these methods is seen in figure 3 which corresponds to $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$ and $\mu = 0.01$. We observe that the h version on a uniform mesh converges at the rate $O\left(N^{-0.7}\right)$, independently of the polynomial degree used, while the p version no longer converges exponentially, but at the algebraic rate $O\left(N^{-1.5}\right)$ which is roughly twice that of the h version.

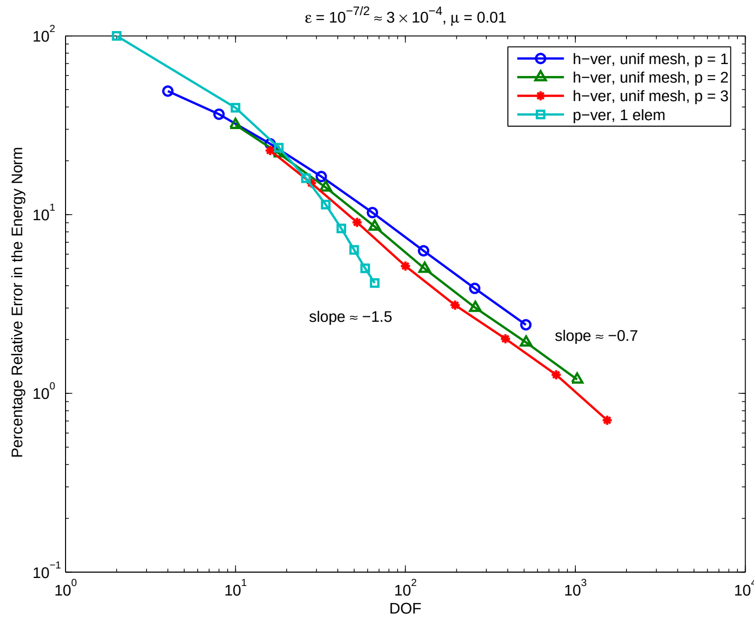

FIGURE 3. Energy norm convergence for $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$ and $\mu = 0.01$.

For these values of ε and μ, we show in figure 4 the behavior of all the methods (showing only the case of $p = 1$ for the uniform h version, based on our previous remark). This figure clearly indicates the robustness of the h version on the Shishkin and exponential meshes, as well as the exponential convergence of the hp version.

Figure 5 shows the behavior of the robust h versions with different polynomial degrees. We see that the exponential mesh produces the optimal algebraic rate $O\left(N^{-p}\right)$, while the Shishkin mesh yields the (usual) quasi-optimal rate $O\left(N^{-p} \ln N\right)$, with the logarithmic term not removeable. For completeness, we have included the hp version to see how it compares with the robust h versions, and we see that only the h version on the exponential mesh is comparable to it.

The situation remains the same as ε and μ decrease even further, as is shown in figure 6, which corresponds to $\varepsilon = 10^{-5}$ and $\mu = 10^{-3}$ (smaller values for ε and μ produce almost identical results).

6.2. The variable coefficient case. Next, we consider a variable coefficient problem, in which

$$A = \begin{bmatrix} 2\left(x+1\right)^2 & -\left(1+x^2\right) \\ -2\cos(\pi x/4) & 2.2e^{1-x} \end{bmatrix}, \overrightarrow{f}(x) = \begin{bmatrix} 2e^x \\ 10x+1 \end{bmatrix}, \overrightarrow{u}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

An exact solution is not available, and for our computations we use a reference solution obtained with polynomials of degree 8 on a very fine mesh which is exponentially

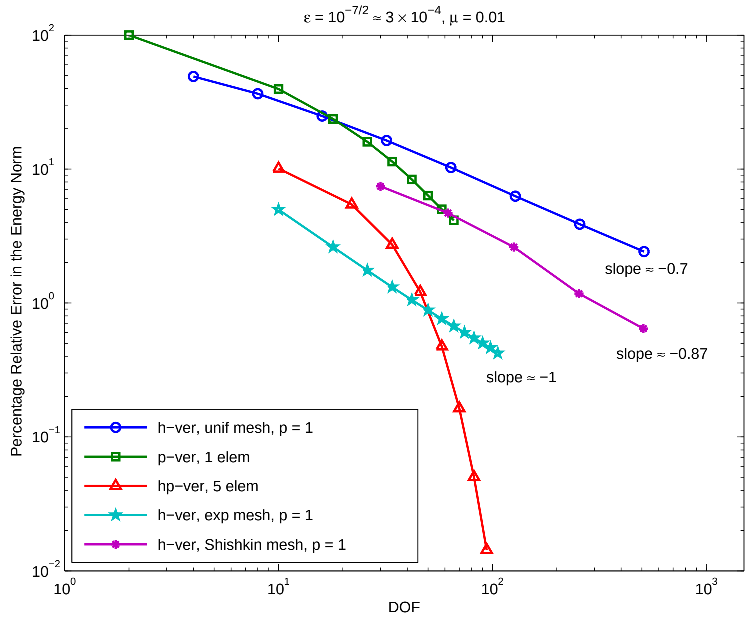

FIGURE 4. Energy norm convergence for $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$ and $\mu = 0.01$.

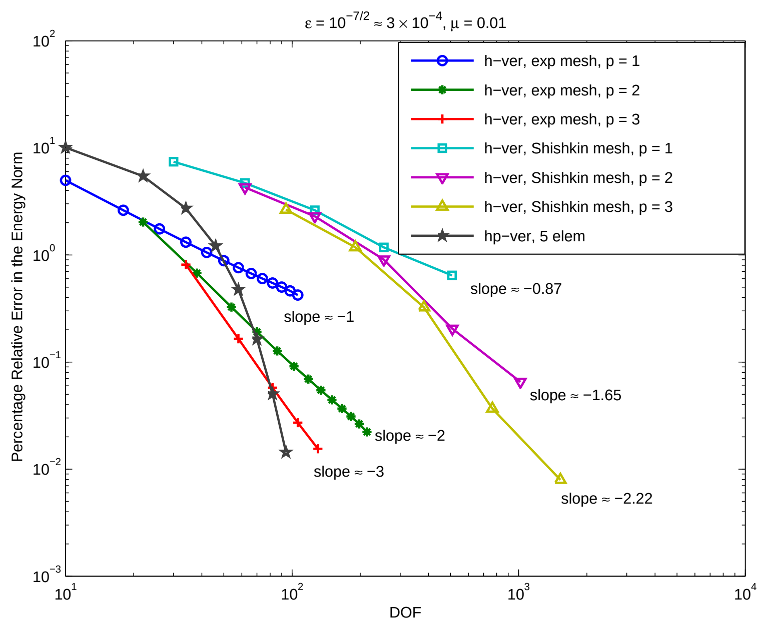

FIGURE 5. Energy norm convergence for $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$ and $\mu = 0.01$.

graded near the boundary and uniform in the middle of the domain. Since this is a variable coefficient problem, in our computations we add uniform refinement in the middle of the domain to the exponential mesh and we refer to it as a "modified exponential mesh."

As in the previous example, we are interested in the percentage relative error in the energy norm (33). However, given the results obtained for the constant coefficient

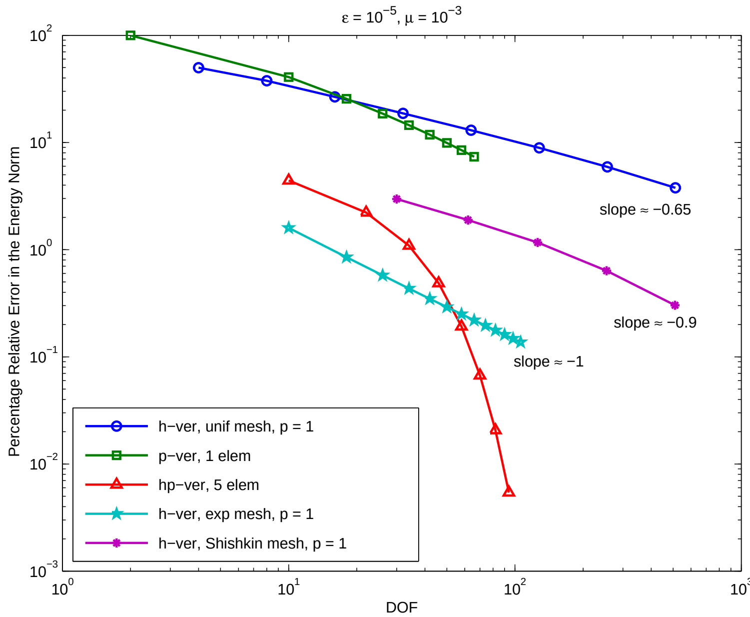

FIGURE 6. Energy norm convergence for $\varepsilon = 10^{-5}$ and $\mu = 10^{-3}$.

case, we now focus our attention only on the methods which are converging uniformly in ε and μ, namely:

- The hp version we are proposing.
- The h version on a Shishkin mesh, with $p = 1, 2$ and 3.
- The h version on the modified exponential mesh,with $p = 1, 2$ and 3.

Figures 7 and 8 show the energy norm convergence of the above three methods for $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$, $\mu = 0.01$ and $\varepsilon = 10^{-5}$, $\mu = 10^{-3}$, respectively. The results are almost identical to the constant coefficient case and they indicate, once more, the robust algebraic convergence rates for the h versions and the exponential convergence rate for the hp version.

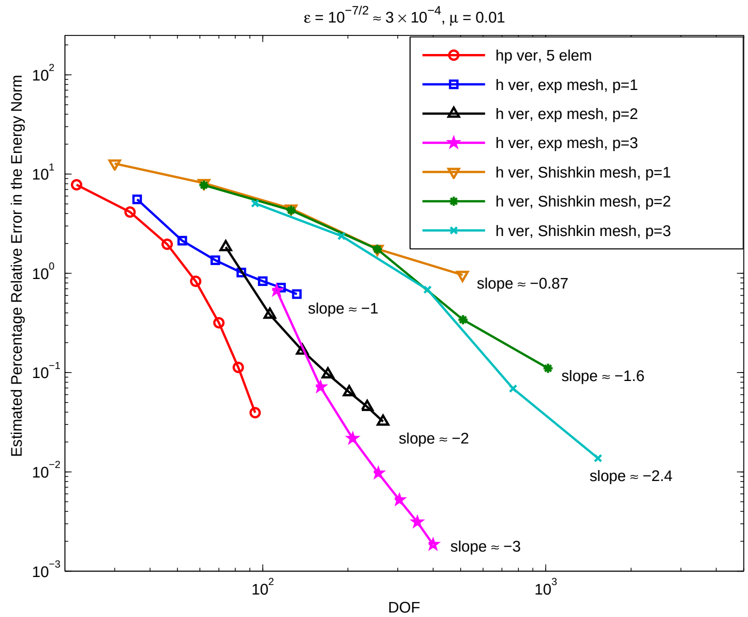

FIGURE 7. Energy norm convergence for $\varepsilon = 10^{-7/2} \approx 3 \times 10^{-4}$ and $\mu = 0.01$.

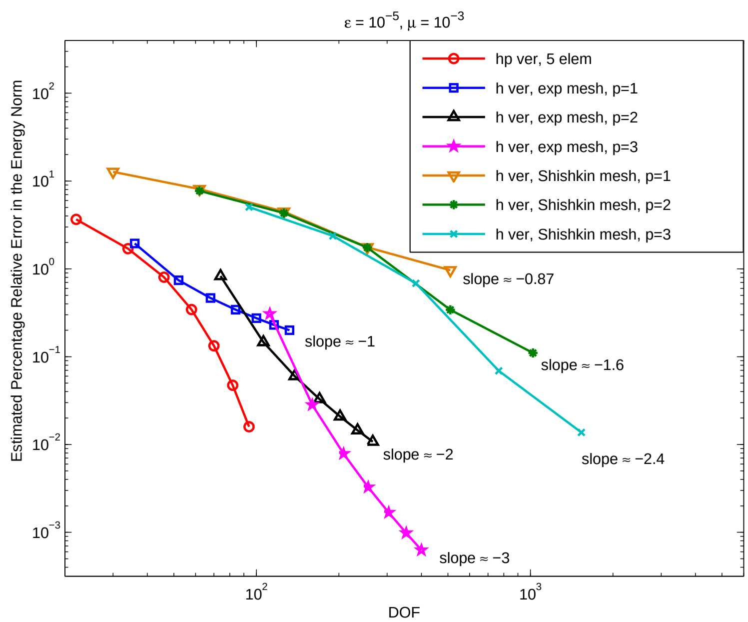

FIGURE 8. Energy norm convergence for $\varepsilon = 10^{-5}$ and $\mu = 10^{-3}$.

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