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AN OPTIMIZED B-SPLINE METHOD FOR SOLVING SINGULARLY PERTURBED DIFFERENTIAL DIFFERENCE EQUATIONS WITH DELAY AS WELL AS ADVANCE

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ABSTRACT. The aim of this work is to present a numerical technique to approximate the solution of boundary value problems for singularly perturbed differential difference equations with delay as well as advance. Such type of problems are the ubiquitous in the mathematical modeling of various practical phenomena in biology and physics, such as in variational problems in control theory and first exit time problems in the modeling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites. Here, we present a second order convergent numerical scheme based on B-spline collocation. Analysis has also been carried out to establish the error estimate which shows that the method converges quadraticly. Finally, to support the predicted theory and to demonstrate the efficiency of the proposed method several numerical experiments are carried out and a comparison is made with existence method.

Key Words. B-spline, differential-difference equation, singularly perturbed problems.

1. INTRODUCTION

The determination of the expected time for the generation of action potentials in nerve cells by random synaptic inputs in the dendrites can be modeled as a first-exit time problem. In Stein's model, the distribution representing inputs is taken as a Poisson process with exponential decay. If in addition, there are inputs that can be modeled as a Wiener process with variance parameter σ and drift parameter μ, then the problem for expected first-exit time y, given initial membrane potential $x \in (x_1, x_2)$, can be formulated as a general boundary value problem for the linear second order differential difference equation [1, 2]

$$\frac{\sigma^2}{2}u''(x) + (\mu - x)u'(x) + \lambda_E u(x + a_E) + \lambda_I u(x - a_I) - (\lambda_E + \lambda_I)u(x) = -1,$$

where the values $x = x_1$ and $x = x_2$ correspond to the inhibitory reversal potential and to the threshold value of membrane potential for action potential generation, respectively. σ and μ are the variance and drift parameters, respectively, u is the expected first-exit time and the first order derivative term $-xu'(x)$ corresponds to exponential decay between synaptic inputs. The undifferentiated terms correspond to excitatory

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and inhibitory synaptic inputs, modeled as a Poisson process with mean rates λ_E and λ_I, respectively and produce jumps in the membrane potential of amounts a_E and a_I, respectively which are small quantities and could depend on the voltage. The boundary condition is

$$u(x) \equiv 0, \quad x \notin (x_1, x_2),$$

This biological problem motivates the study of boundary value problems for singularly perturbed differential difference equations with delay as well as advance. To approximate the solution of this class of differential equation, one encounter with two major difficulties, namely, i) due to presence of the small singular pertubation parameter which is multiplied to the highest order derivative term and ii) due to existence of delay and advance parameters in the argument of reaction terms. To deal with the first difficulty, there are two approaches, namely asymptotic and numerical. Here, we adopt the second approach *i.e.* numerical approach. When the singular perturbation parameter tends to zero, a breakdown occurs and the solution of the singularly perturbed problem often behave analytically quite differently from a solution of the original equation in the narrow region of the domain. The solution changes rapidly and form boundary or transition layers in these narrow regions. Due to the singular behavior of the solution of the problem in these regions, the classical numerical schemes are found to be inadequate to approximate the solution of the singularly perturbed problems. To come out with this problem, there are two approaches in the literature one is fitted operator approach in which one replace the discretize standard operator by an operator which absorb the singular behavior of the solution in the narrow regions where the solution changes exponentially. The second approach is fitted mesh in which to capture the behavior of the solution in these narrow regions, a piecewise uniform is generated in such a way that it is fine in these narrow regions and coarse in the outer region where the solution is smooth. To deal with singular perturbation parameter, we use the second approach because it is always not possible to construct a parameter uniform numerical scheme for each problem based on fitted operator approach. There are some problems for which one can n't construct parameter uniform numerical scheme based fitted operator approach while for the same one can construct parameter uniform scheme based on fitted mesh approach [3]. To tackle with the second difficulty, we use Taylor series approximation for the terms containing delay as well as advance provided the delay and advance parameter are sufficiently small. The numerical study of such type of boundary value problem is initiated in paper[6] where the authors presented a first order convergent numerical scheme based on finite difference. Here, we present a second order convergent scheme based on B-spline collocation method. The proposed method produce a spline function which gives the solution at every point of the domain, whereas the scheme based on finite difference gives solution at the chosen knots. Quintic splines have also been discussed for singularly perturbed problems in [7] . In section 5, a comparative study is made in the form of maximum absolute errors by carrying out several numerical experiments. When the delay is zero, the above boundary value problem reduces to a singularly perturbed ordinary differential equation of the convection-diffusion type and the solution of the differential equation so obtained exhibits layer behavior.

2. DESCRIPTION OF THE PROBLEM

In this section, we state the boundary value problems for a class of singularly perturbed differential difference equations of the convection diffusion type with small delay

$$\varepsilon u''(x) + a(x)u'(x) + \alpha(x)u(x-\delta) + \omega(x)u(x) + \beta(x)u(x+\eta) = f(x), \tag{2.1}$$

$\forall\, x \in \Omega$ and subject to interval conditions

$$\begin{aligned} u(x) &= \phi(x) \quad \text{on } -\delta \le x \le 0, \\ u(x) &= \gamma(x) \quad \text{on } 1 \le x \le 1+\eta, \end{aligned} \tag{2.2}$$

where ε is a small parameter, $0 < \varepsilon \ll 1$ and δ and η are of $o(\varepsilon)$. $a(x)$ $\alpha(x)$, $\beta(x)$, $w(x)$, $f(x)$, $\phi(x)$ $\gamma(x)$ are smooth functions. For a function $u(x)$ to be a smooth solution to the problem (2.1), (2.2), it must satisfy (2.1), (2.2), be continuous on $\overline{\Omega} = [0,1]$ and be continuously differentiable on $\Omega = (0,1)$. It is also assumed that $\alpha(x)$, $\beta(x)$ and $w(x)$ satisfies the condition

$$\alpha(x) + \beta(x) + w(x) \le -\theta < 0 \qquad \forall x \in \overline{\Omega}, \tag{2.3}$$

where θ is a positive constant. Throughout the thesis, C, M and θ denote generic positive constants that are independent of ε and in the case of discrete problems, also independent of the mesh parameter N. Where the value of C may change from result to result but remains constant in each. $\|.\|$ denotes the global maximum norm over the appropriate domain of the independent variable.

3. NUMERICAL METHODS

3.1. **Analytical Results.**

Since the solution of the boundary value problem (2.1), (2.2) is sufficiently differentiable, so by Taylor's series we have

$$\varepsilon u''(x) + (a(x) + \beta(x)\eta - \alpha(x)\delta)u'(x) + (\alpha(x) + \beta(x) + \omega(x))u(x) = f(x), \tag{3.1}$$

$$u(0) = \phi_0, \quad \phi_0 = \phi(0), \tag{3.2a}$$

$$u(1) = \gamma_1, \quad \gamma_1 = \gamma(1), \tag{3.2b}$$

which differ from the original problem (2.1), (2.2) by terms of $O(\delta^2 u'', \eta^2 u'')$. Here, we assume shifts, *i.e.*, δ and η are sufficiently small, so the solution u of the problem (3.1), (3.2) will provide a good approximation to the solution u of the problem (2.1), (2.2). The differential operator L_ε corresponding to the boundary value problem (3.1), (3.2) is defined by

$$L_\varepsilon u(x) = \varepsilon u''(x) + (a(x) + \beta(x)\eta - \alpha(x)\delta)u'(x) + (\alpha(x) + \beta(x) + \omega(x))u(x).$$

Lemma 3.1. *Minimum Principle: Suppose Ψ is a smooth function satisfying $\Psi(0) \geq 0$, $\Psi(1) \geq 0$. Then $L_\varepsilon \Psi(x) \leq 0$ for all $x \in \Omega$ implies $\Psi(x) \geq 0$ for all $x \in \overline{\Omega}$.*

Proof. See [4] □

Lemma 3.2. *Let $u(x)$ be the solution of the problem* (3.1), (3.2) *then we have*

$$\|u\| \leq \theta^{-1}\|f\| + \max(|\phi_0|, |\gamma_1|).$$

Proof. Let us define two barrier functions $\xi^{\pm}$ as

$$\xi^{\pm}(x) = \theta^{-1}\|f\| + \max(|\phi_0|, |\gamma_1|) \pm u(x).$$

Clearly both the barrier functions $\xi^{\pm}(x)$ are non-negative at the boundary (*i.e.*, 0 and 1), now consider

$$\begin{aligned} L_\varepsilon \xi^{\pm}(x) &= (\alpha(x) + \beta(x) + w(x))(\theta^{-1}\|f\| + \max(|\phi_0|, |\gamma_1|)) \pm L_\varepsilon u(x) \\ &= (\alpha(x) + \beta(x) + w(x))[\theta^{-1}\|f\| + \max(|\phi_0|, |\gamma_1|)] \pm f(x) \\ &\leq (-\|f\| \pm f(x)) + (\alpha(x) + \beta(x) + w(x))\max(|\phi_0|, |\gamma_1|) \ \text{ using Inequality (2.3)} \\ &\leq 0, \ \text{ using definition of norm } \|.\|. \end{aligned}$$

Thus, an application of Lemma 3.1 proves the required estimate. □

Theorem 3.3. *Suppose $(a(x) + \beta(x)\eta - \alpha(x)\delta) \leq -M < 0$ where M is a positive constant, then the derivatives of the solution of the problem* (3.1), (3.2) *satisfies the following estimates*

$$|u^k(x)| \leq C[1 + \varepsilon^{-k}E(x, M)], \quad 0 < x < 1, \ \ k = 1, 2, \ldots.$$

$$E(x, M) = \exp\left(\frac{-M(1-x)}{\varepsilon}\right)$$

Proof. The differential equation (3.1) can be written as

$$\varepsilon u''(x) + (a(x) + \beta(x)\eta - \alpha(x)\delta)u'(x) = g(x), \tag{3.3}$$

where $g(x) = h(x) - (\alpha(x) + \beta(x) + \omega(x))u(x)$ On integrating Eq. (3.3) twice, yields

$$u(x) = u_p(x) + \zeta_1 + \zeta_2 \int_x^1 \exp(\varepsilon^{-1}(A(1) - A(z)))dz,$$

where

$$u_p(x) = -\int_x^1 \chi(t)dt,$$
$$\chi(x) = -\int_x^1 \varepsilon^{-1}h(t)\exp(\varepsilon^{-1}(A(t)-A(x)))dt,$$
$$A(x) = \int_0^x (a(t)+\beta(t)\eta-\alpha(t)\delta)dt,$$

and the constants of integration (ζ_1 and ζ_2) may depend on ε. Now, we apply the boundary conditions to find out the constants ζ_1 and ζ_2. The boundary condition $u(1)=\gamma_1$ yields $\zeta_1=\gamma_1$ and the boundary condition $u(0)=\phi_0$ gives

$$\zeta_2\int_0^1 \exp(\varepsilon^{-1}(A(1)-A(t)))dt = -u_p(0)+\gamma_1.$$

The stability estimate (3.2) implies

$$|\chi(x)| \le C\varepsilon^{-1}\int_x^1 \exp(\varepsilon^{-1}(A(t)-A(x)))dt,$$

using the following inequality
$\exp(\varepsilon^{-1}(A(t)-A(x))) \le \exp(-M\varepsilon^{-1}(t-x))$, for $x\le t$;
yields

$$|\chi(x)| \le C\varepsilon^{-1}\int_x^1 \exp(-M\varepsilon^{-1}(t-x))dt \le C.$$

Hence $|u_p(0)|\le C$. Set $\bar{A}=\max_{x\in[0,1]}(a(x)+\beta(x)\eta-\alpha(x)\delta)$. Then

$$\zeta_2\int_0^1 \exp(\varepsilon^{-1}(A(1)-A(t)))dt \ge \int_0^1 \exp(\bar{A}\varepsilon^{-1}(1-t))dt \ge C\varepsilon.$$

It now follows that

$$|\zeta_2| \le \bar{C}\varepsilon^{-1}.$$

where $\bar{C}=C+|\gamma_1|+|\phi_0|$. Thus finally, we obtain

$$u'(x) = \chi(x) - \zeta_2\exp(\varepsilon^{-1}(A(1)-A(x)))$$

which implies

$$|u'(x)| \le \bar{C}\left(1+\varepsilon^{-1}\exp\left(-M\frac{(1-x)}{\varepsilon}\right)\right).$$

The proof for $k>1$ follows by induction and repeated differentiation of equation (3.1) on the similar steps as did by Miller et.al [3]. □

3.2. **Formulation of B-spline Collocation.** In this section, we formulate a B-spline collocation approximation to the solution of the problem (3.1), (3.2). Let $\pi = \{x_0, x_1, x_2, \ldots, x_N\}$ be a partition of the domain $[0,1]$ such that $x_0 = 0$ and $x_N = 1$, where $x_i = ih$ with uniform mesh parameter $h = 1/N$. Let $X_N =$ span $\{B_{-1}, B_0, B_1, \ldots, B_{n+1}\}$, where $B_i(x)$ be a cubic B-splines with knots at $\pi = \{x_0, x_1, x_2, \ldots, x_N\}$. In particular, we introduce four additional knots $x_{i-2} < x_{i-1} < x_0$ and $x_{N+2} > x_{N+1} > x_N$ and functions $B_i(x)$ defined by

$$B_i(x) = \frac{1}{h}\begin{cases} (x-x_{i-2})^3, \quad \text{if } x \in [x_{i-2}, x_{i-1}], \\ h^3 + 3h^2(x-x_{i-1}) + 3h(x-x_{i-1})^2 - (x-x_{i-1})^3, \\ \text{if } x \in [x_{i-1}, x_i], \\ h^3 + 3h^2(x_{i+1}-x) + 3h(x_{i+1}-x)^2 - (x_{i+1}-x)^3, \\ \text{if } x \in [x_i, x_{i+1}], \\ (x_{i+2}-x)^3, \quad \text{if } x \in [x_{i-2}, x_{i-1}], \\ 0, \quad \text{otherwise.} \end{cases} \tag{3.4}$$

From equation (3.4), it is clear that $B_i(x) \in C^2(\mathbb{R})$. Using collocation with these

TABLE 1. Values of $B_j(x)$ and its derivatives at nodal points

	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$B_j(x)$	0	1	4	1	0
$B'_j(x)$	0	$\frac{3}{h}$	0	$\frac{-3}{h}$	0
$B''_j(x)$	0	$\frac{6}{h^2}$	$\frac{-12}{h^2}$	$\frac{6}{h^2}$	0

approximating functions, we seek

$$U_N(x) = c_{-1}B_{-1} + c_0B_0(x) + \cdots + c_{N+1}B_{N+1} \tag{3.5}$$

such that

$$LU_N(x_j) = f(x_j), \quad 0 \le j \le N \tag{3.6}$$

$$U_N(0) = \phi_0 \tag{3.7a}$$

$$U_N(1) = \gamma. \tag{3.7b}$$

We are collocating at $N+1$ knots and that we have introduced two extra splines B_1 and B_{N+1}, to force $U_N(x)$ to satisfies the same boundary data as $u(x)$. Using equation (3.5) in equation (3.6) and the linearity of the operator L, we obtain

$$\sum_{i=-1}^{N+1} Lc_i B_i(x_j) = f(x_j), \quad 0 \le j \le N \tag{3.8}$$

$$\sum_{i=-1}^{1} c_i B_i(x_0) = \phi_0 \tag{3.9a}$$

$$\sum_{i=N-1}^{N+1} B_i(x_N) = \gamma, \tag{3.9b}$$

Equation (3.8) gives

$$\begin{aligned} \epsilon(c_{i-1}B''_{i-1} + c_i B''_i + c_{i+1}B''_{i+1}) + p_i(c_{i-1}B'_{i-1} + c_i B'_i + c_{i+1}B'_{i+1}) \\ q_i(c_{i-1}B_{i-1} + c_i B_i + c_{i+1}B_{i+1}) = f(x_i) \end{aligned} \tag{3.10}$$

where

$$p(x) = (a(x) + \beta(x)\eta - \alpha(x)\delta), \; q(x) = (\alpha(x) + \beta(x) + \omega(x))$$

Simplifying equation (3.10) we get

$$\begin{aligned} c_{i-1}(6\epsilon - 3p_i h + q_i h^2) + c_i(-12\epsilon + 4q_i h^2) \\ + c_{i+1}(6\epsilon + 3p_i h + q_i h^2)) = f_i, \; \forall i \end{aligned} \tag{3.11}$$

Boundary conditions (3.9) becomes

$$c_{-1} + 4c_0 + c_1 = \phi_0 \tag{3.12a}$$

$$c_{n-1} + 4c_n + c_{n+1} = \gamma. \tag{3.12b}$$

Equations (3.11), (3.12) makes the tridiagonal system which can be solved efficiently using standard methods.

3.3. **Mesh Selection.** We decompose the domain $[0, 1]$ into two non overlapping subdomains and then solve the differential equation (3.1–3.1) subject to the different boundary conditions in each sub domain. Using Shishkin mesh [3] strategy we choose transition parameter τ as

$$\tau = \min\{\frac{1}{2}, C\epsilon \ln N\}, \; C \text{ is a constant} \tag{3.13}$$

We decompose the domain into two subdomain regions as $[0, 1-\tau], [1-\tau, 1]$ if boundary layer occurs at $x = 1$. Now we have to solve two differential equation as one in the regular region $[0, 1-\tau]$

$$\varepsilon u''(x) + (a(x) + \beta(x)\eta - \alpha(x)\delta)u'(x) + (\alpha(x) + \beta(x) + \omega(x))u(x) = f(x), \tag{3.14}$$

$$u(0) = \phi_0, \tag{3.15a}$$

$$u(1-\tau) = u_\tau, \quad u_\tau = u(1-\tau), \tag{3.15b}$$

and another in the boundary layer region $[1-\tau, 1]$ as

$$\varepsilon u''(x) + (a(x) + \beta(x)\eta - \alpha(x)\delta)u'(x) + (\alpha(x) + \beta(x) + \omega(x))u(x) = f(x), \tag{3.16}$$

$$u(1-\tau) = u_\tau, \tag{3.17a}$$

$$u(1) = \gamma_1, \tag{3.17b}$$

To find the boundary condition u_τ at the transition point, we use asymptotic expansion for the problem (3.1), (3.2) that is given as

$$u_{as}(x, \epsilon) = \sum_{j=0}^{m} \epsilon^j u_j(x) + \sum_{k=0}^{m} \epsilon^k v_k(\frac{1-x}{\epsilon}) \tag{3.18}$$

where u_j is the solution of the reduced problem and v_k is the corresponding boundary layer correction.

Theorem 3.4. *If the coefficients and the right-hand side of the boundary value problem* (3.1), (3.2) *are sufficiently smooth, then there exists an asymptotic expansion as given* (3.18) *of the solution u such that* [5]

$$|u(x) - u_{as}(x)| \leq \mathcal{C}\epsilon^{m+1} \quad \forall x \in [0, 1] \text{ and } \epsilon \leq \epsilon_0$$

where $\mathcal{C}$ is independent of x and ϵ, and ϵ_0 is a constant.

Here we consider first order asymptotic expansion with $m = 1$.

4. ERROR ESTIMATES

In this section a procedure is described which will calculate the truncation error for the given method over the whole range $0 \leq x \leq 1$. We assume that $u(x)$ be the function with continuous derivatives in the whole range. Using the following facts

$$\begin{aligned} U_N(x_i) &= c_{i-1} + 4c_i + c_{i+1} \\ U_N'(x_i) &= (-3/h)c_{i-1} + (3/h)c_{i+1} \\ U_N''(x_i) &= (6/h^2)c_{i-1} + (-12/h^2)c_i + (6/h^2)c_{i+1} \end{aligned} \tag{4.1}$$

we get the following relationship:

$$\begin{aligned} h[U_N'(x_{i-1}) + 4U_N'(x_i) + U_N'(x_{i+1})] = h[&c_{i-2}(-3/h) + c_i(3/h) \\ &+ 4(c_{i-1}(-3/h) + c_{i+1}(3/h)) + c_i(-3/h) + c_{i+2}(3/h)] \end{aligned} \tag{4.2}$$

which gives

$$h[U_N'(x_{i-1}) + 4U_N'(x_i) + U_N'(x_{i+1})] = 3[u(x_{i+1}) - u(x_{i-1})] \tag{4.3}$$

Similarly

$$h^2 U_N''(x_i) = 6[U_N(x_{i+1}) - U_N(x_i)] - 2h[2U_N'(x_i) + U_N'(x_{i+1})], \tag{4.4}$$

$$h^3 U_N'''(x_{i+}) = 12[U_N(x_i) - U_N(x_{i+1})] + 6h[U_N'(x_i) + U_N'(x_{i+1})], \tag{4.5}$$

and

$$h^3U_N'''(x_{i-}) = 12[U_N(x_{i-1}) - U_N(x_i)] + 6h[U_N'(x_{i-1}) + U_N'(x_i)], \tag{4.6}$$

where $U_N'''(x_{i+})$ denotes the value of $U_N'''(x)$ in (x_i, x_{i+1}). We know the operators $E(u(x_i)) = u(x_{i+1})$, and using operator notation, equation (4.3) may be written in the form

$$(E^{-1} + 4 + E)hS'(x_i) = 3(E - E^{-1})u(x_i)$$

and hence

$$hU_N'(x_i) = \left\{\frac{3(E - E^{-1})}{(E^{-1} + 4 + E)}\right\} u(x_i).$$

If we now put $E = e^{hD}$ and expand in powers of hD, we obtain

$$U_N'(x_i) = u'(x_i) - \frac{1}{180}h^4u^v(x_i) + O(h^6). \tag{4.7}$$

Similarly (4.4) and (4.5) give

$$U_N''(x_i) = u''(x_i) - \frac{1}{12}h^2u^{iv}(x_i) + \frac{1}{360}h^4u^{vi}(x_i) + O(h^6). \tag{4.8}$$

and

$$U_N'''(x_{i+}) = u'''(x_i) + \frac{1}{2}hu^{iv}(x_i) + \frac{1}{12}h^2u^v(x_i) - \frac{1}{360}h^4u^{vii}(x_i) - \frac{1}{1440}h^5u^{viii}(x_i) + O(h^6). \tag{4.9}$$

From (4.9) we now obtain

$$\frac{1}{2}[U_N'''(x_{i+}) + U_N'''(x_{i-})] = u'''(x_i) + \frac{1}{12}h^2u^v(x_i) + O(h^4). \tag{4.10}$$

and

$$U_N'''(x_{i+}) - U_N'''(x_{i-}) = hu^{iv}(x_i) - \frac{1}{720}h^5u^{viii}(x_i) + O(h^7). \tag{4.11}$$

Equation (4.11) gives a very good estimate of $hu^{iv}(x_i)$ (only for interior points) and we have

$$U_N'''(x_{i+}) - U_N'''(x_{i-}) = t_i = hu^{iv}(x_i) + O(h^5) \tag{4.12}$$

We now define $e(x) = U_N(x) - u(x)$ and substitute (4.7), (4.8), (4.10) and (4.11) in the Taylor series expansion of $e(x_i + \vartheta h)$ obtaining

$$e(x_i + \vartheta h) = \frac{\vartheta^2(\vartheta - 1)^2}{24}h^4u^{iv}(x_i) + \frac{\vartheta(\theta^2 - 1)(3\vartheta^2 - 2)}{360}h^5u^v(x_i) + O(h^6). \tag{4.13}$$

where $0 \le \vartheta \le 1$.

5. COMPUTATIONAL RESULTS

Some numerical examples are considered and solved using the methods presented here. The exact solution of the boundary value problem given by Equations (3.1), (3.2) for constant coefficients, forcing term and interval conditions, *i.e.*, $\alpha(x) = \alpha$, $\beta(x) = \beta$, $a(x) = a$, $\omega(x) = \omega$, $f(x) = f$, $\phi(x) = \phi$ and $\gamma(x) = \gamma$ are constants, then the solution z_ε is given by

$$u(x) = c_1 \exp(m_1 x) + c_2 \exp(m_2 x) + f/c$$

where

$$\begin{aligned} c_1 &= [-f + \gamma c + \exp(m2)(f - \phi c)]/[(\exp(m1) - \exp(m2))c] \\ c_2 &= [f - \gamma c + \exp(m1)(-f + \phi c)]/[(\exp(m1) - \exp(m2))c] \\ m_1 &= [-(a - \alpha\delta + \beta\eta) + \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}]/2\varepsilon, \\ m_2 &= [-(a - \alpha\delta + \beta\eta) - \sqrt{(a - \alpha\delta + \beta\eta)^2 - 4\varepsilon c}]/2\varepsilon \\ c &= (\alpha + \beta + \omega). \end{aligned}$$

Example 1. $a(x) = 1$, $\alpha(x) = 2$, $\beta(x) = 0$, $\omega(x) = -3$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = 1$ in boundary-value problem (1.2), (1.3)

Example 2. $a(x) = 1$, $\alpha(x) = 0$, $\beta(x) = 2$, $\omega(x) = -3$, $\phi(x) = 1$, $f(x) = 0$, $\gamma(x) = 1$ in boundary-value problem (1.2), (1.3).

Example 3. $a(x) = 1$, $\alpha(x) = -2$, $\beta(x) = 1$, $\omega(x) = -5$, $\phi(x) = 1$, $f(x) = 0$, $\gamma(x) = 1$ in boundary-value problem (1.2), (1.3).

Example 4. $a(x) = -1$, $\alpha(x) = -2$, $\beta(x) = 0$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

Example 5. $a(x) = -1$, $\alpha(x) = 0$, $\beta(x) = -2$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

Example 6. $a(x) = -1$, $\alpha(x) = -2$, $\beta(x) = -2$, $\omega(x) = 1$, $f(x) = 0$, $\phi(x) = 1$, $\gamma(x) = -1$ in boundary-value problem (1.2), (1.3).

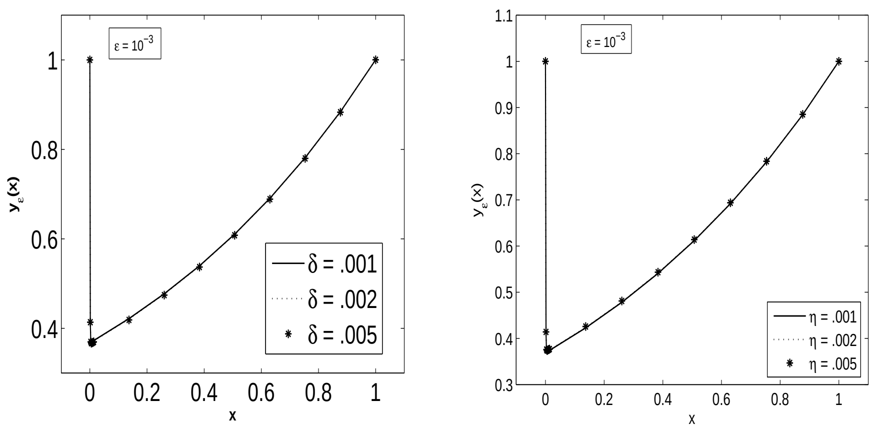
Example 7. $a(x) = 1$, $\alpha(x) = -2$, $\beta(x) = -1$, $\omega(x) = 1$, $f(x) = -1$, $\phi(x) = 1$, $\gamma(x) = 1$ in boundary-value problem (1.2), (1.3).

DISCUSSION

Here we have considered large number of useful examples to demonstrate the efficiency of the cubic B-spline method. First three examples have boundary layer at $x = 0$ and remaining at $x = 1$. Tables 2-7 give the maximum error at different values of parameters. We compare table 3 with table 4 for example 2 and it reveals that present method is much better than the existing one as discussed in [6]. As compared to the results in [6], very small values of parameters (very thin boundary

TABLE 2. Maximum error ($\varepsilon = .001$)

$N \rightarrow$	8	16	64	128
$\delta \downarrow$	Example 1			
.001	0.1288617	0.0663470	0.0082011	0.0027956
.002	0.1290056	0.0661826	0.0081817	0.0027883
.005	0.1283008	0.0656882	0.0081230	0.0027663
$\eta \downarrow$	Example 2			
.001	0.1297049	0.0666750	0.0082399	0.0028102
.002	0.1299369	0.0668387	0.0082592	0.0028175
.005	0.1306291	0.0673282	0.0083169	0.0028393



(a) Numerical solution of example 1 (b) Numerical solution of example 1

FIGURE 1. Approximate sol. for $N = 16$

TABLE 3. Maximum error for example 2 ($\eta = 0.5\varepsilon$)

$\varepsilon \downarrow$	$N \rightarrow 8$	16	32	64	128	256
10^{-3}	0.1295888	0.0665931	0.0260622	0.0082302	0.0028066	0.0009122
10^{-4}	0.1295394	0.0665162	0.0260135	0.0082203	0.0028025	0.0009109
10^{-5}	0.1295344	0.0665085	0.0260086	0.0082193	0.0028021	0.0009107
10^{-6}	0.1295339	0.0665078	0.0260081	0.0082192	0.0028021	0.0009107
10^{-7}	0.1295339	0.0665077	0.0260080	0.0082192	0.0028021	0.0009107
10^{-8}	0.1295339	0.0665077	0.0260080	0.0082192	0.0028021	0.0009107

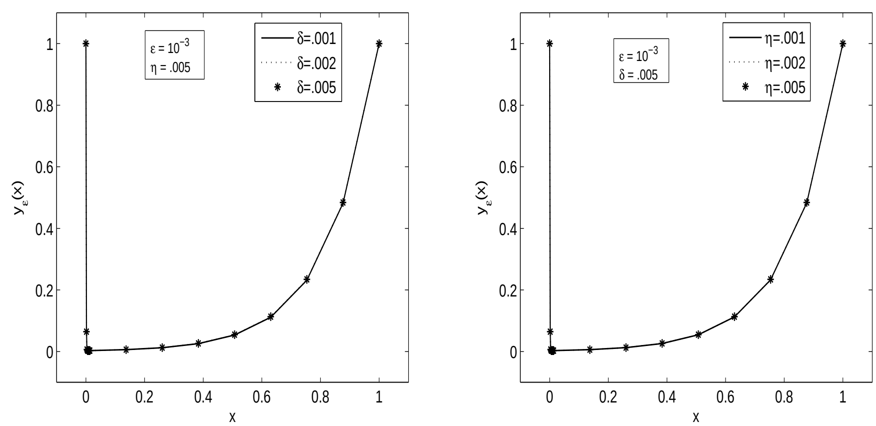
layer) have been taken into consideration. Further Figures 1-4 gives more details about the approximating solution.

6. CONCLUSION

A cubic B-spline approach has been taken into account to approximate the solution of a more general class of singularly perturbed differential difference equations

TABLE 4. Maximum error for example 2 ($\eta = 0.5\varepsilon$) using finite difference [6]

$\varepsilon \downarrow$	$N \to 8$	16	32	64	128	256
10^{-1}	0.10233615	0.06103660	0.03823132	0.02299386	0.01295871	0.00664316
10^{-2}	0.16053996	0.09171283	0.05062424	0.02640865	0.01344656	0.00676030
10^{-3}	0.17511397	0.10213037	0.05896661	0.03133175	0.01623376	0.00825735
10^{-4}	0.17669288	0.10327230	0.05991398	0.03189761	0.01656671	0.00843635
10^{-5}	0.17685213	0.10338763	0.06001002	0.03195506	0.01660057	0.00845456
10^{-6}	0.17686807	0.10339917	0.06001964	0.03196081	0.01660396	0.00845639



(a) Numerical solution of example 3 (b) Numerical solution of example 3

FIGURE 2. Approximate sol. for $N = 16$

TABLE 5. Maximum error for example 3 ($\delta = \eta = 0.5\varepsilon$)

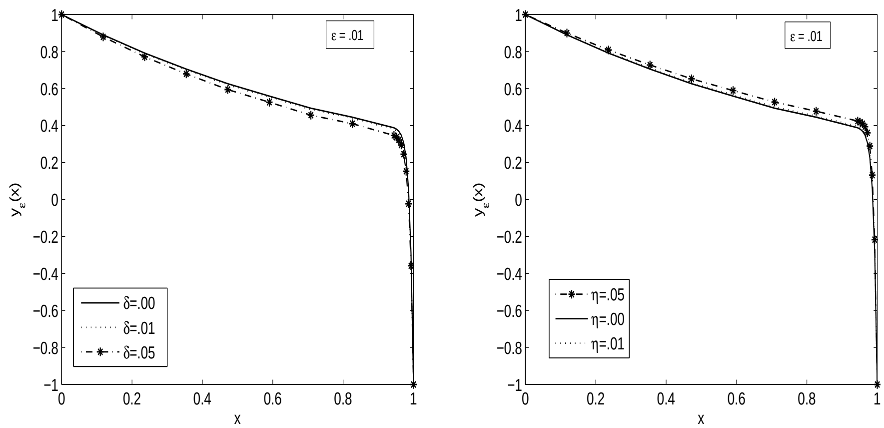
$\varepsilon \downarrow$	$N \to 8$	16	32	64	128	256
10^{-3}	0.2049719	0.1056238	0.0414366	0.0130606	0.0044570	0.0014491
10^{-4}	0.2044679	0.1050201	0.0410817	0.0129795	0.0044254	0.0014384
10^{-5}	0.2044173	0.1049595	0.0410461	0.0129713	0.0044222	0.0014373
10^{-6}	0.2044122	0.1049535	0.0410425	0.0129705	0.0044219	0.0014372
10^{-7}	0.2044117	0.1049529	0.0410422	0.0129704	0.0044219	0.0014372
10^{-8}	0.2044117	0.1049528	0.0410421	0.0129704	0.0044219	0.0014372

which arise in the mathematical modeling of a model of neuronal variability. A numerical scheme is constructed to solve such type of boundary value problems.

A number of numerical experiments are carried out in support of the predicted theory via tabulating the maximum absolute errors in Tables 2-7 for the examples considered and to show the effect of the small shifts on the solution of the problem via plotting the graphs of the solution for different values of negative shift and positive shift for the examples considered, which are reported in the form of Figures 1-4. We

TABLE 6. Maximum error ($\varepsilon = .01$)

	$N \to 8$	32	64	128
$\delta \downarrow$	Example 4			
0.00	0.0470664	0.0080028	0.0028779	0.0009754
0.01	0.0445089	0.0076140	0.0027507	0.0009331
0.05	0.0349893	0.0061692	0.0022623	0.0007074
$\eta \downarrow$	Example 5			
0.00	0.0470664	0.0080028	0.0028779	0.0009714
0.01	0.0496924	0.0083997	0.0030069	0.0010205
0.05	0.0608521	0.0100642	0.0035459	0.0012120



(a) Numerical solution of example 4 (b) Numerical solution of example 5

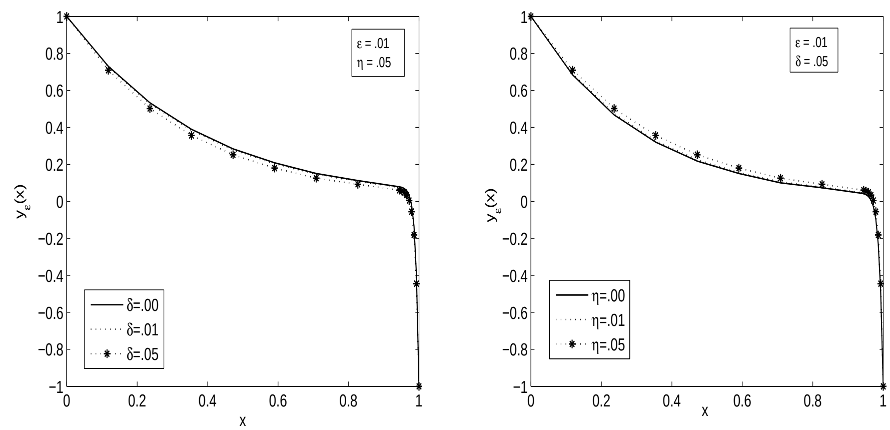
FIGURE 3. Approximate sol. for $N = 16$

TABLE 7. Maximum error for example 6 ($\varepsilon = 0.01$)

	$N \to 8$	32	64	128
$\delta \downarrow$	$\eta = 0.05$			
0.00	0.0474409	0.0077984	0.0027491	0.0009732
0.01	0.0453357	0.0074879	0.0026374	0.0009017
0.05	0.0373800	0.0063012	0.0022552	0.0007652
$\eta \downarrow$	$\delta = 0.05$			
0.00	0.0285293	0.0049505	0.0018119	0.0006161
0.01	0.0301993	0.0052084	0.0018976	0.0006448
0.05	0.0373800	0.0063012	0.0022552	0.0007652

observe from the maximum error tables that the scheme converges quadratically and independently of the singular perturbation parameter.

From Figures 1, 2 illustrate that in the case when the solution of the boundary value problem exhibits layer behavior on the left side, the effect of delay or advance on the solution in the boundary region is negligible while in the outer region it is



(a) Numerical solution of example 6 (b) Numerical solution of example 6

FIGURE 4. Approximate sol. for $N = 16$

TABLE 8. Maximum error for example 7 ($\delta = \eta = 0.5\varepsilon$)

$\varepsilon \downarrow$	$N \rightarrow 8$	16	32	64	128	256
10^{-3}	0.0146404	0.0062139	0.0024983	0.0009001	0.0003052	0.0000996
10^{-4}	0.0145878	0.0061929	0.0024919	0.0008983	0.0003046	0.0000994
10^{-5}	0.0145825	0.0061908	0.0024912	0.0008981	0.0003045	0.0000994
10^{-6}	0.0145820	0.0061906	0.0024912	0.0008981	0.0003045	0.0000994
10^{-7}	0.0145819	0.0061906	0.0024912	0.0008981	0.0003045	0.0000994
10^{-8}	0.0145819	0.0061906	0.0024912	0.0008981	0.0003045	0.0000994

considerable and the change in the advance affects the solution in similar fashion as the change in delay affects, but reversely. Figures 3, 4 illustrate that in the case when the boundary value problem exhibits layer behavior on the sight side, the changes in delay or advance affect the solution in boundary layer region as well as outer region. The thickness of the layer increases as the size of the delay increases while it decreases as the size of the advance increases.

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