Neural, Parallel, and Scientific Computations 16 (2008) 481-496

THE FULLY DISCRETE MULTISTEP CHARACTERISTIC FINITE VOLUME ELEMENT METHODS FOR THE TWO-DIMENSIONAL GENERALIZED NERVE CONDUCTION EQUATION

ZHIYUE ZHANG AND ZHOUFENG WANG

School of Mathematics and Computer Science, Nanjing Normal University,
Nanjing, 210097 zhangzhiyue@njnu.edu.cn

ABSTRACT: In the paper, a fully discrete Multistep characteristic finite volume element method is introduced and analyzed for approximating the solution of a nonlinear-hyperbolic equation in 2-space variables. Piecewise quadratic trial functions and piecewise constant test functions are used to finally obtain error estimate $O(\Delta t^2 + h^2)$. A numerical experiment is given which showed the method is practicable.

Key words: Nonlinear evolution equations, multistep characteristic finite volume element method, error estimate, numerical experiment

1. INTRODUCTION

In this paper, we are concerned with numerical approximation to the following two-dimensional generalized nerve conduction equation:

$$u_{tt} + b(x, u, u_t)\nabla u_t - \triangle u_t - \triangle u = f(u)u_t - g(u), \quad x \in \Omega, \quad t \in J \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u_t}{\partial n} = 0, \quad x \in \partial\Omega, \quad t \in J \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = w_0(x), \quad x \in \Omega \tag{1.3}$$

where $\Omega = [0, 1]^2$, $J = [0, T]$, $\partial\Omega$ denotes the boundary of Ω, $b(x, u, u_t) = \{b_1(x, u, u_t), b_2(x, u, u_t)\}$, u_0 and w_0 are assumed to be enough smooth functions.

We make the following physical assumption (A):

(i) $f(s)$, $g(l)$ and $b_i(x, s, l)$ $(i = 1, 2)$ are bounded, and ε-continuous with respect to s and l respectively. We give the definition of ε-continuous function $f(s)$: When $|s_1 - s_2| \leq \varepsilon$, there exists a positive constant L, such that $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$.

(ii) $u \in C^2(\Omega \times J) \cap L^\infty(W^3_\infty) \bigcap L^2(H^3(\Omega))$, $u_t \in L^2(H^3(\Omega)) \bigcap L^\infty(W^3_\infty)$, $u_{tt} \in L^2(H^3(\Omega)) \bigcap L^2(L^\infty(\Omega))$, $u_{ttt} \in L^2(L^2(\Omega))$

This project is supported by National Natural Science Foundation of China (No.10471067).
Received November 25, 2007 1061-5369 $15.00 ©Dynamic Publishers, Inc.

The finite volume elements, which can be termed as the generalized difference methods, are viewed as a new approach of numerical discretization for partial differential equation [1–2]. Since their constructions are similar to those of some finite difference methods and their convergence can be analyzed in the framework of finite element methods, the finite volume element methods enjoy not only the simplicity of difference methods but also the accuracy of finite elements. Meanwhile, the finite volume element methods maintain the (local) mass conservation law. Consequently they have been widely used in many practical computations and extensively studied in theory. On the other hand, in many case discrete scheme derived in terms of finite volume element methods is asymmetric, it brings us many difficulties in both theoretical research and realistic computations. It is usually necessary for us to seek for some suitable technique that can transform the asymmetric scheme into symmetric one. There are many results about finite volume element methods for elliptic problems and parabolic problems [3–5].

In the process of nerve conduction, nerve conduction signal u and its variability with respect to time and space can be characterized the two-dimensional pseudohyperbolic equation [6] in Mathematics. It is a class of important nonlinear evolution equation of much current interest. There are some results about the equations [7–9]. Since generalized nerve conduction equations can describe lots of physical phenomenons and possess strong physical background, thus it is important for us to develop the studies across-the-board and deeply either from the theoretical point of view or from the numerical analysis and practical point of view.

In the present paper, the generalized nerve conduction equation is regarded as a model problem and characteristic direction method is applied to deal with one-order hyperbolic part of the equation in the process of scheme construction. The trial function space is chosen as the quadratic element space of lagrangian type. The primary advantages of this scheme is that: First, it involves only three time levels for the pseudohyperbolic equation. Second, the estimate of u_t is obtained at the same time. Since u_t is also an important physical parameter in practice, this scheme avoids arising two times error by using the common characteristic difference method to approximate u at first, then to approximate u_t. Finally, we obtain the desired $O(\Delta t^2+h^2)$ error bound. It is important that the accuracy on the temporal direction is improved $O((\Delta t)^2)$ for large scale science and engineering computing problem.

The rest of this paper is organized as follows: In section 2, we present a full-discrete multistep characteristic finite volume element scheme while introducing some notations. In section 3, we give some preliminaries. The error estimates are presented in section 4. In section 5, we carry out numerical experiments to observe the performance of the proposed scheme. The letter c and C will be generic positive constants and may be different each time they are used, ε will be an arbitrarily small positive constant.

2. FULL-DISCRETE MULTISTEP CHARACTERISTIC FINITE VOLUME ELEMENT SCHEME

Let $v = u_t$ in (1.1), in order to construct finite volume element scheme, added $u_t = v$ to the both sides of (1.1). Then (1.1) can be written as

$$v_t + b(x,u,v)\nabla v - \triangle v - \triangle u + u_t = H(u,v), \quad x \in \Omega, \quad t \in J \tag{2.1}$$

$$u_t = v, \quad x \in \Omega, \quad t \in J \tag{2.2}$$

where $H(u,v) = (f(u)+1)v - g(u)$.

The initial and boundary condition are given by

$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial\Omega, \quad t \in J \tag{2.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = w_0(x), \quad x \in \Omega \tag{2.4}$$

At first, in order to attain very high accuracy and use larger time step we apply characteristic direction method to deal with the first two terms of (2.1):

Let $\Psi(x,u,v) = \sqrt{1+|b(x,u,v)|^2}$, here we make a convention that τ is defined as the characteristic direct of $\Psi\frac{\partial}{\partial\tau} = \frac{\partial}{\partial t} + b(x,u,v)\nabla$. We have

$$\Psi\frac{\partial v}{\partial \tau} = \frac{\partial v}{\partial t} + b(x,u,v)\nabla v$$

Then (2.1) can be rewritten as $\Psi\frac{\partial v}{\partial\tau} - \triangle v - \triangle u + u_t = H(u,v)$.

Now, let N denotes a positive integer such that $N\Delta t = T$, $t_n = n\Delta t$ and $\Delta t = t_{j+1} - t_j$, and for a sequence $\varphi_j (j = 1,2,\ldots,N)$, define $\varphi_j = \varphi(t_j)$, $\delta\varphi_j = \varphi_j - \varphi_{j-1}$, $\delta^2\varphi_j = \varphi_j - 2\varphi_{j-1} + \varphi_{j-2}$, $E\varphi_{j+1} = \varphi_{j+1} - \delta^2\varphi_{j+1}$ and $\partial_t\varphi_j = \delta\varphi_j/\Delta t$.

As far as $\Psi\frac{\partial v}{\partial\tau}$ is concerned, we consider the standard multistep backward difference quotient error approximation in the parameter τ [10] along the characteristic direction.

$$\psi(x,u_{n+1},v_{n+1})\frac{\partial v}{\partial\tau}(x,t_{n+1}) \doteq \frac{3v(x,t_{n+1}) - 4v(\widetilde{x},t_n) + v(\widetilde{\widetilde{x}},t_{n-1})}{2\Delta t} \tag{2.5}$$

where $\widetilde{x} = x - b(x,u_{n+1},v_{n+1})\Delta t$, $\widetilde{\widetilde{x}} = x - 2b(x,u_{n+1},v_{n+1})\Delta t$

Subsequently, we need briefly explain some standard notation from this paper. Setting T_h be a quasi-uniform triangulation of $\overline{\Omega}$, T_h consists of finite number of triangular elements K_Q. Q being the barycenter of triangle. Suppose that maximum angle of each element of triangulation T_h is not greater that $\frac{\pi}{2}$, and that the ratio γ of the lengths of two sides of the maximum angle satisfies $\gamma\epsilon[\sqrt{\frac{2}{3}},\sqrt{\frac{3}{2}}]$. The corresponding dual decomposition of T_h is denoted by T_h^*, their detailed construction(see figure 1)is as follows: (i) Construction of $K_{P_0}^*$, suppose that $p_0\epsilon\overline{\Omega}_h$($\overline{\Omega}_h$ denotes the set of the vertexes of all the triangular elements), p_{0i} is a point on $\overline{p_0p_i}$ such that $\overline{p_0p_{0i}} = \frac{1}{3}\overline{p_0p_i}$, connect successively p_{0i} to obtain a polygon K_{p0}^* surrounding p_0;

(ii) Construction of K_m^*, let m be midpoint of a common side of two adjacent triangular elements. A polygon K_m^* surrounding m is obtained by connecting successively $p_{20}Q_{03}Q_2Q_{23}p_{02}Q_{12}Q_1Q_{01}p_{20}$ where Q_{01} denotes the midpoint of $\overline{p_{20}p_{21}}$, Other points are also similar. All the dual elements constitute the dual decomposition T_h^*.

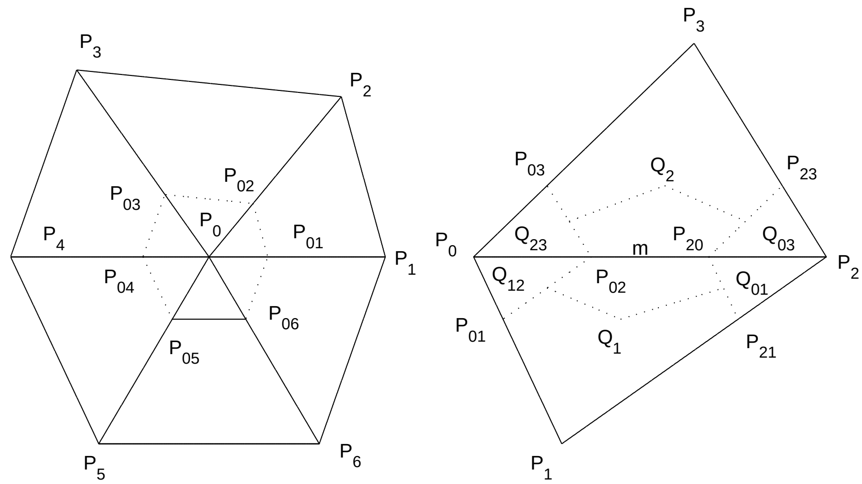

FIGURE 1. Left: Portion of triangulation sharing a common vertex P_0 and its control volume; Right: Portion of two adjacent triangular elements sharing a midpoint of a common side m and its control volume.

The trial function space $\bigsqcup_h$ is chosen as the lagrangian quadratic element space related to T_h, the corresponding basis function are the piecewise quadratic polynomials. The test function space $\bigvee_h$ is taken as the piecewise constant function space on T_h^*.

Let $\pi_h : H^1 \to \bigsqcup_h$, then by the interpolation theory of sobolev spaces [11], we obtain

$$\|u - \pi_h u\|_j \le Ch^{3-j}\|u\|_3 \quad j = 0, 1, 2, \quad u \in H^3(\Omega) \tag{2.6}$$

We define the interpolation operator $\pi_h^* : \bigsqcup_h \longrightarrow \bigvee_h$ by

$$\pi_h^* u_h = \sum_p u_h(p)\chi_p + \sum_m u_h(m)\chi_m \tag{2.7}$$

where χ_p and χ_m are respectively taken as the characteristic function corresponding to K_p^* and K_m^*.

Define $a_K(u, w)$, $A_K(u, v, w)$ as follows:

$$a_K(u, w) = -\sum_{l=i,j,k} [w(p_l) \int_{e_l} \nabla u \cdot nds + w(m_l) \int_{E_l} \nabla u \cdot nds] \tag{2.8}$$

$$A_K(u, v, w) = -\sum_{l=i,j,k} [w(p_l) \int_{e_l} \nabla(u + v) \cdot nds + w(m_l) \int_{E_l} \nabla(u + v) \cdot nds] \tag{2.9}$$

where $w \in \bigvee_h, (u, v) \in H^1(K) \times H^1(K)$

See figure 2, $e_l = \overline{p_{l,l+1}p_{ll+2}}$, $E_l = \overline{p_{l+2l+1}Q_{l+2}QQ_{l+1}p_{l+1l+2}}$ $\forall\ K \epsilon T_h$ where $i+1 = j$, $j+1 = k$, $k+1 = i$, n is the unit outer normal vector on the boundary. Note that $a(u.w) = \sum_K a_K(u, w)$, $A(u, v, w) = \sum_K A_K(u, v, w)$.

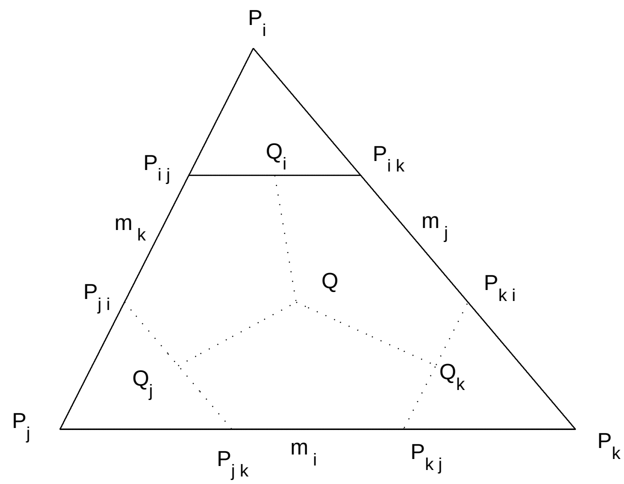

FIGURE 2. triangular element K

Before presenting three discrete norms [11], we introduce two vectors on K related to $u_h \in \bigsqcup_h$.

$$(u_h)_{0,K} = (u_h(p_i), u_h(p_j), u_h(p_k), u_h(m_i), u_h(m_j), u_h(m_k))^\top \tag{2.10}$$

$$\begin{aligned}(u_h)_{1,K} = (&u_h(p_i) - u_h(m_i), u_h(p_j) - u_h(m_j), u_h(p_k) - u_h(m_k),\\ &u_h(m_i) - u_h(m_j), u_h(m_i) - u_h(m_k))^\top\end{aligned} \tag{2.11}$$

$$\|u_h\|_{0,h}^2 = \sum_K \|u_h\|_{0,h,K}^2 = \sum_K \frac{m(K)}{1944}(u_h)_{0,K}^\top B(u_h)_{0,K} \tag{2.12}$$

$$|u_h|_{1,h}^2 = \sum_K |u_h|_{1,h,K}^2 = \sum_K (u_h)_{1,K}^\top (u_h)_{1,K} \tag{2.13}$$

$$\|u_h\|_{1,h}^2 = \|u_h\|_{0,h}^2 + |u_h|_{1,h}^2 \tag{2.14}$$

where $B = (b_{ij})_{6\times 6}$ is a given symmetric positive definite matrix in [12].

Finally, setting $t = t_{n+1}$, then the corresponding variational problem for (2.1) and (2.2) is:

$$\begin{aligned}&((v_t)_{n+1} + b(x, u_{n+1}, v_{n+1}) \cdot \nabla v_{n+1} + (u_t)_{n+1}, \chi)\\ &\quad + A(u_{n+1}, v_{n+1}, \chi) = (H(u_{n+1}, v_{n+1}), \chi)\end{aligned} \tag{2.15}$$

$$((u_t)_{n+1}, \chi) = (v_{n+1}, \chi) \tag{2.16}$$

where $\forall \chi \in \bigvee_h$.

At the same time, one sees that the full discrete multistep characteristic finite volume element schemes for (2.1) and (2.2) read as: find U_{n+1}, $V_{n+1} \in \bigsqcup_h$ $(1 \leq n \leq N-1)$, such that

$$\left(\frac{V_{n+1}-\widehat{V}_n}{\Delta t}, \chi\right) + \frac{2}{3}\left(\frac{3U_{n+1}-4U_n+U_{n-1}}{2\Delta t}, \chi\right) + \frac{2}{3}A(U_{n+1}, V_{n+1}, \chi)$$

$$= \frac{2}{3}(H(U_{n+1}, EV_{n+1}), \chi) + \frac{1}{3}\left(\frac{\widehat{V}_n - \widehat{\widehat{V}}_{n-1}}{\Delta t}, \chi\right) \tag{2.17}$$

$$\left(\frac{3U_{n+1}-4U_n+U_{n-1}}{2\Delta t}, \chi\right) = (V_{n+1}, \chi) \tag{2.18}$$

where $\chi \in \bigvee_h$, $\widehat{V}_n = V_n(\widehat{x}) = V_n(x - b(x, U_{n+1}, EV_{n+1})\Delta t)$, $\widehat{\widehat{V}}_{n-1} = V_{n-1}(\widehat{\widehat{x}}) = V_{n-1}(x - 2b(x, U_{n+1}, EV_{n+1})\Delta t)$.

Assure that U_0 and V_0 respectively denote some approximation of u_0 and v_0 in $\bigsqcup_h$, satisfying:

$$(U_0 - u_0, \chi) = 0, (V_0 - w_0, \chi) = 0 \tag{2.19}$$

It should be pointed out that [7] if $\widehat{x}$ (or $\widehat{\widehat{x}}$) stays out of Ω, then using mirror reflection technique, we can find the symmetric point x^* of $\widehat{x}$ (or $\widehat{\widehat{x}}$) with respect to $\partial\Omega$. At this time, we require $\widehat{V}_n$ (or $\widehat{\widehat{V}}_n$) $= V_n(x^*)$.

3. PRELIMINARIES

Lemma 3.1. $\forall u_h \in \bigsqcup_h$, *there exists positive constants* c_1 *and* c_2 *independent of* h, *such that*

$$c_1\|u_h\|_{0,h} \leq \|u_h\|_0 \leq c_2\|u_h\|_{0,h} \tag{3.1}$$

$$c_1|u_h|_{1,h} \leq |u_h|_1 \leq c_2|u_h|_{1,h} \tag{3.2}$$

$$c_1\|u_h\|_{0,h} \leq \|\pi_h^* u_h\|_0 \leq c_2\|u_h\|_{0,h} \tag{3.3}$$

Lemma 3.2. $\forall u_h \in \bigsqcup_h$, *there exists a positive constant* α, *such that,*

$$a(u_h, \pi_h^* u_h) \geq \alpha|u_h|_{1,h}^2 \tag{3.4}$$

$$a(u_h, \pi_h^* \underline{u}_h) \geq \alpha|u_h|_{1,h}^2 \tag{3.5}$$

where $\underline{u}_h \in \bigsqcup_h$, *we introduce a one-to-one operator mapping* $\underline{u}_h$ *to* u_h, *such that for any triangular element* K, *a relationship between* $\underline{u}_h$ *and* u_h *always holds:* $(\underline{u}_h)_{0,K} = D(u_h)_{0,K}$, $D = (d_{ij})_{6\times 6}$ *is a non-singular matrix defined in [10] and* $\underline{u}_h$ *and* v_h *satisfy* $(v_h, \pi_h^* \underline{u}_h) = \sum\limits_K \frac{m(K)}{1944}(v_h)_{0,K}^\top B(u_h)_{0,K}$.

For the proof of the above two lemma, we can refer to [12] [13].

Theorem 3.3 (the trace theorem [14]). *Suppose that Ω is a bounded region with a lipschitz continuous boundary $\partial\Omega$, then there exists a positive constant C such that,*

$$\|u\|_{L^2(\partial\Omega)} \leq C\|u\|^{\frac{1}{2}}_{L^2(\Omega)} \cdot \|u\|^{\frac{1}{2}}_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega) \tag{3.6}$$

Lemma 3.4 ([15]). $\forall u_h \in \bigsqcup_h$

$$\|\underline{u}_h\|_{0,h} \leq C\|u_h\|_{0,h} \tag{3.7}$$

$$\|\pi_h^*\underline{u}_h\|_0 \leq C\|u_h\|_0 \tag{3.8}$$

$$|\underline{u}_h|_{1,h} \leq C|u_h|_{1,h} \tag{3.9}$$

Proof: Using the relation of u_h and $\underline{u}_h$, the identity (2.12) and the positive definite matrix B, finally we both obtain a positive definite quadratic form related to the vector $(u_h)_{0,K}$. Thus $\|u_h\|_{0,h}$ is equivalent with $\|\underline{u}_h\|_{0,h}$, there exists a positive constant C, such that $\|\underline{u}_h\|_{0,h} \leq C\|u_h\|_{0,h}$. Then $\|\pi_h^*\underline{u}_h\|_0 \leq C\|\underline{u}_h\|_{0,h}$, $\|\underline{u}_h\|_{0,h} \leq C\|u_h\|_{0,h}$. The desired result follows from the two inequality, this completes the proof of (3.8).

By the elliptic condition (3.4), there exists a positive constant M, such that $|\underline{u}_h|^2_{1,h,K} \leq \frac{1}{\alpha}a_K(\underline{u}_h, \pi_h^*\underline{u}_h) = \frac{M}{\alpha m(K)}(u_h)^{\top}_{1,K}G(\underline{u}_h)_{1,K}$ where $G = (h_{ij})_{5\times 5}$ and is a matrix defined in [12], moreover $|h_{ij}| \leq b \cdot m(K)$. Let b be a fixed positive constant, $m(K)$ denotes area of triangular element K. Clearly we use holder inequality to obtain $\frac{M}{\alpha m(K)}(u_h)^{\top}_{1,K}G(\underline{u}_h)_{1,K} \leq C|u_h|_{1,h,K}|\underline{u}_h|_{1,h,K}$

By the above two estimate, it is an easy matter to deduce that $|\underline{u}_h|_{1,h,K} \leq C|u_h|_{1,h,K}$, this completes the proof of (3.9).

Lemma 3.5. $\forall u_h, w_h, \underline{v}_h \in \bigsqcup_h$

$$|A(u, w, \pi_h^*\underline{v}_h) - A(u_h, w, \pi_h^*\underline{v}_h)| \leq C(h^2\|u\|_3 + |\pi_h u - u_h|_1)|v_h|_{1,h} \tag{3.10}$$

$$|A(u, w, \pi_h^*\underline{v}_h) - A(u, w_h, \pi_h^*\underline{v}_h)| \leq C(h^2\|w\|_3 + |\pi_h w - w_h|_1)|v_h|_{1,h} \tag{3.11}$$

Proof: $A_K(u, w, \pi_h^*\underline{v}_h) - A_K(u_h, w, \pi_h^*\underline{v}_h) = -\sum_{l=i,j,k}\Big[(\underline{v}_h(m_{l+2}) - \underline{v}_h(p_l))\int_{\overline{Q_l p_{l l+1}}}\nabla(u - u_h)\cdot nds + (\underline{v}_h(p_l) - \underline{v}_h(m_{l+1}))\int_{\overline{Q_l p_{l,l+1}}}\nabla(u - u_h)\cdot nds + (\underline{v}_h(m_{l+2}) - \underline{v}_h(m_{l+1}))\int_{\overline{QQ_l}}\nabla(u - u_h)\cdot nds\Big]$

Applying holder inequality, the trace theorem (3.6), together with interpolation estimate and inverse property of finite element methods, we deduce that

$$\begin{aligned}
&|\int_{\overline{Q_l p_{l l+1}}}\nabla(u - u_h)\cdot nds| \\
&\leq \int_{\overline{Q_l p_{l l+1}}}|\nabla(u - \pi_h u)\cdot n|ds + \int_{\overline{Q_l p_{l l+1}}}|\nabla(\pi_h u - u_h)\cdot n|ds \\
&\leq h^{\frac{1}{2}}(\|\nabla(u - \pi_h u)\|_{L^2(\overline{Q_l p_{l l+1}})} + \|\nabla(\pi_h u - u_h)\|_{L^2(\overline{Q_l p_{l l+1}})})
\end{aligned}$$

$$\le h^{\frac{1}{2}}(\|u-\pi_h u\|_{H^1(K)}^{\frac{1}{2}} \cdot \|u-\pi_h u\|_{H^2(K)}^{\frac{1}{2}}$$
$$+\|\nabla(\pi_h u - u_h)\|_{L^2(K)}^{\frac{1}{2}} \cdot \|\nabla(\pi_h u - u_h)\|_{H^1(K)}^{\frac{1}{2}})$$
$$\le C(h^2\|u\|_{3,K} + |\pi_h u - u_h|_{1,K})$$

It is obvious that $|\underline{v}_h(m_{l+2}) - \underline{v}_h(p_l)| \le |\underline{v}_h|_{1,h,K} \le C|v_h|_{1,h,K}$. By the similar technique, error estimate of other terms can easily establish, thus we further have the following result: $|A(u,w,\pi_h^*\underline{v}_h) - A(u_h,w,\pi_h^*\underline{v}_h)| \le C(h^2\|u\|_3 + |\pi_h u - u_h|_1)|v_h|_{1,h}$. An argument similar to the one in the above case implies that $|A(u,w,\pi_h^*\underline{v}_h) - A(u,w_h,\pi_h^*\underline{v}_h)| \le C(h^2\|w\|_3 + |\pi_h w - w_h|_1)|v_h|_{1,h}$. This completes the proof.

Lemma 3.6. *Suppose that positive integer R is not greater than N, $\forall\ u \in \bigsqcup_h$, then the following estimate [10] holds:*

$$\Delta t \sum_{l=1}^{R-1} (\partial_t u_l, \pi_h^* \underline{u}_{l+1}) \le \frac{3}{2}\sum_{l=1}^{R-1} \|\delta u_{l+1}\|_{0,h}^2 + \|u_R\|_{0,h}^2 + C(\|u_1\|_{0,h}^2 + \|u_0\|_{0,h}^2) \tag{3.12}$$

4. ERROR ESTIMATES

Note that $U-u = \sigma - \eta$, $V - v = \xi - \theta$, where $\sigma = U - \pi_h u$, $\eta = u - \pi_h u$, $\xi = V - \pi_h v$, $\theta = v - \pi_h v$. Subtracting (2.15) from (2.17) and using (2.16) and (2.18), we have

(4.1)

$$\frac{2}{3}a(\xi_{n+1},\chi) + \left(\frac{\xi_{n+1}-\xi_n}{\Delta t},\chi\right) = \left(\frac{4\widehat{\xi}_n - 3\xi_n - \widehat{\widehat{\xi}}_{n-1}}{3\Delta t},\chi\right) + \left(\frac{3\theta_{n+1} - 4\widehat{\theta}_n + \widehat{\widehat{\theta}}_{n-1}}{3\Delta t},\chi\right)$$
$$-\frac{2}{3}(\xi_{n+1}-\theta_{n+1},\chi) + \left(\frac{2}{3}((v_t)_{n+1} + b(x,U_{n+1},EV_{n+1})\cdot\nabla v_{n+1}) - \frac{v_{n+1}-\widehat{v}_n}{\Delta t}\right.$$
$$\left.+\frac{1}{3}\frac{\widehat{v}_n - \widehat{\widehat{v}}_{n-1}}{\Delta t},\chi\right) + \frac{2}{3}[A(U_{n+1},\theta_{n+1},\chi) + (A(u_{n+1},v_{n+1},\chi) - A(U_{n+1},v_{n+1},\chi))]$$
$$+\frac{2}{3}(H(U_{n+1},EV_{n+1}) - H(u_{n+1},v_{n+1}),\chi) + \frac{2}{3}((b(x,u_{n+1},v_{n+1}) - b(x,U_{n+1},$$
$$EV_{n+1}))\cdot\nabla v_{n+1},\chi) = \sum_{i=1}^{7} T_i^{n+1}(\chi)$$

Subtracting (2.16) from (2.18) yields

(4.2)

$$\frac{3}{2}\left(\frac{\sigma_{n+1}-\sigma_n}{\Delta t},\chi\right) = \frac{1}{2}\left(\frac{\sigma_n - \sigma_{n-1}}{\Delta t},\chi\right) + \frac{3}{2}\left(\frac{\eta_{n+1}-\eta_n}{\Delta t},\chi\right) - \frac{1}{2}\left(\frac{\eta_n - \eta_{n-1}}{\Delta t},\chi\right)$$
$$+(\xi_{n+1}-\theta_{n+1},\chi) - \left(\frac{3u_{n+1} - 4u_n + u_{n-1}}{2\Delta t} - (u_t)_{n+1},\chi\right)$$

The following estimate result, which can be easily proved, will be used in our analysis.

$$\|\frac{w_{n+1}-w_n}{\Delta t}\|_0^2 \le \|\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} w_t dt\|_0^2 \le \frac{1}{\Delta t}\|w_t\|^2_{L^2((t_n,t_{n+1}),L^2)} \tag{4.3}$$

Taking $\chi = \Delta t\pi_h^*\underline{\sigma}_{n+1}$ in (4.2) and summing from $n=1$ to $R-1$, then we will use different technique to deal with every term in the right hand side of the equality (4.2).

For the first term, it follows from lemma(3.6) and initial value condition(B) that

$$\begin{aligned}\frac{1}{2}\sum_{n=1}^{R-1}(\frac{\sigma_n-\sigma_{n-1}}{\Delta t}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) &\le \frac{3}{4}\sum_{n=1}^{R-1}\|\delta\sigma_{n+1}\|_{0,h}^2 + \frac{1}{2}\|\sigma_R\|_{0,h}^2 \\ &+C(h^4+\Delta t^4)\end{aligned} \tag{4.4}$$

By (4.3) and (2.6), we can obtain

$$\frac{3}{2}\sum_{n=1}^{R-1}(\frac{\eta_{n+1}-\eta_n}{\Delta t}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) \le C(h^4+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2) \tag{4.5}$$

$$\frac{1}{2}\sum_{n=1}^{R-1}(\frac{\eta_{n-1}-\eta_n}{\Delta t}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) \le C(h^4+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2) \tag{4.6}$$

Using (2.6), we see that

$$\sum_{n=1}^{R-1}(\xi_{n+1}-\theta_{n+1}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) \le C(h^4+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2) \tag{4.7}$$

In terms of Taylor expansion with integral remainder, we can arrive at:

$$\begin{aligned}-\sum_{n=1}^{R-1}(\frac{3u_{n+1}-4u_n+u_{n-1}}{2\Delta t}-(u_t)_{n+1}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) &= \frac{3}{2}\Delta t\sum_{n=1}^{R-1}((u_t)_{n+1}-\frac{u_{n+1}-u_n}{\Delta t} \\ +\frac{1}{3}(\frac{u_n-u_{n-1}}{\Delta t}-(u_t)_{n+1}), \pi_h^*\underline{\sigma}_{n+1}) &\le C(\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2)\end{aligned} \tag{4.8}$$

In the following, the left hand side of (4.2) is easily shown that

$$\begin{aligned}\frac{3}{2}\sum_{n=1}^{R-1}(\frac{\sigma_{n+1}-\sigma_n}{\Delta t}, \Delta t\pi_h^*\underline{\sigma}_{n+1}) &= \frac{3}{2}\sum_{n=1}^{R-1}(\delta\sigma_{n+1}, \pi_h^*\underline{\sigma}_{n+1}) = \frac{3}{4}\sum_{n=1}^{R-1}[(\delta\sigma_{n+1}, \pi_h^*\underline{\sigma}_{n+1}) \\ +(\sigma_{n+1}, \pi_h^*\delta\underline{\sigma}_{n+1})] &= \frac{3}{4}\sum_{n=1}^{R-1}(\|\delta\sigma_{n+1}\|_{0,h}^2+\|\sigma_{n+1}\|_{0,h}^2-\|\sigma_n\|_{0,h}^2)\end{aligned} \tag{4.9}$$

By virtue of (4.4)-(4.9), we have

$$\|\sigma_R\|_0^2 \le C(h^4+\Delta t^4+\|\sigma_1\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2) \tag{4.10}$$

At the same time, choosing $\chi = \Delta t\pi_h^*\underline{\xi}_{n+1}$ in (4.1), we show the estimate of two-hand sides of (4.1) and sum from $n=1$ to $R-1$.

It can be easily verified [16] that $\|\frac{\xi_k-\widehat{\xi}_k}{\Delta t}\|_0 \le C\|\xi_k\|_1$, $\|\frac{\xi_k-\widehat{\widehat{\xi}}_k}{\Delta t}\|_0 \le C\|\xi_k\|_1$, while using lemma (3.6) and initial value condition (B), we obtain

(4.11)

$$\sum_{n=1}^{R-1} T_1^{n+1}(\Delta t\pi_h^*\underline{\xi}_{n+1}) = \Delta t\sum_{n=1}^{R-1}\left(\frac{4\widehat{\xi}_n-3\xi_n-\widehat{\widehat{\xi}}_{n-1}}{3\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)$$
$$= \Delta t\sum_{n=1}^{R-1}\left[\frac{4}{3}\left(\frac{\widehat{\xi}_n-\xi_n}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)+\frac{1}{3}\left(\frac{\xi_n-\xi_{n-1}}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)+\frac{1}{3}\left(\frac{\xi_{n-1}-\widehat{\widehat{\xi}}_{n-1}}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)\right]$$
$$\le C\Delta t\sum_{n=1}^{R-1}(\|\xi_n\|_1+\|\xi_{n-1}\|_1)\cdot\|\xi_{n+1}\|_0+\frac{1}{2}\sum_{n=1}^{R-1}\|\delta\xi_{n+1}\|_{0,h}^2$$
$$+\frac{1}{3}\|\xi_R\|_{0,h}^2+C(\|\xi_1\|_{0,h}^2+\|\xi_0\|_{0,h}^2)$$
$$\le C\left(h^4+\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2\right)+\varepsilon\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_1^2$$
$$+\frac{1}{2}\sum_{n=1}^{R-1}\|\delta\xi_{n+1}\|_{0,h}^2+\frac{1}{3}\|\xi_R\|_{0,h}^2$$

As for the second term, integrating (4.3), (2.6) with $\|\frac{\theta_{k-1}-\widehat{\widehat{\theta}}_{k-1}}{\Delta t}\|_0 \le C\|\theta_{k-1}\|_1$, we have the following results.

$$\sum_{n=1}^{R-1} T_2^{n+1}(\Delta t\pi_h^*\underline{\xi}_{n+1}) = \Delta t\sum_{n=1}^{R-1}\left(\frac{3\theta_{n+1}-4\widehat{\theta}_n+\widehat{\widehat{\theta}}_{n-1}}{3\Delta t},\pi_h^*\underline{\xi}_{n+1}\right) \tag{4.12}$$
$$= \Delta t\sum_{n=1}^{R-1}(\frac{\theta_{n+1}-\theta_n}{\Delta t},\pi_h^*\underline{\xi}_{n+1})+\frac{4}{3}\Delta t\sum_{n=1}^{R-1}\left(\frac{\theta_n-\widehat{\theta}_n}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)$$
$$+\frac{1}{3}\Delta t\sum_{n=1}^{R-1}\left(\frac{\theta_{n-1}-\theta_n}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)+\frac{1}{3}\Delta t\sum_{n=1}^{R-1}\left(\frac{\widehat{\widehat{\theta}}_{n-1}-\theta_{n-1}}{\Delta t},\pi_h^*\underline{\xi}_{n+1}\right)$$
$$\le C\left(h^4+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2\right)$$

Deduce by an similar estimate in (4.7), we obtain

(4.13)

$$\sum_{n=1}^{R-1} T_3^{n+1}(\Delta t\pi_h^*\underline{\xi}_{n+1}) = -\frac{2}{3}\Delta t\sum_{n=1}^{R-1}(\xi_{n+1}-\theta_{n+1},\pi_h^*\underline{\xi}_{n+1}) \le C\left(h^4+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2\right)$$

Set $\psi(x,U_{n+1},EV_{n+1})=(1+|b(x,U_{n+1},EV_{n+1})|^2)^{\frac{1}{2}}$, the characteristic direction corresponding to $\frac{\partial}{\partial t}+b(x,U_{n+1},EV_{n+1})\cdot\nabla$ is denoted by $\tau(x,U_{n+1},EV_{n+1})$, we can get

$$\psi\frac{\partial v_{n+1}}{\partial\tau}=\frac{\partial v_{n+1}}{\partial t}+b(x,U_{n+1},EV_{n+1})\cdot\nabla v_{n+1}$$

Analogous to (4.8), we consider Taylor expansion and have that

(4.14)

$$\sum_{n=1}^{R-1} T_4^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \Delta t \sum_{n=1}^{R-1}\left(\psi\left(\frac{\partial v_{n+1}}{\partial \tau} - \frac{v_{n+1}-\widehat{v}_n}{\psi \Delta t}\right) - \frac{1}{3}\psi\left(\frac{\partial v_{n+1}}{\partial \tau} - \frac{\widehat{v}_n - \widehat{\widehat{v}}_{n-1}}{\psi \Delta t}\right),\right.$$

$$\left.\pi_h^*\underline{\xi}_{n+1}\right) \leq C\Delta t \sum_{n=1}^{R-1}\left(\|\frac{1}{\Delta t}\int_{(\widehat{x},t_n)}^{(x,t_{n+1})}(|x(\tau)-\widehat{x}|^2 + (t(\tau)-t_n)^2)\frac{\partial^3 v}{\partial \tau^3}d\tau\|_0\right.$$

$$\left.+\|\frac{1}{\Delta t}\int_{(\widehat{\widehat{x}},t_{n-1})}^{(x,t_{n+1})}(|x(\tau)-\widehat{\widehat{x}}|^2 + (t(\tau)-t_{n-1})^2)\frac{\partial^3 v}{\partial \tau^3}d\tau\|_0\right)\cdot\|\xi_{n+1}\|_0$$

$$\leq C\left(\Delta t^4 + \Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2\right)$$

We use lemma (3.5), $\Delta t = O(h^3)$ and inverse property of finite element to conclude that

(4.15)

$$\sum_{n=1}^{R-1} T_5^{n+1}(\Delta t \pi_h^* \underline{\xi}_{n+1}) = \frac{2}{3}\Delta t \sum_{n=1}^{R-1}[A(U_{n+1},\theta_{n+1},\pi_h^*\underline{\xi}_{n+1}) + (A(u_{n+1},v_{n+1},\pi_h^*\underline{\xi}_{n+1})$$

$$- A(U_{n+1},v_{n+1},\pi_h^*\underline{\xi}_{n+1}))] \leq C\left(h^4 + h\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2\right) + \varepsilon\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_1^2$$

In further analysis, we need two hypothesis (C) as follows:

(1) There exists a positive constant M, such that

$$\sup_{1\leq n\leq R-1}\|EV_{n+1}\|_{0,\infty} \leq M$$

(2) $\varepsilon_0 > 0$, when Δt and h are sufficiently small, we have

$$\sup_{1\leq n\leq R-1}\|v_{n+1} - EV_{n+1}\|_{0,\infty} \leq \varepsilon_0$$

We will employ mathematical induction to proof the above (1) and (2). For $n = 1$, combing inverse property and interpolation estimate of finite element method, together with initial value condition (B) and $\Delta t = O(h^3)$, as a result, we can obtain $\|EV_2\|_{0,\infty} \leq 2\|v_1 - V_1\|_{0,\infty} + \|v_0 - V_0\|_{0,\infty} + 2\|v_1\|_{0,\infty} + \|v_0\|_{0,\infty} \leq M$ $\|v_2 - EV_2\|_{0,\infty} \leq C(\|v_2 - 2v_1 + v_0\|_{0,\infty} + 2\|v_1 - V_1\|_{0,\infty} + \|v_0 - V_0\|_{0,\infty} \leq C_1\Delta t^2 + C_2 h + C_3 h \leq \varepsilon_0$

Assume that

$$\sup_{1\leq n\leq R-2}\|EV_{n+1}\|_{0,\infty} \leq M, \sup_{1\leq n\leq R-2}\|v_{n+1} - EV_{n+1}\|_{0,\infty} \leq \varepsilon_0$$

then we give the proof for

$$\sup_{1\leq n\leq R-1}\|EV_{n+1}\|_{0,\infty} \leq M, \sup_{1\leq n\leq R-1}\|v_{n+1} - EV_{n+1}\|_{0,\infty} \leq \varepsilon_0$$

while taking into account error estimates of the other terms.

For sufficiently small Δt and h, a combination of assumption (A) and (C) results in $|H(u_{n+1},v_{n+1})-H(U_{n+1},EV_{n+1})|\leq(1+|f(u_{n+1})|)\cdot|v_{n+1}-EV_{n+1}|+(L_1|EV_{n+1}|+L_2)\cdot|u_{n+1}-U_{n+1}|\leq C(|u_{n+1}-U_{n+1}|+|v_n-V_n|+|v_{n-1}-V_{n-1}|+|v_{n+1}-2v_n+v_{n-1}|)$. This together with initial value condition (B) yields

$$\begin{aligned}\sum_{n=1}^{R-1}T_6^{n+1}(\Delta t\pi_h^*\underline{\xi}_{n+1}) &= \Delta t\sum_{n=1}^{R-1}(H(U_{n+1},EV_{n+1})-H(u_{n+1},v_{n+1}),\pi_h^*\underline{\xi}_{n+1})\\ &\leq C(h^4+\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2)\end{aligned}\tag{4.16}$$

Noticing that $b_i(x,l,s)$ is $\varepsilon-$continuous with respect to l and s respectively. It leads to $|b(x,u_{n+1},v_{n+1})-b(x,U_{n+1},EV_{n+1})|\leq L_1|u_{n+1}-U_{n+1}|+L_2|v_{n+1}-EV_{n+1}|\leq C(|u_{n+1}-U_{n+1}|+|v_n-V_n|+|v_{n-1}-V_{n-1}|+|v_{n+1}-2v_n+v_{n-1}|)$ which together with (2.6) and initial value condition (B) implies

$$\begin{aligned}\sum_{n=1}^{R-1}T_7^{n+1}(\Delta t\pi_h^*\underline{\xi}_{n+1}) &= \Delta t\sum_{n=1}^{R-1}(b(x,u_{n+1},v_{n+1})-b(x,U_{n+1},EV_{n+1}),\pi_h^*\underline{\xi}_{n+1})\\ &\leq C\left(h^4+\Delta t^4+\Delta t\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2\right)\end{aligned}\tag{4.17}$$

Now we turn to the error estimate in the left-hand side terms of (4.1).

To estimate the first term, note that poincare inequality and (3.5), we obtain

$$\frac{2}{3}\Delta t\sum_{n=1}^{R-1}a(\xi_{n+1},\pi_h^*\underline{\xi}_{n+1})\geq C\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_1^2\tag{4.18}$$

The proof of the second term parallels to that of (4.9), we have

$$\sum_{n=1}^{R-1}(\frac{\xi_{n+1}-\xi_n}{\Delta t},\Delta t\pi_h^*\underline{\xi}_{n+1})=\frac{1}{2}[\sum_{n=1}^{R-1}\|\delta\xi_{n+1}\|_{0,h}^2+\|\xi_R\|_{0,h}^2-\|\xi_1\|_{0,h}^2]\tag{4.19}$$

Collecting (4.11)-(4.19) and (4.10), let Δt and h be sufficiently small, we can get

$$\begin{aligned}\|\sigma_R\|_0^2+\|\xi_R\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_1^2 &\leq C(h^4+\Delta t^4+\|\xi_1\|_0^2+\|\sigma_1\|_0^2\\ &+(h+\Delta t)\sum_{n=1}^{R-1}\|\sigma_{n+1}\|_0^2+\Delta t\sum_{n=1}^{R-1}\|\xi_{n+1}\|_0^2)\end{aligned}\tag{4.20}$$

Consequently, the discrete Gronwall inequality argument and initial value condition (B) produce

$$\|\sigma_R\|_0^2+\|\xi_R\|_0^2+\Delta t\sum_{n=0}^{R}\|\xi_n\|_1^2\leq C(\Delta t^4+h^4)\tag{4.21}$$

Combining (4.21) with (2.6) leads to

$$\sup_{0\le n\le R}(\|u_n - U_n\|_0^2 + \|v_n - V_n\|_0^2) + \Delta t\sum_{n=0}^{R}\|v_n - V_n\|_1^2 \le C(h^4 + \Delta t^4) \tag{4.22}$$

where the positive integer R is not greater that N. It remains to check the induction hypothesis (1) and (2).

For $n = R-1$, $\|EV_R\|_{0,\infty} \le 2\|v_{R-1} - V_{R-1}\|_{0,\infty} + \|v_{R-2} - V_{R-2}\|_{0,\infty} + 2\|v_{R-1}\|_{0,\infty} + \|v_{R-2}\|_{0,\infty} \le C_1 + C_2 h \le M$ $\|v_R - EV_R\|_{0,\infty} \le C(\|v_R - 2v_{R-1} + v_{R-2}\|_{0,\infty} + 2\|v_{R-1} - V_{R-1}\|_{0,\infty} + \|v_{R-2} - V_{R-2}\|_{0,\infty} \le C_1\Delta t^2 + C_2 h \le \varepsilon_0$

Theorem 4.1. *Let u, v be the solutions to problem* (2.1)–(2.4), $\{U_k\}_{k=0}^N, \{V_k\}_{k=0}^N$ *to the multistep finite volume element scheme* (2.17)–(2.19). *Suppose that initial value U_i and V_i $(i = 0, 1)$ satisfy the conditions (B), i.e.,*

$$\sum_{i=0}^{1}(\|U_i - \pi_h u_i\|_0 + \|V_i - \pi_h v_i\|_0 + \Delta t^{\frac{1}{2}}\|V_i - \pi_h v_i\|_1) \le C(\Delta t^2 + h^2)$$

if partition parameters satisfy $\Delta t = O(h^3)$, provided initial assumption (A) is satisfied, then for sufficiently small Δt and h, the following error estimate holds

$$\sup_{0\le k\le N}(\|u_k - U_k\|_0 + \|v_k - V_k\|_0) + (\Delta t\sum_{k=0}^{N}\|v_k - V_k\|_1^2)^{\frac{1}{2}} \le C(\Delta t^2 + h^2) \tag{4.23}$$

5. NUMERICAL EXPERIMENT

The multistep finite volume element method is used to approximate the following nonlinear hyperbolic equation :

$$u_{tt} - \triangle u_t - \triangle u = r(x,t), x \in \Omega = (0,\pi)\times(0,\pi), \quad t \in \left(0, \frac{1}{2}\right]$$

$$u(x,0) = \sin 2x_1 \sin x_2, u_t(x,0) = -\sin 2x_1 \sin x_2, \quad x \in \Omega$$

$$u = 0, \quad u_t = 0, \quad x \in \partial\Omega, \quad t \in \left(0, \frac{1}{2}\right]$$

where the true solution $u = e^{-t}\sin 2x_1 \sin x_2$, $r(x,t) = u = e^{-t}\sin 2x_1 \sin x_2$.

To obtain numerical solution of this problem, we place over $\overline{\Omega} = [0,\pi]\times[0,\pi]$ $10 \times 10 = 100$ uniform squares, ending up with a square mesh; then we further decompose it into right triangulation by drawing the dragonal of each small square. $\frac{\pi}{10}$ and 0.1 respectively denote space mesh size h and time step size Δt.

Three methods are used to solve that problem.

(1) The multistep characteristic finite volume element method on triangular meshes denoted by CMTFVM.

(2) The one-step characteristic finite volume element method on triangular meshes denoted by COTFVM

(3) The multistep bilinear finite volume element method along characteristics on quadrilateral networks denoted by CQFVM.

The numerical results are presented in the following tables.

Table 1. The comparison among CMTFVM, COTFVM and CQFVM about u, $T = 0.3$

	CMTFVM U_h	COTFVM U_h	CQFVM U_h	TS u
$(\frac{4}{5}\pi, \frac{1}{5}\pi)$	-0. 4131718	-0. 4178679	-0. 3753615	-0. 4141300
$(\frac{4}{5}\pi, \frac{2}{5}\pi)$	-0. 6691440	-0. 6774786	-0. 6073465	-0. 6700764
$(\frac{4}{5}\pi, \frac{3}{5}\pi)$	-0. 6692150	-0. 6776754	-0. 6073467	-0. 6700764
$(\frac{4}{5}\pi, \frac{4}{5}\pi)$	-0. 4136360	-0. 4188983	-0. 3753621	-0. 4141301
$(\frac{2}{5}\pi, \frac{1}{10}\pi)$	0. 1337142	0. 1346008	0. 1219627	0. 1345590
$(\frac{2}{5}\pi, \frac{3}{10}\pi)$	0. 3516002	0. 3555177	0. 3193021	0. 3522801
$(\frac{2}{5}\pi, \frac{1}{2}\pi)$	0. 4348727	0. 4403433	0. 3946794	0. 4354421
$(\frac{2}{5}\pi, \frac{7}{10}\pi)$	0. 3519777	0. 3567838	0. 3193008	0. 3522801

Table 2. The comparison among CMTFVM, COTFVM and CQFVM about v, $T = 0.3$

	CMTFVM V_h	COTFVM V_h	CQFVM V_h	TS v
$(\frac{4}{5}\pi, \frac{1}{5}\pi)$	0. 4208816	0. 4297562	0. 3682060	0. 4141300
$(\frac{4}{5}\pi, \frac{2}{5}\pi)$	0. 6750816	0. 6894825	0. 5957708	0. 6700764
$(\frac{4}{5}\pi, \frac{3}{5}\pi)$	0. 6743754	0. 6886085	0. 5957703	0. 6700764
$(\frac{4}{5}\pi, \frac{4}{5}\pi)$	0. 4165091	0. 4253623	0. 3682060	0. 4141301
$(\frac{2}{5}\pi, \frac{1}{10}\pi)$	-0. 1413925	-0. 1442637	-0. 1196371	-0. 1345590
$(\frac{2}{5}\pi, \frac{3}{10}\pi)$	-0. 3566366	-0. 3654419	-0. 3132144	-0. 3522801
$(\frac{2}{5}\pi, \frac{1}{2}\pi)$	-0. 4383393	-0. 4476261	-0. 3871548	-0. 4354421
$(\frac{2}{5}\pi, \frac{7}{10}\pi)$	-0. 3532283	-0. 3597593	-0. 3132153	-0. 3522801

Table 3. The comparison e of maximum absolute error and average absolute error

	maximum absolute error	average absolute error
CMTFVM(u)	6. 5546110E-03	8. 9920021E-04
COTFVM(u)	9. 6043348E-03	3. 1863260E-03
CQFVM(u)	8. 1847012E-02	4. 1811626E-02
CMTFVM(v)	1. 9632958E-02	4. 0403814E-03
COTFVM(v)	2. 9337764E-02	9. 8494338E-03
CQFVM(v)	8. 5374057E-02	4. 9320929E-02

Tables 1 and 2 give the numerical results and their corresponding true solution (TS) about u and u_t, respectively. The maximum absolute error and the average absolute error is also provided in Table 3. From the Table 1, it is easy to see that the finite volume element method on the triangular mesh behaves better than the one on the quadrilateral mesh, but the algorithm on triangulation is slightly more complicated than the case that the mesh is quadrilateral. One can also find that the accuracy of multistep on triangular grid is much better than that of one-step. For the case of V_h, we have the similar results in terms of Table 2. From Table 3, for either U_h or V_h, we can see that the maximum absolute error and the average absolute error of CMTFVM is the smallest among CMTFVM, COTFVM and CQFVM.

REFERENCES

[1] LeVeque, R.J., 2002: Finite Volume Methods for Hyperbolic Problems, Cambridge University Press, London.

[2] Ewing, R.E., T. Lin and Y. Lin, 2002: On the accuracy of the finite volume element method based piecewise linear polynomials, SIAM J. Numer. Anal., 39: 1865–1888.

[3] Chatzipanteidis P. and R. D. Lazarov, 2005: Error estimates for finite volume element method for elliptic PDE's in nonconvex polygonal domains, SIAM J. Numer.Anal., 42(5): 1932–1958.

[4] Carstensen, C., R.D. Lazarov, and S. Tomov, 2005: Explicit and averaging a posteriori error estimates for adaptive finite volume methods, SIAM J. Numer.Anal. 42(6): 2496–2521.

[5] Wang, H., W.D. Zhao, 2003: An upwind finite volume scheme and its Maximumu-Principle-Preserving ADI splitting for unsteady-stat advection-diffusion equations, Numer. Mathods for PDE, 19:211–226.

[6] Pao, C.V., 1975: An mixed initial boundary value problem arising in Neurophusicology, J. Math. Anal. Appl., 52:105–119.

[7] Zhang, Z. Y., 2002: The new ADI characteristics finite element methods for the three-dimensional generalized nerve conduction equation, N. P. Sci. comp., 10: 355–370.

[8] Zhang, Z. Y. and H. Wei, 2003: A multisep characteristic finite difference method for a class of nerve conduction equations, N. P. Sci. comp., 11:315–323.

[9] Zhang, Z. Y., 2004: An economical difference scheme for heat transport equation at the microscale, Numer. Meth. for PDEs, 20: 855–863.

[10] Yang M. and Y. R. Yuan, 2004: A multistep finite volume element scheme along characteristics for nonlinear convection diffusion problems, Math. Numer. Sinica, 4: 484–496.

[11] Li, R.Z. Chen and W. Wu, 2000: Generalized difference methods for differential equations, Volume 226 of Monographs and Textbooks in Pure and Applied Mathematics, Numerical analysis of finite volume methods. Marcel Dekker Inc., New York.

[12] Chen, Z. Y., 1990: A generalized difference method for the equations of heat conduction, Acta. Sci. Natur. Univ. Sunyaseni, 1: 6–13.

[13] Tian, M. and Z. Y. Chen, 1991: Quadratic element generalized difference methods for elliptic equation, Numer. Math. J. Chinese universities, 2: 99–113.

[14] Brenner, S. C. and L. R. Scott, The mathematical theory of finite element methods, Springer Verlag, New York, 1994

[15] M. Yang, 2004: Quadratic finite volume element methods for nonlinear parabolic equation, Numer. Math. J. Chinese universities, 3: 257–266.

[16] F. R. Thomas, Time stepping along characteristics with incomplete iteration for a Garlerkin approximation of miscible displacement in porous media, SIAM. J. Numer. Anal, 5 (1985) 970–1013