Neural, Parallel, and Scientific Computations 16 (2008) 579-592

Numerical Investigation of a PH/PH/1 Inventory System with Positive Service Time and Shortage

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Abstract: The aim of this paper is to numerically investigate a PH/PH/1 inventory model with reneging of customers and finite shortage of items. We assume that arrivals occur according to a phase type renewal process. The associated phase type distribution has representation $(\boldsymbol{\alpha}, U)$. The service times are identically and independently distributed random variables having common phase type distribution with representation $(\boldsymbol{\beta}, V)$. The lead-time is zero. Costumers renege from the system at a constant rate γ. Shortage is permitted and hence shortage cost is finite. We perform the steady state analysis of the inventory model using Matrix analytic method. A suitable cost function is defined and analyzed numerically. The optimal shortage level is numerically evaluated. Some measures of the system performance in the steady state are also derived.

Key words: Inventory, Reneging, Retrial, Phase type distribution, Matrix analytic method, Cost analysis.

AMS 1991 Subject classification: 90B05, 90B22.

1. Introduction

Phase type (*PH*) distribution, which was introduced by Neuts [8], generalizes the conventional approach employing only exponential distributions and other distributions such as Erlang, Generalized Erlang, and Coxian. In contrast to exponential distribution (which is closed under minimum only), the class of Phase-type distributions has very strong closure properties: they are closed under maximum, minimum and convolution (see Neuts [9]). Some work related to discrete PH distributions could be found in Alfa [1]; those related to continuous PH distributions could be found in Chakravarthy [4], Chakravarthy, Krishnamoorthy and Joshua [5].

Received June 1, 2007

The first published work on stochastic inventory with service time is due to Parthasarathy and Vijayalakshmi [10]. They provide the transient analysis of an (S-1, S) inventory model with positive service time. Artalejo, Krishnamoorthy and Lopez Herrero [2] were the first to study inventory policies with positive lead-time and retrial of customers who could not get the item during their earlier attempts to access the service station. Where as their approach was algorithmic, the paper by Ushakumari [12] is purely analytical. Recently, Schwarz *et al.* [11] discussed M/M/1 Queueing systems with inventory where service times and lead times are exponentially distributed. They derived stationary distribution of joint queue length and inventory processes in explicit product form and proved that the limiting distribution of the queue length process is same as that in classical M/M/1/∞ system. The product form solution could be arrived at solely because of the assumption that customers do not join the system when inventory level is zero.

In this paper, we consider the inter-arrival time distribution of customers to be phase type with representation $(\boldsymbol{\alpha}, U)$. The service times have common phase type distribution with representation $(\boldsymbol{\beta}, V)$. The lead-time is assumed to be zero. Customers join the system and tend to leave from it with positive probability without getting any service. We compute the long run behavior of the system in the steady sate. A number of descriptors of the system are provided. A cost function associated with the model is numerically investigated. When the shortage cost is finite and relatively less than the holding cost of the inventoried items, the system orders for replenishment of the item only on accumulation of a certain number of customers. The shortage cost can be measured in terms the waiting cost of customers in the absence of inventory. In this paper, we assume that shortage cost is less than holding cost. Thus when the number of customers in the system accumulates to say K, an order is placed for replenishment and received instantly since the lead time is zero. However, customers tend to renege from the system. Here we assume the reneging rate to be the constant γ, irrespective of the number of customers waiting (i.e., excluding one in service). An objective of this paper is to compute the optimal value of K. Apart from this we also investigate the optimal number of items (S) to be purchased at each replenishment epoch. Since the model is quite complex one can not expect analytical tractability of the problem. Hence we approach the problem algorithmically.

This paper is organized as follows. Section 2 deals with mathematical description of the model. Section 3 presents stability condition. Section 4 describes algorithmic analysis. Section 5 gives performance measures of the system. Finally, cost analysis and numerical results are included in section 6.

2.Mathematical Description of the Model

The following Assumptions and Notations are used for the analysis of the model.

Assumptions

(i) Maximum inventory level is S .

(ii) Inter-arrival distribution is phase type with representation $(\boldsymbol{\alpha}, U)$.

(iii) Lead-time is zero.

(iv) Service times have phase type distribution with representation $(\boldsymbol{\beta}, V)$.

(v) The reneging rate is a constant γ, when there are $i\ (\geq 2)$ customers in the system.

(vi) A maximum of $K(>0)$ shortages is allowed in the system.

Notations

$N(t)$: Number of customers in the system at time t .

$I(t)$: Inventory level at the time t .

$J_1(t)$: Phase of the arrival process at time t .

$J_2(t)$: Phase of the service process at time t , if a service is going on at that time.

e : $(1,1,1,.....1)'$, column vector of 1's of appropriate order.

Let $I(t)$, $N(t)$, $J_1(t)$ and $J_2(t)$ be respectively the inventory level , number of customers in the system, phase of the arrival process and phase of the service process at time t . Write $X(t)=\{(N(t),I(t),J_1(t),J_2(t));t\geq 0\}$. Then $\{X(t),t\geq 0\}$ is a level independent quasi-birth death process (LIQBD) on the state space

$$\{(0,j,k,0);1\leq j\leq S-1;1\leq k\leq m_1\}\bigcup\{(i,0,k,0);1\leq i\leq K-1;1\leq k\leq m_1\}\bigcup$$
$$\{(i,j,k,l);i\geq 1,1\leq j\leq S;1\leq k\leq m_1,1\leq l\leq m_2\}.$$

Here the value 0 in the last coordinate indicates that no service is going on due to either the absence of customers or zero inventory level or both. Also it may be noted that if the inventory level is reduced to zero at a customer's departure epoch, then an order for replenishment will be placed only on accumulation of K customers. This is done to ensure that at least certain minimum number of customers is served continuously in a busy cycle. This is because the fixed cost of ordering can be very high compared to the holding cost of customers.

The infinitesimal generator Q of the process is a block tri-diagonal matrix given by

$$Q=\begin{pmatrix} B_0 & A_{0,0} \\ A_{2,1} & A_{1,1} & A_{0,1} \\ & A_{2,2} & A_{1,2} & A_{0,2} \\ & & \ddots & \ddots & \ddots \\ & & & A_{2,K-1} & A_{1,K-1} & A_{0,K-1} \\ & & & & A_{2,K} & A_1 & A_0 \\ & & & & & A_2 & A_1 & A_0 \\ & & & & & & A_2 & A_1 & A_0 \\ & & & & & & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1}$$

where $B_0 = I_S \otimes U$, a square matrix of order $m_1 S$; $A_{0,0} = diag\left(U^0\alpha, I_{S-1} \otimes (U^0\alpha \otimes \beta)\right)$, which is a matrix of order $m_1 S \times (m_1 m_2 S + m_1)$; $A_{0,1} = diag\left(\mathbf{U}^0\boldsymbol{\alpha}, I_S \otimes (U \otimes I_{m_2})\right)$,

$A_{1,1} = diag\left(U - \gamma I_{m_1}, I_S \otimes (U \oplus V)\right)$

and $A_{1,2} = diag\left((U-\gamma I_{m_1}), I_S \otimes (U \oplus V - \gamma I_{m_1 m_2}\right)$. Also, $A_{2,1}$ is a matrix with entries γI_{m_1} , $I_{m_1} \otimes \mathbf{V}^0$ and $I_{S-1} \otimes \left(I_{m_1} \otimes \mathbf{V}^0\right)$ respectively on first position in first row, first position in second row and last in last row; all other entries are zero. $A_{2,2}$ is a matrix with all entries on the main diagonal are $\gamma I_{m_1 m_2}$ except γI_{m_1} in the first position and all entries on the lower diagonal are equal to $I_{m_1} \otimes \mathbf{V}^0\boldsymbol{\beta}$ except $I_{m_1} \otimes \mathbf{V}^0$ in the first position; entries other than along main and lower diagonals are zero.

In the above the notations $\otimes$, $\oplus$ stand for Kronecker product and sum, respectively. For details about the Kronecker operations on matrices, we refer the reader to Bellman [3]. Note that

$$A_{0,2} = A_{0,3} = ... = A_{0,K-3} = A_{0,K-2} = A_{0,1},$$
$$A_{1,3} = A_{1,4} = ... = A_{1,K-2} = A_{1,K-1} = A_{1,2},$$
$$A_{2,3} = A_{2,4} = ... = A_{2,K-2} = A_{2,K-1} = A_{2,2}, \text{ where}$$

$A_{0,i}\,(1 \le i \le K-2)$, $A_{1,i}\,(2 \le i \le K-1)$ and $A_{2,i}\,(2 \le i \le K-1)$ are square matrices of the same order $(m_1 m_2 S + m_1)$. We note that $A_{0,K-1}$ is a matrix with entries $(\mathbf{U}^0\boldsymbol{\alpha}) \otimes \boldsymbol{\beta}$, $I_{S-1} \otimes \left((\mathbf{U}^0\boldsymbol{\alpha}) \otimes I_{m_2}\right)$ and $(\mathbf{U}^0\boldsymbol{\alpha}) \otimes I_{m_2}$ respectively on first position in first row, first position in second row and last in last row; all other entries are zero. $A_{2,K}$ is a rectangular matrix with entries $I_{m_1} \otimes \mathbf{V}^0, I_{m_1} \otimes \mathbf{V}^0\boldsymbol{\beta}$ and $\gamma I_{m_1 m_2}$ respectively on (1,1)th , (i, i) th and $(i, i+1)$ th positions; all other entries are zero. Also $A_0 = I_S \otimes \left((\mathbf{U}^0\boldsymbol{\alpha}) \otimes I_{m_2}\right)$, $A_1 = I_S \otimes \left((U \oplus V) - \gamma I_{m_1 m_2}\right)$ and

A_2 is a matrix with all entries on the main diagonal $\gamma I_{m_1m_2}$; entries on the lower diagonal and in the upper right hand corner are equal to $I_{m_1} \otimes \mathbf{V}^0\boldsymbol{\beta}$ and all other entries are zero.

3. Stability Condition

Define the generator A as $A = A_0 + A_1 + A_2$. Then A is a matrix with all entries on the main diagonal $U \oplus V + (\mathbf{U}^0\boldsymbol{\alpha}) \otimes Im_2$; entries on the lower diagonal and in the upper right hand corner are $I_{m_1} \otimes \mathbf{V}^0\boldsymbol{\beta}$ and all other entries are zero.

Now we can represent A as $A = A_U + A_V$, where $A_U = diag\left((U + \mathbf{U}^0\boldsymbol{\alpha}) \otimes Im_2\right)$ and A_V is a matrix with all entries on the main diagonal $I_{m_2} \otimes V$; entries on the lower diagonal and in the upper right hand corner are $I_{m_1} \otimes \mathbf{V}^0\boldsymbol{\beta}$ and all other entries are zero.

Theorem 3.1: The system is stable if and only if

$$\rho < 1 \tag{2}$$

where $\rho = \left(\tilde{\boldsymbol{\pi}}\mathbf{U}^0 / (\gamma + \tilde{\tilde{\boldsymbol{\pi}}}\mathbf{V}^0)\right)$ with $\tilde{\boldsymbol{\pi}}$ satisfying $\tilde{\boldsymbol{\pi}}(U + \mathbf{U}^0\boldsymbol{\alpha}) = 0$, $\tilde{\boldsymbol{\pi}}\mathbf{e} = 1$; and $\tilde{\tilde{\boldsymbol{\pi}}}$ satisfying $\tilde{\tilde{\boldsymbol{\pi}}}(V + \mathbf{V}^0\boldsymbol{\beta}) = 0$, $\tilde{\tilde{\boldsymbol{\pi}}}\mathbf{e} = 1$.

Proof: From the well-known result due to Neuts [8] we have Q is positive recurrent iff

$$\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e} \tag{3}$$

where $\boldsymbol{\pi}$ is the steady state probability vector of A.

That is,

$$\boldsymbol{\pi}A = 0 \tag{4}$$

and

$$\boldsymbol{\pi}\mathbf{e} = 1. \tag{5}$$

Now $\boldsymbol{\pi} = \frac{1}{S}\left(\mathbf{e}_S' \otimes (\tilde{\boldsymbol{\pi}} \otimes \tilde{\tilde{\boldsymbol{\pi}}})\right)$ satisfies (4) and (5) where $\tilde{\boldsymbol{\pi}}$ is such that $\tilde{\boldsymbol{\pi}}(U + \mathbf{U}^0\boldsymbol{\alpha}) = 0$, $\tilde{\boldsymbol{\pi}}\mathbf{e} = 1$ and $\tilde{\tilde{\boldsymbol{\pi}}}$ is such that $\tilde{\tilde{\boldsymbol{\pi}}}(V + \mathbf{V}^0\boldsymbol{\beta}) = 0$, $\tilde{\tilde{\boldsymbol{\pi}}}\mathbf{e} = 1$. Substituting $\boldsymbol{\pi}$ in (3) we get (2). This completes the proof.

3.1 Steady State Probability Vector

Let $\mathbf{x} = \left(x_0, x_1, ...x_K, x_{K+1}, ...\right)$ be the steady state probability vector of Q. Under the stability condition (2), x_is$'$ $(i \geq N)$ are given by

$$x_{K+r} = x_K R^r \, (r \geq 1)$$

where R is the unique non- negative solution of the equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

in which the spectral radius is less than one and the vectors $x_0, x_1 ......, x_K$ are given by solving the following equations

$$\left.\begin{aligned} & x_0 B_0 + x_1 A_{2,1} = 0 \\ & x_{i-1} A_{0,i-1} + x_i A_{1,i} + x_{i+1} A_{2,i+1} = 0, (1 \leq i \leq K-1) \\ & x_{K-1}(A_{0,K-1} + RA_1 + R^2 A_2) = 0 \end{aligned}\right\} \tag{6}$$

subject to the normalizing condition

$$\left(\sum_{i=1}^{K-1} x_i + x_K (I - R)^{-1}\right)\mathbf{e} = 1 \, . \tag{7}$$

4. Algorithmic Analysis

4.1 Evaluation of the Rate Matrix R

To find the rate matrix R we use the relation

$$R = A_0(-A_1 - A_0 G)^{-1},$$

where the matrix G is the minimal nonnegative solution of the matrix quadratic equation

$$A_2 + A_1 G + A_0 G^2 = 0 \, .$$

The matrix G will be stochastic if $sp(R) < 1$. The logarithmic reduction algorithm due to Ramaswami (see Latouche and Ramaswami [7]) can be used to evaluate *R*..

4.2 Computation of the Boundary Probabilities

Let $\mathbf{x}^*$ be the partitioned vector $(x_0, x_1 ......, x_K)$ corresponding to the boundary portion of Q as in (1). Then $\mathbf{x}^*$ is the stationary vector normalized by (7) of the infinitesimal generator T shown below

$$T = \begin{pmatrix} B_0 & A_{0,0} & & & & \\ A_{2,1} & A_{1,1} & A_{0,1} & & & \\ & A_{2,2} & A_{1,2} & A_{0,2} & & \\ & & \ddots & \ddots & \ddots & \\ & & & A_{2,K-1} & A_{1,K-1} & A_{0,K-1} \\ & & & & A_{2,K} & A_1 + RA_2 \end{pmatrix}.$$

Now the system (6) can be written as $\mathbf{x}^*T = 0$. To solve this system, we use the block Gauss-Seidel iterative scheme. The vectors $x_0, x_1, \ldots\ldots x_K$ in the $(n+1)$th iteration are given by

$$x_0(n+1) = x_1(n)A_{2,1}B_0^{-1}$$
$$x_i(n+1) = [x_{i+1}(n)A_{2,i+1} + x_{i-1}(n+1)A_{0,i-1}]A_{1,i}^{-1}\ , \quad (1 \le i \le (K-1))$$
$$x_K(n+1) = -x_{K-1}(n+1)A_{0,K-1}(A_1 + RA_2)^{-1}.$$

After each iteration, the elements of $\mathbf{x}^*$ may be scaled to satisfy (7).

5. System Performance Measures

The components of the steady state probability vector $\mathbf{x}=(x_0,x_1,\ldots\ldots,x_{K-1},x_K,\ldots\ldots)$ can be partitioned as

$$x_0 = \left(y_{0,j,k,0}\right)\ ,\ \ 0 \le j \le (S-1) \text{ and } 1 \le k \le m_1;$$

for $1 \le i \le K-1$, $x_i = \left(y_{i,j,k,l}\right)$, $0 \le j \le S$, $1 \le k \le m_1$, $1 \le l \le m_2$, with $l = 0$ when $j = 0$; for $i \ge K$, $x_i = \left(y_{i,j,k,l}\right)$, $1 \le j \le S$, $1 \le k \le m_1$ and $1 \le l \le m_2$. Then we have

(i) Expected re-order rate, ERO, is given by

$$ERO = \sum_{k=1}^{m_1}\left((y_{K-1,0,k,0})\left(\mathbf{U}^0(k)\right)\right)$$

(ii) Expected inventory level, EI, is given by

$$EI = \sum_{j=0}^{S-1} j\left(\sum_{k=1}^{m_1} y_{0,j,k,0}\right) + \sum_{j=1}^{S} j\left(\sum_{i=1}^{\infty}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2} y_{i,j,k,l}\right)$$

(iii) Expected number of departures after receiving service/unit time, EDS, is given by

$$EDS = \sum_{i=1}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}\left((y_{i,j,k,l})(\mathbf{V}^0(l))\right)$$

(iv) Expected number of departures due to reneging of customers/unit time, EDR, is given by

$$EDR = \gamma\left(\left(\sum_{i=2}^{\infty}\sum_{j=1}^{S}\sum_{k=1}^{m_1}\sum_{l=1}^{m_2}(y_{i,j,k,l})\right) + \left(\sum_{i=1}^{K-1}\sum_{k=1}^{m_1}(y_{i,0,k,0})\right)\right)$$

(v) Expected number of customers, EC, in the system is given by

$$EC = \left(\sum_{i=1}^{\infty} ix_i\right)\mathbf{e}$$
$$= \left((\sum_{i=1}^{K-1} ix_i) + x_K\left(K(I-R)^{-1} + R(I-R)^{-2}\right)\right)\mathbf{e}$$

(vi) Expected shortages, ES, in the system is given by

$$ES = \left(\sum_{i=1}^{K-1} ix_i\right)\mathbf{e}$$

6. Cost Analysis and Numerical Results

6.1 Cost Analysis

In order to construct a cost function explicitly, we define the following costs

C = fixed cost

c_1 = procurement cost/unit

c_2 = holding cost of inventory /unit /unit time

c_3 = service cost/unit/unit time

c_4 = loss due to reneging of customers/ unit /unit time

c_5 = holding cost of customers / unit /unit time

c_6 = shortage cost/unit/unit time

c_7 = revenue (profit) due to service / unit /unit time

The expected total cost (ETC) of the system/unit time is given by

$$ETC = (C + c_1 S)\,ERO + c_2 EI + (c_3 - c_7) EDS + c_4 EDR + c_5 EC + c_6 ES \,.$$

6.2 Numerical results

In the following tables and graphs, we provide numerical values of different performance measures and expected total cost of the system. In this, the input parameters are

$$m_1 = 2,\; m_2 = 2, \boldsymbol{\alpha} = (0.5, 0.5),\; \boldsymbol{\beta} = (0.5, 0.5)\,,$$

$$\mathbf{U}^0 = \begin{pmatrix} 1.5 \\ 1.5 \end{pmatrix},\; \mathbf{V}^0 = \begin{pmatrix} 4.5 \\ 3.5 \end{pmatrix},\; U = \begin{pmatrix} -2.0 & 0.5 \\ 0.5 & -2.0 \end{pmatrix},\; V = \begin{pmatrix} -5.0 & 0.5 \\ 0.5 & -4.0 \end{pmatrix}$$

The numerical values in the tables and total expected cost of the system are obtained using computations with FORTRAN program. The procedure of the program can be summarized in the following five steps.

Step 1: *Input the matrices* A_0 *,* A_1 *,* A_2 *and boundary matrices.*

Step 2: Check *the stability condition (2).*

Step 3: *Obtain the rate matrix* R *using the logarithmic reduction algorithm due to Latouche*

and Ramswami [7].

Step 4: *Evaluate the steady sate probability vector* $\mathbf{x}^*$.

Step 5: *Calculate the performance measures and total expected cost of the system.*

TABLE 6.1. Variations in no. shortages K. S =20; $\gamma = 1$.

K	EI	EC	ROR	EDS	EDR	ES
5	5.76406	1.15563	0.05797	1.08832	0.40018	1.01679
6	4.97724	1.49983	0.05392	1.01043	0.47531	1.35370
7	4.31124	1.89804	0.05019	0.94015	0.54324	1.74284
8	3.75169	2.34127	0.04685	0.87744	0.60395	2.17749
9	3.28344	2.82212	0.04388	0.82168	0.65798	2.65072
10	2.89269	3.33463	0.04123	0.77213	0.70602	3.15654
11	2.56752	3.87409	0.03888	0.72805	0.74877	3.69009
12	2.29793	4.43691	0.03678	0.68878	0.78685	4.24761
13	2.07554	5.02055	0.03491	0.65380	0.82075	4.82643
14	1.89333	5.62357	0.03325	0.62269	0.85086	5.42493

TABLE 6.2. Variations in maximum S. *K*=10; $\gamma = 1$.

S	EI	EC	ROR	EDS	EDR	ES
13	1.24635	4.01872	0.05171	0.63077	0.84182	3.79552
14	1.43158	3.89957	0.04977	0.65350	0.82012	3.68468
15	1.63534	3.78931	0.04804	0.67555	0.79899	3.58186
16	1.85657	3.68660	0.04647	0.69675	0.77863	3.48592
17	2.09410	3.59044	0.04502	0.71702	0.75913	3.39601
18	2.34672	3.50010	0.04367	0.73633	0.74053	3.31148
19	2.61328	3.41499	0.04241	0.75469	0.72284	3.23180
20	2.89269	3.33463	0.04123	0.77213	0.70602	3.15654
21	3.18396	3.25861	0.04012	0.78870	0.69005	3.08533
22	3.48616	3.18658	0.03906	0.80443	0.67487	3.01785

TABLE 6.3. Variations in reneging rate γ. K=10; S = 10.

γ	EI	EC	ROR	EDS	EDR	ES
0.5	1.39624	5.75632	0.10177	0.99360	0.47910	5.00223
0.6	1.26978	5.54461	0.09297	0.90076	0.56936	4.85897
0.7	1.14748	5.33238	0.08433	0.81103	0.65742	4.70760
0.8	1.02940	5.11857	0.07589	0.72454	0.74308	4.54734
0.9	0.91572	4.90179	0.06768	0.64150	0.82609	4.37713
1.0	0.80678	4.68051	0.05973	0.56224	0.90606	4.19591
1.1	0.70311	4.45327	0.05212	0.48720	0.98247	4.00295
1.2	0.60547	4.21906	0.04491	0.41694	1.05464	3.79831
1.3	0.51474	3.97773	0.03819	0.35210	1.12181	3.58313
1.4	0.43182	3.73029	0.03202	0.29329	1.18324	3.35986

$C = 100$: $S = 20$; $\gamma = 1$; $c_1 = 50$; $c_2 = 20$;
$c_3 = 10$; $c_4 = 30$; $c_5 = 10$; $c_6 = 10$; $c_7 = 20$.

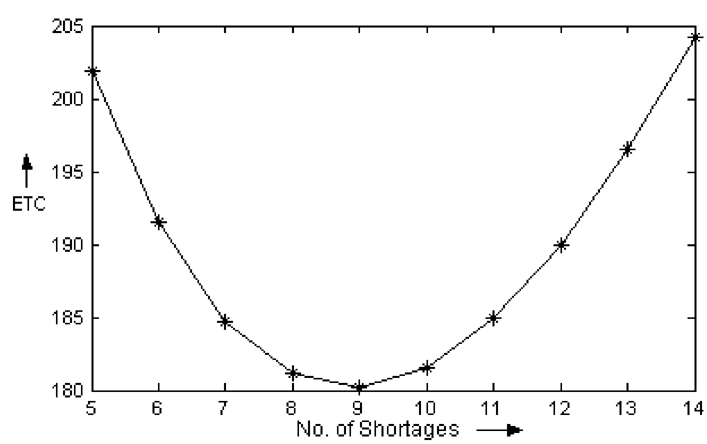

Fig.6.1. **Number of shortage (K) versus ETC**

C =100; K=10; $\gamma = 1$; c_1 =50; c_2 =20;
c_3 =10; c_4 =30; c_5 =10; c_6 =10; c_7 =200.

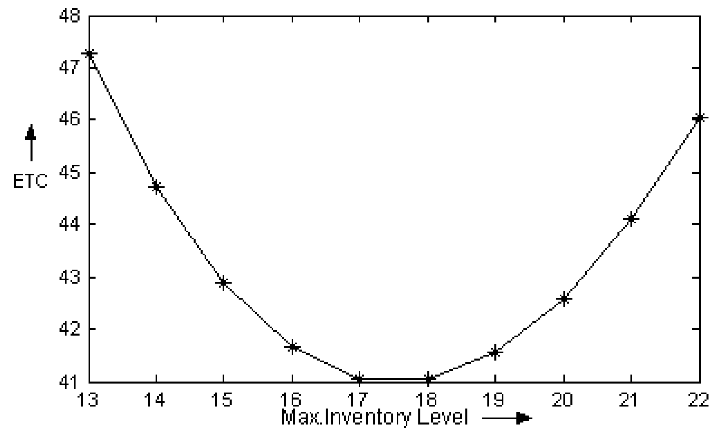

Fig.6.2.**Maximum inventory level (S) versus ETC**

C =100; K=10; $S = 10$; c_1 =50; c_2 =20;
c_3 =10; c_4 =100; c_5 =10; c_6 =10; c_7 =200.

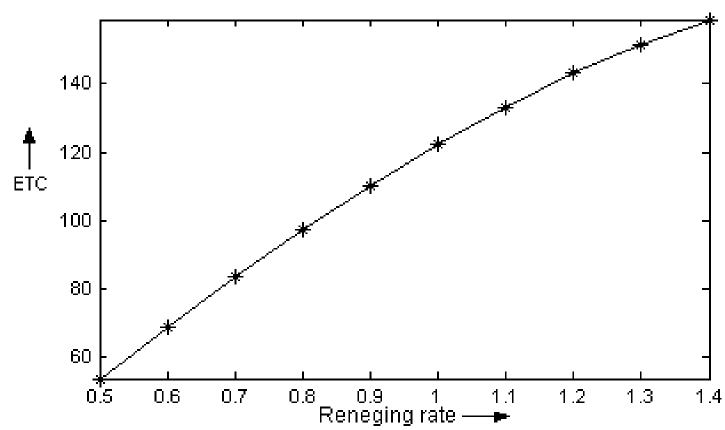

Fig.6.3.**Reneging rate (γ) versus ETC**

6.3. Interpretation of the Numerical Results in the tables

As the number of shortage K is increased, the values of expected number of customers EC, expected system abandonment due to reneging EDR and expected number of shortage ES are also increased (see table 6.1). Again from the same table we note that expected number of customers increases with increasing value of K. Further reorder rate, expected number of services per unit time and expected inventory level are seen to vary inversely with K. With the increase in maximum inventory level S, expected inventory level EI and expected departure due to service EDS will increase (see table 6.2). Also expected number of customers varies inversely with S; this is the case with reorder rate, expected reneging and expected shortage. Table 6.3 indicates that as the reneging rate γ increases, the expected departure due to reneging EDR also increases. However measures such as expected inventory level, expected number of customers in the system, expected number of departures on service completion and expected number of shortages tend to decrease with increasing value of γ.

6.4. Interpretation of the Graphs and Concluding Remarks

In graphical illustrations, we computed the optimal value of the total expected cost per unit time by varying the parameters one at a time and keeping other parameters fixed. By fixing the vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{U}^0, \mathbf{V}^0$ and the matrices U, V and all parameters except the 'number of shortages K', it is clear from the fig.6.1 that the cost function is convex in K and attains its optimum (minimum) value 180.18780 at $K = 9$. As the maximum inventory level S increases (keeping other parameters fixed), the cost function is again seen to be convex. For given parameter values this function attains its optimum value 41.03711 at $S = 18$ (see fig.6.2). As the reneging rate γ increases (keeping other parameters fixed), the expected total cost increases monotonically. (see fig.6.3). It may be noted that because it is not possible to get the closed form expression for the system state probability distribution, the cost function that we constructed cannot be proved to be convex. However, our numerical experiments are indicative of the cost function being either strictly convex or monotone.

Finally, we compare the results obtained here with one of the existing models. Since inventory with positive service time has not been discussed quite extensively, we compare our present results with the results in Krishnamoorthy and Jose [6]. In that paper, lead time is assumed to be positive. Further customers encountering a busy server, will have to proceed to an orbit from where they retry to access the server. The reneging takes place when retrial customers find the busy sever or inventory level zero. In that model, the total expected cost of the system is seen to be either convex or monotone increasing which is the case with the problem investigated in the present paper as well.

Acknowledgement

Krishnamoorthy acknowledges the financial support of Indian Science Lab (General Motors) research project.

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