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Taylor's Decomposition at Several Points for Odd Order Ordinary DEs

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Abstract: *Taylor's decomposition at $2n$ points is presented. $(2n-1)$-step difference schemes generated by Taylor's decomposition at $2n$ points for the numerical solutions of an initial-value problem, a boundary value problem and a nonlocal boundary value problem for a $(2n-1)$-th order differential equation are constructed. Numerical examples are given.*

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1 INTRODUCTION

Modern computers allow the implementation of highly accurate difference schemes for differential equations. Hence, a task of current interest is the construction and investigation of highly accurate difference schemes for differential equations. Difference schemes generated by Taylor's decomposition at several points for the numerical solutions of the linear differential equations of second, third, fourth orders have been studied extensively (see [1]-[3] and the references therein).

The present paper is devoted to the construction and investigation of the new high order of accuracy difference schemes for the approximate solutions of the boundary-value problem of the form

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + a(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = y_0^i,\ i = 0, 1, \ldots, n-1, \\ y^{(i)}(T) = y_T^i,\ i = 0, 1, \ldots, n-2 \end{cases} \tag{1.1}$$

and of the initial-value problem of the form

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + a(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = y_0^i, i = 0, 1, \ldots, 2n-2 \end{cases} \tag{1.2}$$

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and of the nonlocal boundary value problem of the form

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + a(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = y^{(i)}(T), i = 0, 1, \ldots, 2n-2 \end{cases} \tag{1.3}$$

assuming $a(t)$ and $f(t)$ to be such that problems (1.1), (1.2) and (1.3) have a unique smooth solutions defined on $[0, T]$.

The uniform grid space

$$[0, T]_\tau = \{t_k = k\tau, 0 \leq k \leq N, N\tau = T\}$$

is considered for the construction of the $(2n-1)$-step difference schemes for the approximate solutions of problems (1.1), (1.2) and (1.3). It is well known that applying the approximation of $\frac{d^{2n-1}y(t)}{dt^{2n-1}}$on $2n$ points $t_{k+n}, t_{k\pm i} \in [0, T]_\tau$, $i = 0, 1, \ldots, n-1$ we cannot construct better than the following first order of accuracy difference scheme

$$\begin{cases} \tau^{-(2n-1)} \sum\limits_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} + a(t_k)u_k = f(t_k), \\ n-1 \leq k \leq N-n, u_N = y(T),\ \tau^{-j} \sum\limits_{i=0}^{j} u_i(-1)^i C_i^j \\ = y^{(j)}(0),\ j = 0, 1, \ldots, n-1, \tau^{-j} \sum\limits_{i=0}^{j} u_{N-i}(-1)^i C_i^j \\ = y^{(j)}(T),\ j = 0, 1, \ldots, n-2 \end{cases} \tag{1.4}$$

for the numerical solution of the boundary-value problem (1.1), the following first order of accuracy difference scheme

$$\begin{cases} \tau^{-(2n-1)} \sum\limits_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} + a(t_k)u_k = f(t_k), \\ n-1 \leq k \leq N-n,\ \tau^{-j} \sum\limits_{i=0}^{j} u_i(-1)^i C_i^j = \\ y^{(j)}(0), j = 0, 1, \ldots, 2n-2 \end{cases} \tag{1.5}$$

for the numerical solution of the initial-value problem (1.2) and the following first order of accuracy difference scheme

$$\begin{cases} \tau^{-(2n-1)} \sum\limits_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} + a(t_k)u_k = f(t_k), \\ n-1 \leq k \leq N-n, \tau^{-j} \sum\limits_{i=0}^{j} u_i(-1)^i C_i^j = \\ \tau^{-j} \sum\limits_{i=0}^{j} u_{N-i}(-1)^i C_i^j,\ j = 0, 1, \ldots, 2n-2 \end{cases} \tag{1.6}$$

for the numerical solution of the nonlocal boundary-value problem (1.3). Here $C_i^n = \begin{pmatrix} n \\ i \end{pmatrix} = \frac{n!}{(n-i)!i!}$.

In the present paper Taylor's decomposition at $2n$ points is presented. The use of this formula gives the $(2n-1)$-step difference schemes of $2n$-th order of accuracy for the approximate solutions of problems (1.1)-(1.3). Numerical examples are given.

2 TAYLOR'S DECOMPOSITION AT $2n$ POINTS

The utilization of Taylor's decomposition at $2n$ points in the construction of the $(2n-1)$-step difference schemes of the $2n$-th order of accuracy for the approximate solutions of problems (1.1), (1.2) and (1.3) is based on the following theorem.

Theorem 2.1 *Let the function $v(t)$ $(0\le t\le T)$ have a $(4n-2)$-th continuous derivative and $t_{k+n},t_{k\pm i}\in[0,T]_\tau$ for $i=0,1,\dots,n-1$ and α_i, $i=0,1,\dots,2n-1$ be a unique solution for the following system*

$$\sum_{i=0}^{2n-1}\frac{(n-i)^j}{j!}\alpha_i=-\sum_{i=0}^{2n-1}\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^iC_i^{2n-1},\ j=0,1,\dots,2n-1. \tag{2.1}$$

Then the following relation holds:

$$\tau^{-(2n-1)}\sum_{i=0}^{2n-1}v(t_{k+n-i})(-1)^iC_i^{2n-1}+\sum_{j=0}^{2n-1}v^{(2n-1)}(t_{k+n-j})\alpha_j=o(\tau^{2n}). \tag{2.2}$$

Proof. First, we will prove the uniqueness of (2.1). The corresponding matrix form of (2.1) is

$$\begin{aligned} A\alpha &= B \text{ where } A=(a_{ji}) \text{ with } a_{ji}=(n-i)^j,\ B=(b_j) \text{ with}\\ b_j &= -j!\sum_{i=0}^{2n-1}\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^iC_i^{2n-1} \text{ and } \alpha=(\alpha_j),\ j=0,1,\dots,2n-1. \end{aligned}$$

It is clear that A is a Vandermonde type of matrix.Thus,

$$\det A=\prod_{s=0}^{2n-2}\prod_{j=s+1}^{2n-1}(s-j)\neq 0.$$

Then A is invertible and above system has a unique solution.

Second, using Taylor's formula, we get

$$\begin{aligned} &\tau^{-(2n-1)}\sum_{i=0}^{2n-1}v(t_{k+n-i})(-1)^iC_i^{2n-1}\\ &\quad=\tau^{-(2n-1)}\sum_{i=0}^{2n-1}\sum_{j=0}^{4n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^j}{j!}(-1)^iC_i^{2n-1}+o(\tau^{2n}) \end{aligned}$$

and

$$\sum_{j=0}^{2n-1}v^{(2n-1)}(t_{k+n-j})\alpha_j=\sum_{j=0}^{2n-1}\sum_{m=0}^{2n-1}\frac{v^{(2n-1+m)}(t_k)(n-j)^m\tau^m}{m!}\alpha_j+o(\tau^{2n}).$$

The use of Taylor's formula gives

$$\tau^{-(2n-1)}\sum_{i=0}^{2n-1} v(t_{k+n-i})(-1)^i C_i^{2n-1} + \sum_{j=0}^{2n-1} v^{(2n-1)}(t_{k+n-j})\alpha_j$$

$$= \tau^{-(2n-1)}\sum_{i=0}^{2n-1}\sum_{j=0}^{4n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^j}{j!}(-1)^i C_i^{2n-1} + o(\tau^{2n})$$

$$+\sum_{j=0}^{2n-1}\sum_{m=0}^{2n-1}\frac{v^{(2n-1+m)}(t_k)(n-j)^m\tau^m}{m!}\alpha_j + o(\tau^{2n})$$

$$=\sum_{i=0}^{2n-1}[\sum_{j=0}^{4n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{m=0}^{2n-1}\frac{v^{(2n-1+m)}(t_k)(n-i)^m\tau^m}{m!}\alpha_i\ ] + o(\tau^{2n})$$

$$=\sum_{i=0}^{2n-1}[\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{j=2n-1}^{4n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{m=0}^{2n-1}\frac{v^{(2n-1+m)}(t_k)(n-i)^m\tau^m}{m!}\alpha_i\ ] + o(\tau^{2n})$$

$$=\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{i=0}^{2n-1}[\sum_{j=2n-1}^{4n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{m=0}^{2n-1}\frac{v^{(2n-1+m)}(t_k)(n-i)^m\tau^m}{m!}\alpha_i\ ] + o(\tau^{2n})$$

$$=\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1}\tau^j v^{(2n-1+j)}(t_k)\left[\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^i C_i^{2n-1} + \frac{(n-i)^j}{j!}\alpha_i\right]$$

$$+o(\tau^{2n}) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^i C_i^{2n-1}$$

$$+\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-1}\tau^j v^{(2n-1+j)}(t_k)\left[\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^i C_i^{2n-1} + \frac{(n-i)^j}{j!}\alpha_i\right]$$

$$+o(\tau^{2n}) = \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^iC_i^{2n-1}$$
$$+\sum_{j=0}^{2n-1}\tau^j v^{(2n-1+j)}(t_k)\sum_{i=0}^{2n-1}\left[\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^iC_i^{2n-1}+\frac{(n-i)^j}{j!}\alpha_i\right]+o(\tau^{2n})$$

Using system (2.1), we get

$$\tau^{-(2n-1)}\sum_{i=0}^{2n-1}v(t_{k+n-i})(-1)^iC_i^{2n-1}+\sum_{j=0}^{2n-1}v^{(2n-1)}(t_{k+n-j})\alpha_j \tag{2.3}$$

$$\begin{aligned}
&= \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^iC_i^{2n-1}\\
&\quad+\sum_{j=0}^{2n-1}\tau^j v^{(2n-1+j)}(t_k)\left[\sum_{i=0}^{2n-1}\frac{(n-i)^{j+2n-1}}{(j+2n-1)!}(-1)^iC_i^{2n-1}+\sum_{i=0}^{2n-1}\frac{(n-i)^j}{j!}\alpha_i\right]+o(\tau^{2n})\\
&= \sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^iC_i^{2n-1}+o(\tau^{2n}).
\end{aligned}$$

We claim that

$$\begin{aligned}
&\sum_{i=0}^{2n-1}\sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)(n-i)^j\tau^{j-(2n-1)}}{j!}(-1)^iC_i^{2n-1}\\
&= \sum_{j=0}^{2n-2}\frac{v^{(j)}(t_k)\tau^{j-(2n-1)}}{j!}\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}(n-i)^j=0.
\end{aligned} \tag{2.4}$$

In order to prove above claim it is enough to prove the following relation

$$\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}(n-i)^a=0 \text{ for all } a=0,1,\ldots,2n-2.$$

$$\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}(n-i)^a=\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}\sum_{j=0}^{a}(-1)^jC_j^a n^{a-j}i^j$$
$$=\sum_{j=0}^{a}(-1)^jC_j^a n^{a-j}\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}i^j=0 \text{ for all } a=0,1,\ldots,2n-2.$$

Therefore, it suffices to show

$$\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}i^j=0 \text{ for all } j=0,1,\ldots,2n-2. \tag{2.5}$$

Consider the function

$$f(x)=(1+x)^{2n-1}=\sum_{i=0}^{2n-1}C_i^{2n-1}x^i.$$

It is clear that formula (2.5) for $j=0$ follows from $f(-1)=0$. Using the formula

$$xf'(x)=x(2n-1)(1+x)^{2n-2}=\sum_{i=0}^{2n-1} iC_i^{2n-1}x^i$$

for $x=-1$, we can obtain formula (2.5) for $j=1$. The proof of (2.5) for $j\geq 2$ is based on the strong mathematical induction. Actually, we will prove (2.5) for $k+1$ under the assumption that (2.5) holds for $j=1,\dots,k<2n-2$. We have that

$$[f(x)]^{(k)}=\left[(1+x)^{2n-1}\right]^{(k)}=\frac{(2n-1)!}{(2n-1-k)!}(1+x)^{2n-1-k}$$
$$=\sum_{i=0}^{2n-1}\prod_{j=0}^{k}(i-j)C_i^{2n-1}x^{i-k-1}.$$

From that it follows

$$\frac{(2n-1)!}{(2n-1-k)!}(1+x)^{2n-1-k}x^{k+1}=\sum_{i=0}^{2n-1}\prod_{j=0}^{k}(i-j)C_i^{2n-1}x^i.$$

Applying the formula $\prod\limits_{j=0}^{k}(i-j)=i^{k+1}+\sum\limits_{m=0}^{k}i^m z_m$, we get

$$\frac{(2n-1)!}{(2n-1-k)!}(1+x)^{2n-1-k}x^{k+1}=\sum_{i=0}^{2n-1}i^{k+1}C_i^{2n-1}x^i+\sum_{m=0}^{k}z_m\sum_{i=0}^{2n-1}i^mC_i^{2n-1}x^i.$$

Putting $x=-1$, in the last formula, we get

$$0=\sum_{i=0}^{2n-1}i^{k+1}C_i^{2n-1}(-1)^i+\sum_{m=0}^{k}z_m\sum_{i=0}^{2n-1}i^mC_i^{2n-1}(-1)^i.$$

Then, from

$$\sum_{i=0}^{2n-1}(-1)^iC_i^{2n-1}i^m=0 \text{ for } m=0,1,\dots,k$$

it follows

$$\sum_{i=0}^{2n-1}i^{k+1}C_i^{2n-1}(-1)^i=0.$$

So, formula (2.5) is proved for any $j=0,1,\dots,2n-1$. Formula (2.2) follows from (2.4) and (2.3). Theorem 2.1 is proved.

3 $2n$-th ORDER OF APPROXIMATION DIFFERENCE SCHEME OF THE BOUNDARY-VALUE PROBLEM (1.1)

The construction of the $(2n-1)$-step difference scheme of the $2n$-th order of accuracy for the approximate solution of problem (1.1) is based on Taylor's decomposition at $2n$ points (2.2) and on the $2n$-th order of approximation for $v^{(s)}(0)$ for $s=0,1,\dots,2n-3$.

Theorem 3.1 *Let the function $v(t)$ have a $(2n+s)-th$ continuous derivative, λ_j be defined as*

$$\lambda_j = -\sum_{i=0}^{2n-2} B_i \frac{i^j}{j!} \text{ for } j = 2n-1, 2n, \ldots, 2n-1+s, \tag{3.1}$$

where B_i, $i = 0, 1, \ldots, 2n-2$ and C_i, $i = 0, 1, \ldots, s$ be the unique solutions of the following systems

$$\sum_{i=0}^{2n-2} B_i i^j = \left\{ \begin{array}{ll} 0 & \text{for } j = 0, 1, \ldots, s-1, s+1, \ldots, 2n-2, \\ -s! & \text{for } j = s, \end{array} \right\} \tag{3.2}$$

$$\sum_{j=0}^{s} C_j j^i = i!\lambda_{2n-1+i} = \frac{-i!}{(2n-1+i)!} \sum_{k=0}^{2n-2} B_k (k)^{2n-1+i} \text{ for } i = 0, 1, \ldots, s. \tag{3.3}$$

Then the following relation holds:

$$v^{(s)}(0) + \tau^{-s} \sum_{i=0}^{2n-2} B_i v(i\tau) + \tau^{2n-1-s} \sum_{i=0}^{s} v^{(2n-1)}(i\tau) C_i = o(\tau^{2n}). \tag{3.4}$$

Proof. It is clear that (3.2) and (3.3) are systems with Vandermonde matrix.Thus, there are a unique solution of systems (3.2) and (3.3). Consider the following formula

$$\tau^s v^{(s)}(0) + \sum_{i=0}^{2n-2} B_i v(i\tau) + \sum_{i=0}^{s} v^{(2n-1+i)}(0) \lambda_{2n-1+i} \tau^{2n-1+i}.$$

Using Taylor's formula, we obtain

$$\begin{aligned}
&\tau^s v^{(s)}(0) + \sum_{i=0}^{2n-2} B_i v(i\tau) + \sum_{i=0}^{s} v^{(2n-1+i)}(0) \lambda_{2n-1+i} \tau^{2n-1+i} \\
&= \tau^s v^{(s)}(0) + \sum_{i=0}^{2n-2} B_i \sum_{j=0}^{2n-1+s} \frac{v^{(j)}(0)(i)^j \tau^j}{j!} + o(\tau^{2n+s}) \\
&+ \sum_{i=0}^{s} v^{(2n-1+i)}(0) \lambda_{2n-1+i} \tau^{2n-1+i} \\
&= \sum_{i=0}^{2n-2} B_i \sum_{j=0}^{s-1} \frac{v^{(j)}(0)(i)^j \tau^j}{j!} + \tau^s v^{(s)}(0) \left[1 + \sum_{i=0}^{2n-2} B_i \frac{i^s}{s!} \right] \\
&+ \sum_{i=0}^{2n-2} B_i \sum_{j=s+1}^{2n-2} \frac{v^{(j)}(0)(i)^j \tau^j}{j!} + \left[\sum_{i=0}^{2n-2} B_i \sum_{j=2n-1}^{2n-1+s} \frac{v^{(j)}(0)(i)^j \tau^j}{j!} + \sum_{j=2n-1}^{2n-1+s} v^{(j)}(0) \tau^j \lambda_j \right] \\
&+ o(\tau^{2n+s}) = \tau^s v^{(s)}(0) \left[1 + \sum_{i=0}^{2n-2} B_i \frac{i^s}{s!} \right] + \sum_{j=0}^{s-1} v^{(j)}(0) \tau^j \sum_{i=0}^{2n-2} B_i \frac{(i)^j}{j!} \\
&+ \sum_{j=s+1}^{2n-2} v^{(j)}(0) \tau^j \sum_{i=0}^{2n-2} B_i \frac{(i)^j}{j!} + \sum_{j=2n-1}^{2n-1+s} v^{(j)}(0) \tau^j \left[\lambda_j + \sum_{i=0}^{2n-2} B_i \frac{(i)^j}{j!} \right] + o(\tau^{2n+s}).
\end{aligned}$$

Applying (3.2) and (3.1), we get

$$\tau^s v^{(s)}(0)+\sum_{i=0}^{2n-2} B_i v(i\tau)+\sum_{i=0}^{s} v^{(2n-1+i)}(0)\lambda_{2n-1+i}\tau^{2n-1+i}=o(\tau^{2n+s}). \qquad (3.5)$$

Now, we will show the following identity

$$\sum_{j=0}^{s}\tau^{2n-1+j}v^{(2n-1+j)}(0)\lambda_{2n-1+j} \;=\sum_{j=0}^{s}\tau^{2n-1}v^{(2n-1)}(j\tau)C_j+o(\tau^{2n+s})$$
$$\text{for } j=2n-1,2n,\dots,2n-1+s.$$

Using Taylor's formula, we obtain

$$\begin{aligned}
&\sum_{j=0}^{s}\tau^{2n-1+j}v^{(2n-1+j)}(0)\lambda_{2n-1+j} \;-\sum_{j=0}^{s}\tau^{2n-1}v^{(2n-1)}(j\tau)C_j\\
=&\;\sum_{j=0}^{s}\tau^{2n-1+j}v^{(2n-1+j)}(0)\lambda_{2n-1+j}\\
&-\sum_{j=0}^{s}\sum_{i=0}^{s}\frac{\tau^{2n-1+i}v^{(2n-1+i)}(0)}{i!}C_j(j)^i+o(\tau^{2n+s})\\
&=\sum_{j=0}^{s}\tau^{2n-1+j}v^{(2n-1+j)}(0)\lambda_{2n-1+j}\\
&\quad-\sum_{i=0}^{s}\tau^{2n-1+i}v^{(2n-1+i)}(0)\sum_{j=0}^{s}\frac{C_j(j)^i}{i!}+o(\tau^{2n+s})\\
&=\sum_{i=0}^{s}\tau^{2n-1+i}v^{(2n-1+i)}(0)\left[\lambda_{2n-1+i}-\sum_{j=0}^{s}\frac{C_j(j)^i}{i!}\right]+o(\tau^{2n+s}).
\end{aligned}$$

Using (3.1), we get

$$\sum_{j=0}^{s}\tau^{2n-1+j}v^{(2n-1+j)}(0)\lambda_{2n-1+j} \;-\sum_{j=0}^{s}\tau^{2n-1}v^{(2n-1)}(j\tau)C_j=o(\tau^{2n+s})$$
$$\text{for } j=2n-1,2n,\dots,2n-1+s.$$

Then from the last formula and (3.5), it follows (3.4). Theorem 3.1 is proved.

Theorem 3.2 *Let the function $v(t)$ have a $(2n+s)-th$ continuous derivative, L_j be defined as*

$$L_{2n-1+j}+\sum_{i=0}^{2n-2}M_i\frac{(-i)^j}{j!}=0 \quad \textit{for } j=2n-1,2n,\dots,2n-1+s, \qquad (3.6)$$

where M_i, $i=0,1,\dots,2n-2$ and N_i, $i=0,1,\dots,s$ be the unique solutions of the following systems

$$\sum_{i=0}^{2n-2}M_i i^j=\left\{\begin{array}{l} 0 \quad \textit{for } j=0,1,\dots,s-1,s+1,\dots,2n-2\,, \\ -s!(-1)^s \textit{ for } j=s, \end{array}\right\} \qquad (3.7)$$

$$\sum_{j=0}^{s} N_j(-j)^i \ = i!L_{2n-1+i} = \frac{-i!}{(2n-1+i)!}\sum_{k=0}^{2n-2} M_k(-k)^{2n-1+i} \textit{ for } i=0,1,\ldots,s. \quad (3.8)$$

Then the following relations hold:

$$v^{(s)}(T) + \tau^{-s}\sum_{i=0}^{2n-2} M_i v(T-i\tau) + \tau^{2n-1-s}\sum_{i=0}^{s} v^{(2n-1)}(T-i\tau)N_i = o(\tau^{2n}). \quad (3.9)$$

Proof. In a similar manner proof of the theorem 3.1 systems (3.7) and (3.8) are with Vandermonde matrix. Thus, systems (3.7) and (3.8) have a unique solution.

Consider the following formula

$$\tau^s v^{(s)}(T) + \sum_{i=0}^{2n-2} M_i v(T-i\tau) + \sum_{i=0}^{s} v^{(2n-1+i)}(T)L_{2n-1+i}\tau^{2n-1+i}.$$

Using Taylor's formula, we obtain

$$\begin{aligned}
&\tau^s v^{(s)}(T) + \sum_{i=0}^{2n-2} M_i \sum_{j=0}^{2n-1+s} \frac{v^{(j)}(T)(-i)^j\tau^j}{j!} + o(\tau^{2n+s}) \quad (3.10)\\
&+\sum_{i=0}^{s} v^{(2n-1+i)}(T)L_{2n-1+i}\tau^{2n-1+i}\\
&= \sum_{i=0}^{2n-2} M_i \sum_{j=0}^{s-1} \frac{v^{(j)}(T)(-i)^j\tau^j}{j!} + \tau^s v^{(s)}(T)\left[1+\sum_{i=0}^{2n-2} M_i\frac{(-i)^s}{s!}\right]\\
&+\sum_{i=0}^{2n-2} M_i \sum_{j=s+1}^{2n-2} \frac{v^{(j)}(T)(-i)^j\tau^j}{j!}\\
&+\left[\sum_{i=0}^{2n-2} M_i \sum_{j=2n-1}^{2n-1+s} \frac{v^{(j)}(T)(-i)^j\tau^j}{j!} + \sum_{j=2n-1}^{2n-1+s} v^{(j)}(T)\tau^j L_{2n-1+j}\right]\\
&+o(\tau^{2n+s}) = \tau^s v^{(s)}(T)\left[1+\sum_{i=0}^{2n-2} M_i\frac{(-i)^s}{s!}\right] + \sum_{j=0}^{s-1} v^{(j)}(T)\tau^j \sum_{i=0}^{2n-2} M_i\frac{(-i)^j}{j!}\\
&+\sum_{j=s+1}^{2n-2} v^{(j)}(T)\tau^j \sum_{i=0}^{2n-2} M_i\frac{(-i)^j}{j!} + \sum_{j=2n-1}^{2n-1+s} v^{(j)}(T)\tau^j\left[L_{2n-1+j}+\sum_{i=0}^{2n-2} M_i\frac{(-i)^j}{j!}\right]\\
&+o(\tau^{2n+s}).
\end{aligned}$$

Applying (3.6) and (3.7), we obtain

$$\tau^s v^{(s)}(T) + \sum_{i=0}^{2n-2} M_i v(T-i\tau) + \sum_{i=0}^{s} v^{(2n-1+i)}(T)L_{2n-1+i}\tau^{2n-1+i} = o(\tau^{2n+s}). \quad (3.11)$$

Now, we will show the following identity

$$\sum_{j=0}^{s} \tau^{2n-1+j}v^{(2n-1+j)}(T)L_{2n-1+j} \ = \sum_{j=0}^{s} \tau^{2n-1}v^{(2n-1)}(T-j\tau)N_j + o(\tau^{2n+s}) \quad (3.12)$$

$$\text{for } j = 2n-1, 2n, \ldots, 2n-1+s.$$

Using Taylor's formula, we obtain

$$\sum_{j=0}^{s} \tau^{2n-1+j} \upsilon^{(2n-1+j)}(T) L_{2n-1+j} \; - \sum_{j=0}^{s} \tau^{2n-1} \upsilon^{(2n-1)}(T - j\tau) N_j + o(\tau^{2n+s})$$
$$= \sum_{j=0}^{s} \tau^{2n-1+j} \upsilon^{(2n-1+j)}(T) L_{2n-1+j}$$

$$- \sum_{j=0}^{s} \sum_{i=0}^{s} \frac{\tau^{2n-1+i} \upsilon^{(2n-1+i)}(T)}{i!} N_j (-j)^i + o(\tau^{2n+s})$$

$$= \sum_{j=0}^{s} \tau^{2n-1+j} \upsilon^{(2n-1+j)}(T) L_{2n-1+j} - \sum_{i=0}^{s} \tau^{2n-1+i} \upsilon^{(2n-1+i)}(T) \sum_{j=0}^{s} \frac{N_j (-j)^i}{i!}$$

$$+ o(\tau^{2n+s}) = \sum_{i=0}^{s} \tau^{2n-1+i} \upsilon^{(2n-1+i)}(T) \left[L_{2n-1+i} - \sum_{j=0}^{s} \frac{N_j (-j)^i}{i!} \right] + o(\tau^{2n+s}).$$

Using (3.8), we get (3.12). Formula (3.9) follows from (3.11) and (3.12). Theorem 3.2 is proved.

Applying Taylor's decomposition at $2n$ points (2.2), formula (3.4) and equation (1.1), we get

$$\tau^{-(2n-1)} \sum_{i=0}^{2n-1} y(t_{k+n-i})(-1)^i C_i^{2n-1} - \sum_{j=0}^{2n-1} a(t_{k+n-j}) y(t_{k+n-j}) \alpha_j$$

$$= - \sum_{j=0}^{2n-1} f(t_{k+n-j}) \alpha_j + o(\tau^{2n}), n-1 \le k \le N-n,$$

$$y^{(s)}(0) + \tau^{-s} \sum_{i=0}^{2n-2} B_i v(i\tau) - \tau^{2n-1-s} \sum_{i=0}^{s} a(i\tau) y(i\tau) C_i$$

$$= -\tau^{2n-1-s} \sum_{i=0}^{s} f(i\tau) C_i + o(\tau^{2n}).$$

Neglecting the last small terms in the last two formulas, we obtain the $2n-$th order of accuracy difference scheme

$$\begin{cases} \sum\limits_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} - \tau^{2n-1}\sum\limits_{j=0}^{2n-1} a(t_{k+n-j})u_{k+n-j}\alpha_j \\ = -\tau^{2n-1}\sum\limits_{j=0}^{2n-1} f(t_{k+n-j})\alpha_j, t_k = k\tau, n-1 \le k \le N-n,\ u_0 = y(0), u_N = y(T) \\ \tau^{2n-1}\sum\limits_{i=0}^{s} C_i a(i\tau)u_i - \sum\limits_{i=0}^{2n-2} B_i u_i = \tau^s y^{(s)}(0) + \tau^{2n-1}\sum\limits_{i=0}^{s} C_i f(i\tau) \\ \text{for } s = 1,2,\ldots,n-1, \\ \tau^s y^{(s)}(T) + \tau^{2n-1}\sum\limits_{i=0}^{s} f(T-i\tau)N_i = \tau^{2n-1}\sum\limits_{i=0}^{s} u_{N-i}a(T-i\tau)N_i \\ -\sum\limits_{i=0}^{2n-2} M_i u_{N-i} \text{ for } s = 1,2,\ldots,n-2 \end{cases} \tag{3.13}$$

for the approximate solution of problem (1.1). For numerical analysis we consider the boundary-value problem

$$\begin{cases} \frac{d^5y(t)}{dt^5} + (80t^4 + 280t^2 + 74)y(t) = (32t^5 + 240t^3 + 236t)\exp(t(t-1)) \\ = f(t), \\ 0 < t < 1, y(0) = 1, y'(0) = -1, y''(0) = 3, y(1) = 1, y'(1) = 1 \end{cases}$$

for the fifth order differential equations with the exact solution

$$y(t) = \exp(t(t-1)).$$

For approximate solutions of this boundary-value problem, we use the first order of accuracy difference scheme (1.4) and the sixth order of accuracy difference scheme with different values for τ, namely $\tau = \frac{1}{30}, \frac{1}{50}, \frac{1}{70}$. The sixth order of accuracy difference scheme is

$$\begin{cases} u_{k+3} + u_{k+2}(-5 + \frac{1}{24}\tau^5 a(t_{k+2})) + u_{k+1}(10 + \frac{11}{24}\tau^5 a(t_{k+1})) \\ +u_k(-10 + \frac{11}{24}\tau^5 a(t_k)) + u_{k-1}(5 + \frac{1}{24}\tau^5 a(t_{k-1})) \\ -u_{k-2} = \tau^5(\frac{1}{24}f(t_{k+2}) + \frac{11}{24}f(t_{k+1}) + \frac{11}{24}f(t_k) + \frac{1}{24}f(t_{k-1})), \\ t_k = k\tau, 2 \le k \le N-3, u_0 = 0,\ u_N = 1,\ u_0(\frac{-25}{12} + \frac{2\tau^5}{15}a(0)) + u_1(4 - \frac{\tau^5}{3}a(\tau)) \\ -3u_2 + \frac{4}{3}u_3 - \frac{1}{4}u_4 = \tau^5(f(0)\frac{2}{15} - f(\tau)\frac{1}{3}), \\ u_0(\frac{35}{12} + \frac{8}{180}\tau^5 a(0)) + u_1(-\frac{26}{3} + \frac{23}{90}\tau^5 a(\tau)) + u_2(\frac{19}{2} + \frac{96}{180}\tau^5 a(2\tau)) \\ -u_3\frac{14}{3} + \frac{11}{12}u_4 = \tau^5(\frac{8}{180}f(0) + \frac{23}{90}f(\tau) + \frac{96}{180}f(2\tau)), \\ (\frac{25}{12} + \frac{2}{15}\tau^5 a(T))u_N + (-4 - \frac{1}{3}\tau^5 a(T-\tau))u_{N-1} + 3u_{N-2} - \frac{4}{3}u_{N-2} + \frac{1}{4}u_{N-3} \\ = \tau^5(\frac{2}{15}f(T) - \frac{1}{3}f(T-\tau)). \end{cases} \tag{3.14}$$

The errors of the numerical solutions defined as

$$E_N = \max_{0 \le k \le N} |y(t_k) - u_k|$$

are given in the following table.

Difference schemes	E_{30}	E_{50}	E_{70}
Difference scheme (1.4)	0.00124321	0.00077113	0.00023673
Difference scheme (3.14)	0.00000106	0.00000078	0.00000026

Thus, the sixth order of accuracy difference scheme (3.14) is more accurate comparing with the first order of accuracy difference scheme (1.4).

4 $2n-th$ ORDER OF APPROXIMATION DIFFERENCE SCHEME OF THE INITIAL-VALUE PROBLEM (1.2)

The construction of the $(2n-1)$-step difference scheme of the $2n$-th order of accuracy for the approximate solution of problem (1.2) is based on Taylor's decomposition at $2n$ points (2.2), on the formula (3.4) and on the $2n$-th order of approximation for $v^{(i)}(0)$ for $i=0,1,\ldots,2n-2$

$$\begin{cases} \tau^{-(2n-1)}\sum\limits_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} - \sum\limits_{j=0}^{2n-1} a(t_{k+n-j})u_{k+n-j}\alpha_j = -\sum\limits_{j=0}^{2n-1} f(t_{k+n-j})\alpha_j, \\ t_k = k\tau, n-1 \le k \le N-n,\ u_0 = y(0),\ \tau^{2n-1-s}\sum\limits_{i=0}^{s} C_i a(i\tau)u_i \\ -\tau^{-s}\sum\limits_{i=0}^{2n-2} B_i u_i = y^{(s)}(0) + \tau^{2n-1-s}\sum\limits_{i=0}^{s} C_i f(i\tau) \text{ for } s=1,\ldots,2n-2 \end{cases}$$

for the approximate solution of problem (1.2).

For numerical analysis we consider the initial-value problem

$$\begin{cases} \frac{d^5 y(t)}{dt^5} + (32t^5 + 120t)y(t) = \exp(-t^2)160t^3 \\ = f(t), \\ 0<t<1, y(0)=1, y'(0)=0, y''(0)=-2, y'''(0)=0, y^{(4)}(0)=12 \end{cases}$$

for the fifth order differential equations. The exact solution of this problem is

$$y(t) = \exp(-t^2).$$

The sixth order of accuracy difference scheme is

$$\begin{cases} u_{k+3} + u_{k+2}(-5 + \frac{1}{24}\tau^5 a(t_{k+2})) + u_{k+1}(10 + \frac{11}{24}\tau^5 a(t_{k+1})) \\ +u_k(-10 + \frac{11}{24}\tau^5 a(t_k)) + u_{k-1}(5 + \frac{1}{24}\tau^5 a(t_{k-1})) \\ -u_{k-2} = \tau^5(\frac{1}{24}f(t_{k+2}) + \frac{11}{24}f(t_{k+1}) + \frac{11}{24}f(t_k) + \frac{1}{24}f(t_{k-1})), \\ t_k = k\tau, 2 \le k \le N-3, u_0 = 0,\ u_0(\frac{-25}{12} + \frac{2\tau^5}{15}a(0)) + u_1(4 - \frac{\tau^5}{3}a(\tau)) \\ -3u_2 + \frac{4}{3}u_3 - \frac{1}{4}u_4 = \tau^5(f(0)\frac{2}{15} - f(\tau)\frac{1}{3}), u_0(\frac{35}{12} + \frac{8}{180}\tau^5 a(0)) + u_1(-\frac{26}{3} + \frac{23}{90}\tau^5 a(\tau)) \\ +u_2(\frac{19}{2} + \frac{96}{180}\tau^5 a(2\tau)) - u_3\frac{14}{3} + \frac{11}{12}u_4 = \tau^5(\frac{8}{180}f(0) + \frac{23}{90}f(\tau) + \frac{96}{180}f(2\tau)), \\ u_0(\frac{-5}{2} - \frac{3}{40}\tau^5 a(0)) + u_1(9 - \frac{11}{12}\tau^5 a(\tau)) + u_2(-12 - \frac{83}{120}\tau^5 a(2\tau)) + u_3(7 - \frac{1}{15}\tau^5 a(3\tau)) \\ -\frac{3}{2}u_4 = -\tau^5(\frac{3}{40}f(0) + f(\tau)\frac{11}{12} + f(2\tau)\frac{83}{120} + f(3\tau)\frac{1}{15}), \end{cases} \tag{4.1}$$

$$u_0(1+\tau^5\frac{95}{288}a(0))+u_1(-4+\tau^5\frac{307}{240}a(\tau))+u_2(6+\tau^5\frac{4}{15}a(2\tau))$$
$$+u_3(-4+\frac{103}{720}\tau^5a(3\tau))+u_4(1-\frac{3}{160}\tau^5a(4\tau))$$
$$=\tau^5(\frac{95}{288}f(0)+\frac{307}{240}f(\tau)+\frac{4}{15}f(2\tau)+\frac{103}{720}f(3\tau)-\frac{3}{160}f(4\tau)).$$

For approximate solutions of this initial-value problem, we use the first order of accuracy difference scheme (1.5) and the sixth order of accuracy difference scheme (3.14) with different values for τ, namely $\tau=\frac{1}{40},\frac{1}{60},\frac{1}{80}$. The errors of the numerical solutions are given in the following table.

Difference schemes	E_{40}	E_{60}	E_{80}
Difference scheme (1.5)	0.07492100	0.066832000	0.048370000
Difference scheme (4.1)	0.00040349	0.000003211	0.000000351

Thus, the sixth order of accuracy difference scheme (4.1) is more accurate comparing with the first order of accuracy difference scheme (1.5).

5 $2n-th$ ORDER OF APPROXIMATION DIFFERENCE SCHEME OF THE NONLOCAL BOUNDARY VALUE PROBLEM (1.3)

The construction of the $(2n-1)$-step difference scheme of the $2n$-th order of accuracy for the approximate solution of problem (1.3) is based on Taylor's decomposition at $2n$ points (2.2), on the formulas (3.4) and on the $2n$-th order of approximation for $v^{(i)}(T)$ where $i=0,1,\ldots,2n-2$.

Applying formulas (3.4), (3.9) and (2.2), we obtain

$$\tau^{-(2n-1)}\sum_{i=0}^{2n-1}y(t_{k+n-i})(-1)^iC_i^{2n-1}-\sum_{j=0}^{2n-1}a(t_{k+n-j})y(t_{k+n-j})\alpha_j$$
$$=-\sum_{j=0}^{2n-1}f(t_{k+n-j})\alpha_j+o(\tau^{2n}),1\leq k\leq N-2,$$
$$y^{(s)}(0)=y^{(s)}(T)\Rightarrow\tau^{-s}\sum_{i=0}^{2n-2}\left[M_iv(T-i\tau)-B_iy(i\tau)\right]$$
$$+\tau^{2n-1-s}\sum_{i=0}^{s}\left[y^{(2n-1)}(T-i\tau)N_i-y^{(2n-1)}(T-i\tau)C_i\right]$$
$$=\tau^{-s}\sum_{i=0}^{2n-2}\left[M_iv(T-i\tau)-B_iy(i\tau)\right]+\tau^{2n-1-s}\sum_{i=0}^{s}\{-y(T-i\tau)a(T-i\tau)N_i$$
$$+f(T-i\tau)N_i+C_ia(i\tau)y(i\tau)-C_if(i\tau)\}=o(\tau^{2n}).$$

Neglecting the last small terms in the last three formulas, we obtain the $2n$-th order of accuracy difference scheme

$$\begin{cases} \tau^{-(2n-1)} \sum_{i=0}^{2n-1} u_{k+n-i}(-1)^i C_i^{2n-1} - \sum_{j=0}^{2n-1} a(t_{k+n-j}) u_{k+n-j} \alpha_j \\ = - \sum_{j=0}^{2n-1} f(t_{k+n-j}) \alpha_j, t_k = k\tau, 1 \leq k \leq N-n,\ u_0 = u_N, \\ \sum_{i=0}^{2n-2} (M_i u_{N-i} - B_i u_i) + \tau^{2n-1} \sum_{i=0}^{s} (-u_{N-i} a(T-i\tau) N_i \\ +C_i a(i\tau) u_i) = \tau^s \sum_{i=0}^{s} (C_i f(i\tau) - f(T-i\tau) N_i) \text{ for } s = 1, \dots, 2n-2 \end{cases} \tag{5.1}$$

for the approximate solution of problem (1.3) .

Now, for numerical analysis we consider the nonlocal boundary value problem

$$\begin{cases} \frac{d^3 y(t)}{dt^3} + \exp(t^{10}) y(t) = t^3(t-1)^3 \exp(t^{10}) + 120t^3 - 180t^2 + 72t - 6, \\ 0 < t < 1, y(0) = y(1), y'(0) = y'(1), y''(0) = y''(1) \end{cases}$$

for the third order differential equations. The exact solution of this problem is

$$y(t) = t^3(1-t)^3.$$

The fourth order of accuracy difference scheme is

$$\begin{cases} \tau^{-3}(u_{k+2} - 3u_{k+1} + 3u_k - u_{k-1}) + \frac{1}{2} a(t_k) u_k + \frac{1}{2} a(t_{k+1}) u_{k+1} \\ = \frac{1}{2} f(t_k) + \frac{1}{2} f(t_{k+1}), 1 \leq k \leq N-2,\ u_0 = u_N, \\ (-\frac{3}{2} - \frac{1}{12}\tau^3 a(0)) u_0 + (2 - \frac{1}{4}\tau^3 a(\tau)) u_1 - \frac{1}{2} u_2 \\ -(\frac{3}{2} - \frac{1}{12}\tau^3 a(T)) u_N - (-2 - \frac{1}{4}\tau^3 a(T-\tau)) u_{N-1} - \frac{1}{2} u_{N-2} \\ = \frac{1}{12}\tau^3 (f(T) - f(0)) + \frac{1}{4}\tau^3 (f(T-\tau) - f(\tau)), \\ (1 + \frac{3}{8}\tau^3 a(0)) u_0 + (-2 + \frac{2}{3}\tau^3 a(\tau)) u_1 + (1 - \frac{1}{24}\tau^3 a(2\tau)) u_2 \\ -(1 - \frac{3}{8}\tau^3 a(T)) u_N - (-2 - \frac{2}{3}\tau^3 a(T-\tau)) u_{N-1} \\ -(1 + \frac{1}{24}\tau^3 a(T-2\tau)) u_{N-2} = \tau^3 \{ \frac{3}{8}(f(T) + f(0)) \\ + \frac{2}{3}(f(T-\tau) + f(\tau)) - \frac{1}{24}(f(T-2\tau) + f(2\tau)) \}. \end{cases} \tag{5.2}$$

For approximate solutions of this nonlocal boundary-value problem, we use the first order of accuracy difference scheme (1.6) and the fourth order of accuracy difference scheme (5.2) with different values for τ, namely $\tau = \frac{1}{20}, \frac{1}{40}, \frac{1}{60}$. The errors of the numerical solutions are given in the following table.

Difference schemes	E_{20}	E_{40}	E_{60}
Difference scheme (1.6)	1.6191000000	0.74250000000	0.475800000000
Difference scheme (5.2)	0.0000029109	0.00000018214	0.000000035982

Hence, the fourth order of accuracy difference scheme (5.2) is more accurate when compared with the first order of accuracy difference scheme (1.6).

It is known that (see, for example, [4]-[7]) various boundary value problems for the partial differential equations can be reduced to the boundary value problem

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + A(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = y_0^i, i = 0, 1, \ldots, n-1, \\ y^{(i)}(T) = y_T^i, i = 0, 1, \ldots, n-2 \end{cases} \tag{5.3}$$

and the initial-value problem

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + A(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = y_0^i, \ i = 0, 1, \ldots, 2n-2 \end{cases} \tag{5.4}$$

and the nonlocal boundary value problem

$$\begin{cases} \frac{d^{2n-1}y(t)}{dt^{2n-1}} + A(t)y(t) = f(t), 0 < t < T, \\ y^{(i)}(0) = \ y^{(i)}(T) \text{ for } i = 0, 1, \ldots, 2n-2 \end{cases} \tag{5.5}$$

for the differential equations in a Hilbert space H, with the self-adjoint positive definite operators $A(t)$. The use of Taylor's decomposition at $2n$ points permits to construct the difference schemes of the $2n$-th order of accuracy for the approximate solutions of problems (5.3)-(5.5). Operator method of [8]-[11] permits to establish the stability of these difference schemes.

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