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A MATRIX PENCIL APPROACH FOR THE SOLUTION OF LINEAR SYSTEMS WITH RECTANGULAR OR SINGULAR COEFFICIENT MATRICES

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Abstract: We describe a new approach for the classical problem of solving ordinary systems of linear equations with rectangular (or singular) coefficient matrices. Using the complex Kronecker canonical form, the solution is analytically derived. This approach succeeds in decreasing the importance of arbitrary-unknown elements. Moreover, we have identified the necessary and required mathematical condition in order to be able to obtain the solutions of the subject system.

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1. Introduction - Preliminaries Results

For every finite matrix $A \in \mathbb{C}^{m\times n}$, there is a unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the four equations.

(1) $AXA = A,$

(2) $XAX = X,$

(3) $(AX)^* = AX,$

(4) $(XA)^* = XA,$

where A^* denotes the conjugate transpose of A.

For any matrix $A \in \mathbb{C}^{m\times n}$, let $A\{1,\ldots,4\}$ denotes the set $X \in \mathbb{C}^{n\times m}$ which satisfies equations $1,\ldots,4$ from among the above equations (1) - (4). A matrix $X \in A\{1,2,3,4\}$ is called a $\{1,2,3,4\}$-inverse of A, and also denoted by $A^{(1,2,3,4)}$.

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One of the principal applications of $\{1\}$-inverses (i.e. it satisfies only the equation (1)) is to the solution of ordinary systems of linear equations (1.1), see for instance [1], [2], [4] etc. Let $A \in \mathbb{C}^{m\times n}$, $\underline{b} \in \mathbb{C}^{m}$, then the equation

$$A\underline{x} = \underline{b} \tag{1.1}$$

is *consistent* if and only if for some $A^{(1)}$ (i.e. $AGA = A$, which some time is called a g-inverse),

$$AA^{(1)}\underline{b} = \underline{b}$$

in which case the general solution of (1.1) is given by

$$\underline{x} = A^{(1)}\underline{b} + \left(I - A^{(1)}A\right)z, \tag{1.2}$$

for arbitrary $z \in \mathbb{C}^{n}$, (i.e. n unknown elements).

In another words by using a classical terminology of linear systems, it is known that a system of *m* linear equations in *n* unknowns is said to be a ***consistent*** system if it possesses at least one solution. If there are no solutions, then the system is called ***inconsistent,*** see [8]**.**

To describe the set of all possible solutions of a consistent non-homogeneous rectangular system, see Eq. (1.1), we construct a general solution as follows; consult [8]. Definitely, this approach is very classical and well known. However, for reasons of complicity and comparison, it is described briefly in the next few lines.

Algorithm: Let $\left[A\,|\,\underline{b}\right]$ be the augmented matrix for a consistent $m\times n$ non-homogeneous system in which *rank* (A) = *r*.

- Reducing $\left[A\,|\,\underline{b}\right]$ to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the ***general solution***

$$\underline{x} = \underline{p} + x_{f_1}\underline{h}_1 + x_{f_2}\underline{h}_2 + \ldots + x_{f_{n-r}}h_{n-r}\,. \tag{1.3}$$

- As the free variables x_{f_i} range over all possible values, this general solution generates all possible solutions of the system.
- Column $\underline{p}$ is a particular solution of the non-homogeneous system.

- The expression $x_{f_1}\underline{h}_1 + x_{f_2}\underline{h}_2 + \ldots + x_{f_{n-r}}\underline{h}_{n-r}$ is the general solution of the associated homogeneous system.

- Column $\underline{p}$ as well as the columns $\underline{h}_i$ are independent of the row echelon form to which $\left[A \,|\, \underline{b}\right]$ is reduced.

- The system possesses a ***unique*** solution if and only if any of the following is true.
  - $rank\,(A) = n$ = number of unknowns.
  - There are no free variables.
  - The associated homogeneous system possesses only the trivial solution.

Now, we return to the terminology of generalized inverses, i.e. Eq. (1.2) provides us with the solution of system (1.1). However, the generalized inverses are very useful in formulating theoretical statements such as those above, but just as in the case of the ordinary inverse, generalized inverses are not practical computational tools. In addition to being computationally inefficient, serious numerical problems result from the fact that $A^{(1)}$ need **not** be a continuous function of the entries of A, for further details see [8], pp. 424. For example, consider

$$A(x) = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \Rightarrow A^{(1)}(x) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1/x \end{bmatrix}, & for\ x \neq 0 \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & for\ x = 0 \end{cases}.$$

Thus, we can conclude that not only $A^{(1)}(x)$ is discontinuous in the sense that

$$\lim_{x \to 0} A^{(1)}(x) \neq A^{(1)}(0),$$

but it is discontinuous in the worst way because as $A(x)$ comes closer to $A(0)$ the matrix $A^{(1)}(x)$ moves farther away from $A^{(1)}(0)$.

In the present study, we provide a new approach based on matrix pencil theory, which minimizes naturally and sufficiently the arbitrary elements of vector $z \in \mathbb{C}^n$. This method follows a completely different way. Thus, it is based on the deeper knowledge of the *structure* of matrices, since the complex Kronecker canonical form is being used. However, before we discuss analytically system (1.1), we shall begin with the ordinary system with rectangular (or singular) coefficient matrix,

$$AX = B, \tag{1.4}$$

where $A, B \in \mathbb{C}^{m\times n}$, (with $\det A = 0$ when $m = n$), and $X \in \mathbb{C}^{n\times n}$. Now, with the given constant matrices A, B and an indeterminate s, the pencil $sA - B$ is called singular when $m \neq n$ (or $m = n$ and $\det(sA - B) \equiv 0$). It is well known that the pencil $sA - B$ is said to be strictly equivalent to the pencil $sA_1 - B_1$ if and only if there exist invertible matrices $P \in \mathbb{C}^{m\times m}$, $Q \in \mathbb{C}^{n\times n}$, such as $P(sA - B)Q = sA_1 - B_1$. The class of strict equivalence, i.e. $\mathcal{E}_s(sA - B)$, is characterized by a uniquely defined element, known as a ***complex Kronecker canonical form*** (**KCF**), $sA_k - B_k$, see [3], which is specified by the complete set of invariants of $\mathcal{E}_s(sA - B)$. The characterization of singular pencils requires the definition of additional sets of invariants known as the minimal indices. Now, let us assume that $r = rank_{\mathbb{C}(s)}(sA - B) \leq \min\{m, n\}$, where $\mathbb{C}(s)$ denotes the field of rational functions in s having complex coefficients. Then equations,

$$(sA - B)\underline{x}(s) = \underline{0} \text{ and } \underline{\psi}^t(s)(sA - B) = \underline{0}^t,$$

have solutions $\underline{x}(s)$ and $\underline{\psi}(s)$, which are vectors in the rational vector spaces

$$\mathcal{N}_r(s) \triangleq \mathcal{N}_r(sA - B) \text{ and } \mathcal{N}_l(s) \triangleq \mathcal{N}_l(sA - B),$$

respectively, where

$$\mathcal{N}_r(s) \triangleq \left\{\underline{x}(s) \in \mathbb{C}^n(s) : (sA - B)\underline{x}(s) = \underline{0}\right\},$$

and

$$\mathcal{N}_l(s) = \left\{\underline{\psi}(s) \in \mathbb{C}^m(s) : \underline{\psi}^t(s)(sA - B) = \underline{0}^t\right\}.$$

Obviously, $\mathcal{N}_r(s)$ and $\mathcal{N}_l(s)$ are vector spaces over $\mathbb{C}(s)$ with

$$\dim \mathcal{N}_r(s) = n - r \text{ and } \dim \mathcal{N}_l(s) = m - r.$$

It is also known that $\mathcal{N}_r(s)$ and $\mathcal{N}_l(s)$ are spanned by minimal polynomial bases $\{\underline{x}_i(s), i = 1, 2, \ldots, n - r\}$ and $\{\underline{\psi}_j^t(s), i = 1, 2, \ldots, m - r\}$ of minimal degrees, correspondingly, see [6], with

$$\left\{v_1 = v_2 = \cdots = v_g = 0 < v_{g+1} \leq v_{g+2} \leq \cdots \leq v_{n-r}\right\} \tag{1.5}$$

and

$$\left\{u_1 = u_2 = \cdots = u_h = 0 < u_{h+1} \leq u_{h+2} \leq \cdots \leq u_{m-r}\right\}. \tag{1.6}$$

The sets of the minimal degrees $\left\{v_i,\ 1 \leq i \leq n-r\right\}$ and $\left\{u_j,\ 1 \leq j \leq m-r\right\}$ are known by [3] as *column minimal indices* (**c.m.i.**) and *row minimal indices* (**r.m.i.**) of $sA-B$, respectively. Moreover, we have elementary divisors (**e.d.**) of the following type:

- **e.d.** of the type s^d, $d \in \mathbb{N}$, are called *zero finite elementary divisors* (**z. f.e.d.**)
- **e.d.** of the type $(s-a)^c$, $a \neq 0$, $c \in \mathbb{N}$ are called *non-zero finite elementary divisors* (**nz. f.e.d.**)
- **e.d.** of the type $\hat{s}^q$ are called infinite elementary divisors (**i.e.d**)
- **c.m.i.** of the type $v \in \mathbb{N} \bigcup \{0\}$ are called *column minimal indices* (**c.m.i.**) deduced from the column degrees of minimal polynomial bases of the maximal sub-module $\mathcal{M}_{\mathcal{N}}$ embedded in $\mathcal{N}_r(s)$ with a free $\mathbb{C}(s)$-module structure.
- **r.m.i.** of the type $u \in \mathbb{N} \bigcup \{0\}$ are called *row minimal indices* (**r.m.i**.) deduced from the row degrees of minimal polynomial bases of the maximal sub module $\mathcal{M}_{\mathcal{N}}$ embedded in $\mathcal{N}_l(s)$ with a free $\mathbb{C}(s)$-module structure.

For further details, see [5], [6] and [7].

Thus, there exists $P \in \mathbb{C}^{m\times m}$ and $Q \in \mathbb{C}^{n\times n}$ such that the *complex Kronecker form* $sA_k - B_k$ of the singular pencil $sA-B$ is defined as follows.

$$\begin{aligned} P(sA-B)Q &= sA_k - B_k \\ &\triangleq block\ diag\left\{\mathbb{O}_{h,g}, s\Lambda_v - \lambda_v, s\Lambda_u^t - \lambda_u^t, sI_p - J_p, sH_q - I_q\right\}. \end{aligned} \tag{1.7}$$

Note that the, matrix $\mathbb{O}_{h,g}$ is uniquely defined by the sets $\underbrace{\{0,0,\ldots,0\}}_{g}$ and $\underbrace{\{0,0,\ldots,0\}}_{h}$ of zero column and row minimal indices, respectively.

The second normal block $s\Lambda_v - \lambda_v$ is uniquely defined by the *set of non-zero column minimal indices* (a new arrangement of the indices of v must be noted in order to simplify the notation) $\{v_{g+1} \le v_{g+2} \le \cdots \le v_{n-r}\}$ of $sA - B$ and has the form

$$s\Lambda_v - \lambda_v \triangleq block\ diag\left\{s\Lambda_{v_{g+1}} - \lambda_{v_{g+1}}, \dots, s\Lambda_{v_i} - \lambda_{v_i}, \dots, s\Lambda_{v_{n-r}} - \lambda_{v_{n-r}}\right\}, \quad (1.8)$$

where $\Lambda_{v_i} = \left[I_{v_i} \vdots \underline{0}\right] \in \mathbb{C}^{v_i \times (v_i+1)}$, $\lambda_{v_i} = \left[H_{v_i} \vdots \underline{\varepsilon}_{v_i}\right] \in \mathbb{C}^{v_i \times (v_i+1)}$ for every $i = g+1, g+2, \dots, n-r$, and I_{v_i} and H_{v_i} denote the $v_i \times v_i$ identity and the nilpotent (with annihilation index v_i) matrix, respectively. Also, $\underline{0}$ and $\underline{\varepsilon}_{v_i} = \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix}^t \in \mathbb{C}^{v_i}$ are the zero column and the column with element 1 at the v_i-place, respectively.

The third normal block $s\Lambda_u^t - \lambda_u^t$ is uniquely determined by the *set of non-zero row minimal indices* (a new arrangement of the indices of u must be noted in order to simplify the notation) $\{u_{h+1} \le u_{h+2} \le \cdots \le u_{m-r}\}$ of $sF - G$ and has the form

$$s\Lambda_u^t - \lambda_u^t \triangleq block\ diag\left\{s\Lambda_{u_{h+1}}^t - \lambda_{u_{h+1}}^t, \dots, s\Lambda_{u_j}^t - \lambda_{u_j}^t, \dots, s\Lambda_{u_{m-r}}^t - \lambda_{u_{m-r}}^t\right\}, \quad (1.9)$$

where $\Lambda_{u_j}^t = \begin{bmatrix}\underline{e}_{u_j}^t \\ \cdots \\ H_{u_j}\end{bmatrix} \in \mathbb{C}^{(u_j+1)\times u_j}$, $\lambda_{u_j}^t = \begin{bmatrix}\underline{0}^t \\ \cdots \\ I_{u_j}\end{bmatrix} \in \mathbb{C}^{(u_j+1)\times u_j}$ for every $j = h+1, h+2, \dots, m-r$, and I_{u_j} and H_{u_j} denote the $u_j \times u_j$ identity and nilpotent (with annihilation index u_j) matrix and the zero column matrix, respectively. Furthermore, $\underline{0}$ and $\underline{e}_{u_j} = \begin{bmatrix}1 & \cdots & 0 & 0\end{bmatrix}^t \in \mathbb{C}^{u_j}$ are the zero column and the column with element 1 at the first place, respectively.

The forth and the fifth normal matrix block is the complex Weierstrass form $sA_w - B_w$ of the regular pencil $sA - B$ which is defined by

$$sA_w - B_w \triangleq block\ diag\left\{sI_p - J_p, sH_q - I_q\right\}, \quad (1.10)$$

where the first normal Jordan type block $sI_p - J_p$ is uniquely defined by the set of f.e.d.,

$$(s - a_1)^{p_1}, \dots (s - a_\nu)^{p_\nu}, \ \sum_{j=1}^{\nu} p_j = p \quad (1.11)$$

of $sA - B$ and has the form

$$sI_p - J_p \triangleq block\ diag\left\{sI_{p_1} - J_{p_1}\left(a_1\right), \ldots, sI_{p_\nu} - J_{p_\nu}\left(a_\nu\right)\right\}. \tag{1.12}$$

The q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.,

$$\left(\hat{s}\right)^{q_1}, \ldots, \left(\hat{s}\right)^{q_\sigma},\ \sum\nolimits_{j=1}^{\sigma} q_j = q \tag{1.13}$$

of $sA - B$ and has the form

$$sH_q - I_q \triangleq block\ diag\left\{sH_{q_1} - I_{q_1}, \ldots, sH_{q_\sigma} - I_{q_\sigma}\right\}. \tag{1.14}$$

Thus, the H_q is a nilpotent matrix of index $\tilde{q} = \max\left\{q_j : j = 1, 2, \ldots, \sigma\right\}$, where

$$H_q^{\tilde{q}} = \mathbb{O}\ . \tag{1.15}$$

We proceed to identify the matrices I_{p_i}, $J_{p_i}\left(a_i\right)$, H_{q_i} as follows:

$$I_{p_i} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{p_i \times p_i},$$

$$J_{p_i}\left(a_i\right) = \begin{bmatrix} a_i & 1 & 0 & \cdots & 0 \\ 0 & a_i & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_i & 1 \\ 0 & 0 & 0 & 0 & a_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i} \text{ and } H_{q_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{q_i \times q_i}. \tag{1.16}$$

Lemma 1. System (1.4) may be decomposed into the equivalent set of subsystems

$$\mathbb{O}_{h,g} Y_{g,n} = D_{h,n}, \text{ where } Y_{g,n} \in \mathbb{C}^{g \times n} \tag{1.17}$$

$$\Lambda_{v_i} Y_{v_i+1,n} = D_{v_i,n}, \text{ where } Y_{v_i+1,n} \in \mathbb{C}^{(v_i+1)\times n} \text{ for } i = g+1, g+2, \ldots, n-r, \tag{1.18}$$

$$\Lambda_{u_j}^t Y_{u_j,n} = D_{u_j+1,n}, \text{ where } Y_{u_j,n} \in \mathbb{C}^{u_j \times n} \text{ for } j = h+1, h+2, \ldots, m-r, \tag{1.19}$$

$$I_p Y_{p,n} = D_{p,n}, \text{ where } Y_{p,n} \in \mathbb{C}^{p \times n} \tag{1.20}$$

and

$$H_{q_j} Y_{q_j,n} = D_{q_j,n}, \text{ for } j = 1, 2, \ldots, \sigma, \text{ where } Y_{q,n} \in \mathbb{C}^{q \times n}. \tag{1.21}$$

Proof. Consider the transformation

$$X = QY\ . \tag{1.22}$$

Substituting the previous expression into (1.4) we obtain

$$AQY = B\ .$$

Whereby, multiplying by P, we arrive at $A_k Y = PB \in \mathbb{C}^{m\times n}$.

Now, by denoting $D = PB$, we can conclude that $A_k Y = D$. After that, we use the complex Kronecker canonical form, i.e.

$$A_k \triangleq block\ diag\left\{\mathbb{O}_{h,g}, \Lambda_v, \Lambda_u^t, I_p, H_q\right\}.$$

Furthermore, by writing

$$Y = \begin{bmatrix} Y_{g,n}^t & Y_{v,n}^t & Y_{u,n}^t & Y_{p,n}^t & Y_{q,n}^t \end{bmatrix}^t \text{ and } D = \begin{bmatrix} D_{h,n}^t & D_{v,n}^t & D_{u,n}^t & D_{p,n}^t & D_{q,n}^t \end{bmatrix}^t$$

and taking into account the previous analysis, we can easily arrive at (1.17) - (1.21).

The following Theorem is very important, because it provides the baseline for Theorem 2, which follows in the 2nd section. Under the results of this Theorem, we have an understanding of the consistency of the solution of system (1.3).

Theorem 1. By solving systems (1.17) - (1.21), we have the following results.

a) We obtain a consistent solution, when the row vectors $\underline{d}_{i,n} = \underline{0}$ ($\underline{d}$ is a row vector of $D = PB$), for every

$$i = 1,2,\dots,h, h+\sum_{i=g+1}^{n-r} v_i + u_{h+1}+1, h+\sum_{i=g+1}^{n-r} v_i + u_{h+1}+u_{h+2}+2,\dots, h+\sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j$$
$$+m-h-r, h+\sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r+\sum_{i=1}^{v} p_i + q_1, h+\sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r$$
$$+\sum_{i=1}^{v} p_i + q_1 + q_2,\dots, h+\sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r+\sum_{i=1}^{v} p_i + \sum_{j=1}^{\sigma} q_j.$$

b) We obtain specified arbitrarily vectors, i.e. $\underline{y}_j \in \mathbb{C}^{1\times n}$, for every

$$j = 1,2,\dots,g, g+v_{g+1}+1,\ g+v_{g+1}+v_{g+2}+2,\ \dots,\ g+\sum_{i=g+1}^{n-r} v_i + n-r-g, g+\sum_{i=g+1}^{n-r} v_i$$
$$+n-r-g+\sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{v} p_i + 1, g+\sum_{i=g+1}^{n-r} v_i + n-r-g+\sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{v} p_i + q_1 + 1,\dots,$$
$$g+\sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + n-r-g+\sum_{i=1}^{v} p_i + \sum_{j=1}^{\sigma-1} q_j + 1\ .$$

c) All the other elements are known, since they are derived by the corresponding elements of $D = PB$.

Proof.

- System (1.17) has a consistent solution when the first hth rows of matrix D are equal to zero. Moreover, the first gth rows of matrix Y are arbitrary.
- Systems (1.18), i.e. $\Lambda_{v_i} Y_{v_i+1,n} = D_{v_i,n}$, for $i = g+1, g+2, \ldots, n-r$, can be written as follows

$$\left[I_{v_i} \,\vdots\, \underline{0} \right]_{v_i \times (v_i+1)} \begin{bmatrix} \underline{y}_{1,n} \\ \underline{y}_{2,n} \\ \vdots \\ \underline{y}_{v_i+1,n} \end{bmatrix}_{(v_i+1)\times n} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{v_i,n} \end{bmatrix}_{v_i \times n} \Leftrightarrow \begin{bmatrix} \underline{y}_{1,n} \\ \underline{y}_{2,n} \\ \vdots \\ \underline{y}_{v_i,n} \end{bmatrix} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{v_i,n} \end{bmatrix}.$$

Consequently, $\underline{y}_{l,n} = \underline{d}_{l,n} \in \mathbb{C}^{1\times n}$ for every $l = 1, 2, \ldots, v_i$ (note that l defines the place of vector instead of the dimension), and every $i = g+1, g+2, \ldots, \ n-r$. Furthermore, the row vector $\underline{y}_{v_i+1,n}$ is arbitrarily chosen.

Note that arbitrary row vectors are at $g + v_{g+1} + 1$, $g + v_{g+1} + v_{g+2} + 2$, . . . , $g + \sum_{i=g+1}^{n-r} v_i + n - r - g$ -place of matrix Y.

- Systems (1.19), i.e. $\Lambda_{u_j}^t Y_{u_j,n} = D_{u_j+1,n}$, for $j = h+1, h+2, \ldots, m-r$, can be written as follows

$$\begin{bmatrix} \underline{e}_{u_j}^t \\ \cdots \\ H_{u_j} \end{bmatrix}_{(u_j+1)\times u_j} \begin{bmatrix} \underline{y}_{1,n} \\ \underline{y}_{2,n} \\ \vdots \\ \underline{y}_{u_j,n} \end{bmatrix}_{u_j \times n} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{u_j+1,n} \end{bmatrix}_{(u_j+1)\times n} \Leftrightarrow \begin{bmatrix} \underline{y}_{1,n} \\ \underline{y}_{2,n} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{u_j+1,n} \end{bmatrix}.$$

Consequently, $\underline{y}_{l,n} = \underline{d}_{l,n} \in \mathbb{C}^{1\times n}$ for every $l = 1, 2, \ldots, u_j$,

and every $j = h+1, h+2, \ldots, m-r$. Furthermore, in order to obtain consistent solutions, the row vectors $\underline{d}_{u_j+1,n}$ are equal to zero. Also the zero row vectors are at $h + \sum_{i=g+1}^{n-r} v_i + u_{h+1} + 1$, $h + \sum_{i=g+1}^{n-r} v_i + u_{h+1} + u_{h+2} + 2$, . . . , $h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m$ $-h - r$ -place of matrix D.

- From system (1.20), we derive that $I_p Y_{p,n} = D_{p,n} \Rightarrow Y_{p,n} = D_{p,n}$.
- Systems (1.21), i.e. $H_{q_j} Y_{q_j,n} = D_{q_j,n}$, for $j = 1, 2, \ldots, \sigma$, can be written as follows

$$H_{q_j} \begin{bmatrix} \underline{y}_{1,n} \\ \underline{y}_{2,n} \\ \vdots \\ \underline{y}_{q_j,n} \end{bmatrix}_{q_j \times n} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{q_j,n} \end{bmatrix}_{q_j \times n} \Leftrightarrow \begin{bmatrix} \underline{y}_{2,n} \\ \underline{y}_{3,n} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \underline{d}_{1,n} \\ \underline{d}_{2,n} \\ \vdots \\ \underline{d}_{q_j,n} \end{bmatrix}.$$

Consequently, $\underline{y}_{l,n} = \underline{d}_{l-1,n} \in \mathbb{C}^{1\times n}$ for every $l = 2, \ldots, q_j$ and every $j = 1, 2, \ldots, \sigma$.

The row vector $\underline{y}_{1,n}$ is arbitrarily chosen. Furthermore, in order to obtain consistent solution, the row vectors $\underline{d}_{q_j,n}$ are equal to zero.

Note that arbitrary row vectors are at

$$g + \sum_{i=g+1}^{n-r} v_i + n - r - g + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + 1,$$

$$g + \sum_{i=g+1}^{n-r} v_i + n - r - g + \sum_{i=1}^{\nu} p_i + q_1 + 1, \ldots,$$

$$g + \sum_{i=g+1}^{n-r} v_i + n - r - g + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j + 1$$ -place of matrix Y.

Finally, the zero row vectors are at

$$h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m - h - r + \sum_{i=1}^{\nu} p_i + q_1,$$

$$h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m - h - r + \sum_{i=1}^{\nu} p_i + q_1 + q_2, \ldots,$$

$$h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m - h - r + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma} q_j$$ -place of matrix D.

The above Theorem is a very interesting and straightforward result of KCF. Thus, we have succeeded in having a better understanding of the structure of system (1.3).

Remarks a) In order to obtain consistent solutions of system (1.4), $m - r + \sigma$ rows of matrix D should be equal to zero, see Theorem 1 (a).

b) Moreover, Y has $n - r + \sigma$ are arbitrarily chosen row vectors, see Theorem 1 (b).

In what follows with study further the results of Theorem 1.

2. Main Results

References [1], [2] and [4] have pointed out the importance of the pseudo-inverses to the solution of a general system of equation (1.1). In this section, we study the solution of system (1.1) fully by using a different approach, which is based on matrix pencil theory. However, in order to use the results of Section 1, we shall use the following definitions.

First, consider

$$A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix} \in \mathbb{C}^{m \times n},$$

where $\underline{a}_i \in \mathbb{C}^m$ for $i = 1, 2, \dots, n$ are column vectors of matrix A. Moreover, we define the $n \times n$ matrix as

$$X = \begin{bmatrix} \underline{x} & \underline{e}_2 & \underline{e}_3 & \dots & \underline{e}_n \end{bmatrix} \in \mathbb{C}^{n \times n},$$

where $\underline{e}_i = [0 \quad \dots \quad 0 \quad \underbrace{1}_{i-place} \quad 0 \quad \dots \quad 0]^t$, $i = 2, 3, \dots, n$. Finally, we define the matrix B by

$$B = \begin{bmatrix} \underline{b} & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix} \in \mathbb{C}^{m \times n},$$

where $\underline{b} \in \mathbb{C}^m$. It is profound that $A\underline{e}_i = \underline{a}_i$ for every $i = 2, 3, \dots, n$.

Consequently, we obtain the equivalent system

$$AX = B \Leftrightarrow A\begin{bmatrix} \underline{x} & \underline{e}_2 & \underline{e}_3 & \dots & \underline{e}_n \end{bmatrix} = \begin{bmatrix} \underline{b} & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix}. \tag{2.1}$$

In order to have the singular case apart from the rectangular matrices, i.e. when $m \neq n$; for $m = n$, we need to prove that the $\det(sA - B) = 0$ is always true (considering the notion of matrix pencil theory). Thus, the following theorem is more than important. Note that A, B are not arbitrary matrices, but they are having a very concrete expression.

Theorem 2. Consider the pencil $sA - B$, where the singular matrix $A = \begin{bmatrix} \underline{a}_1 & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix}$ and the matrix $B = \begin{bmatrix} \underline{b} & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix}$ are square, i.e. $m = n$, and the linear system $A\underline{x} = \underline{b}$ is consistent then $\det(sA - B) = 0$ is always true.

Proof. Considering the matrix pencil

$$sA - B = \begin{bmatrix} s\underline{a}_1 - \underline{b} & (s-1)\underline{a}_2 & \cdots & (s-1)\underline{a}_n \end{bmatrix}, \tag{2.2}$$

then

$$\det(sA-B)=\left|s\underline{a}_1-\underline{b}\quad (s-1)\underline{a}_2\quad \cdots\quad (s-1)\underline{a}_n\right|=$$
$$=\left|s\underline{a}_1\quad (s-1)\underline{a}_2\quad \cdots\quad (s-1)\underline{a}_n\right|+\left|-\underline{b}\quad (s-1)\underline{a}_2\quad \cdots\quad (s-1)\underline{a}_n\right|$$
$$=s(s-1)^{n-1}\left|\underline{a}_1\ \underline{a}_2\cdots\underline{a}_n\right|-(s-1)^{n-1}\left|\underline{b}\ \underline{a}_2\cdots\underline{a}_n\right|.$$

However, since we have assumed that $\det A=\left|\underline{a}_1\ \underline{a}_2\cdots\underline{a}_n\right|=0$, when $m=n$, we obtain

$$\det(sA-B)=-(s-1)^{n-1}\left|\underline{b}\ \underline{a}_2\cdots\underline{a}_n\right|.$$

Suppose that $\det(sA-B)\neq 0$. We can show that

$$\left|\underline{b}\ \underline{a}_2\cdots\underline{a}_n\right|\neq 0 .$$

This is true if $\underline{b}\notin\langle\underline{a}_2\cdots\underline{a}_n\rangle$ and $\underline{a}_2,\ldots,\underline{a}_n$ are linear independent.

Then, the matrix pencil $sA-B$ is regular. Consequently, there are invertible matrices P,Q such that

$$P(sA-B)Q=sA_w-B_w,$$

where sA_w-B_w is the *complex Weierstrass canonical form* of pencil $sA-B$.

It is known that

$$PAQ=A_w=\begin{bmatrix} I_p & \mathbb{O}_{p,q}\\ \mathbb{O}_{q,p} & H_q\end{bmatrix}.$$

Now, using the transformation (1.21) and multiplying by P, the system (2.1) can be written

$$AQY=B\Rightarrow PAQY=PB\Rightarrow A_wY=D,\text{ where } D=PB.$$

Moreover,

$$A_wY=D\Leftrightarrow\begin{bmatrix} I_p & \mathbb{O}_{p,q}\\ \mathbb{O}_{q,p} & H_q\end{bmatrix}\begin{bmatrix}\mathbf{y}_p\\ \mathbf{y}_q\end{bmatrix}=\begin{bmatrix}\mathbf{d}_p\\ \mathbf{d}_q\end{bmatrix},$$

where $\mathbf{y}_p\in\mathbb{C}^{p\times n}$, $\mathbf{y}_q\in\mathbb{C}^{q\times n}$; $\mathbf{d}_p\in\mathbb{C}^{p\times n}$, $\mathbf{d}_q\in\mathbb{C}^{q\times n}$ and $n=p+q$.

Then two subsystems are derived, that is

$$\begin{cases} \mathbf{y}_p = \mathbf{d}_p \\ H_q \mathbf{y}_q = \mathbf{d}_q \end{cases}. \tag{2.3}$$

Matrix $B = \begin{bmatrix} \underline{b} & \underline{a}_2 & \dots & \underline{a}_n \end{bmatrix}$ has nonzero determinant. Also, every row vector of matrix B is nonzero. Additionally, since P is invertible, the matrix $D = PB$ is also invertible. Consequently, the last vector $\underline{d}_n \in \mathbb{C}^{1\times n}$ is also nonzero.

Now, taking the second part of (2.3), we have derived that

$$H_q \mathbf{y}_q = \mathbf{d}_q \Leftrightarrow \begin{bmatrix} y_{p+2,1} & y_{p+2,2} & \cdots & y_{p+2,n} \\ \vdots & \vdots & \dots & \vdots \\ y_{p+q,1} & y_{p+q,2} & \cdots & y_{p+q,n} \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} \underline{d}_{p+1} \\ \vdots \\ \underline{d}_{p+q-1} \\ \underline{d}_n \end{bmatrix}.$$

Consequently, $\underline{d}_n = \underline{0}$ which is not accepted. Thus, $\det(sA - B) = 0$, and the required result is proven.

We proceed to continue the study of the results presented in Section 1 in more details. We can write (1.22) as

$$X = QY \Leftrightarrow \begin{bmatrix} \underline{x} & \underline{e}_2 & \underline{e}_3 & \dots & \underline{e}_n \end{bmatrix} = \begin{bmatrix} Q_{i,j} \end{bmatrix}_{i,j=1,2,\dots n} \begin{bmatrix} \underline{y}_1 & \underline{y}_2 & \underline{y}_3 & \cdots & \underline{y}_n \end{bmatrix} \tag{2.4}$$

where $\underline{y}_i \in \mathbb{C}^n$ for $i = 1, 2, \dots, n$, are row vectors of matrix Y. Then

$$\underline{x} = \begin{bmatrix} \sum_{j=1}^{n} Q_{1,j} y_{j,1} \\ \sum_{j=1}^{n} Q_{2,j} y_{j,1} \\ \vdots \\ \sum_{j=1}^{n} Q_{n,j} y_{j,1} \end{bmatrix} \tag{2.5}$$

and

$$\underline{e}_i = \begin{bmatrix} \sum_{j=1}^{n} Q_{1,j} y_{j,i} \\ \sum_{j=1}^{n} Q_{2,j} y_{j,i} \\ \vdots \\ \sum_{j=1}^{n} Q_{n,j} y_{j,i} \end{bmatrix} \text{ for every } i = 2,3,\ldots,n \tag{2.6}$$

where $Y = \left[y_{i,j} \right]_{i,j=1,2,\ldots,n}$. However, some of the y_{j1}, for $j = 1,2,\ldots,n$ are arbitrarily chosen and some other are already known. More precisely, see also Theorem 1,

- $y_{1,1}, y_{2,1}, \ldots, y_{g,1}$,
- $y_{g+v_{g+1}+1,1}$, $y_{g+v_{g+1}+v_{g+2}+2,1}, \ldots, y_{\sum_{i=g+1}^{n-r} v_i + n-r,1}$,

and

- $y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + 1,1}, y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + q_1 + 1,1}, \ldots,$
  $y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j + 1,1}$

are arbitrarily chosen. Note that all the other elements are known.

Thus, each element of $\underline{x}$ is given by the following expression, for $l = 1,2,\ldots,n$

$$x_l = \sum_{\substack{s=g+1, \\ s \neq g+v_{g+1}+1,\, g+v_{g+1}+v_{g+2}+2,\, \ldots,\, \sum_{i=g+1}^{n-r} v_i + n-r \\ s \neq \sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + 1, \\ \sum_{i=g+1}^{n-r} v_i + n-r + \sum_{i=1}^{\nu} p_i + q_1 + 1, \ldots, \\ \sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j + 1,1}}^{n} Q_{l,s} d_{s,1} + \sum_{s=1}^{g} Q_{l,s} y_{s,1} + \sum_{s=1}^{n-r-g} Q_{l,\mathrm{g}+\sum_{\tau=1}^{j} v_{g+\tau}+j}\, y_{\mathrm{g}+\sum_{\tau=1}^{s} v_{g+\tau}+s,1}$$

$$+ \sum_{s=1}^{\sigma} Q_{l,\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{\tau=1}^{s-1} q_\tau + 1}\, y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{\tau=1}^{s-1} q_\tau + 1,1}.$$

Furthermore, we can write it in vector form as

$$\underline{x} = Q \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_{g+1,1} \\ \vdots \\ d_{g+v_{g+1},1} \\ 0 \\ d_{g+v_{g+1}+2,1} \\ \vdots \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r-1,1} \\ 0 \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r+1,1} \\ \vdots \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i,1} \\ 0 \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + 2,1} \\ \vdots \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j,1} \\ 0 \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j + 2,1} \\ \vdots \\ d_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j,1} \end{bmatrix} + Q \underbrace{\begin{bmatrix} y_{1,1} \\ \vdots \\ y_{g,1} \\ 0 \\ \vdots \\ 0 \\ y_{g+v_{g+1}+1,1} \\ 0 \\ \vdots \\ 0 \\ y_{\sum_{i=g+1}^{n-r} v_i + n-r,1} \\ 0 \\ \vdots \\ 0 \\ y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + 1,1} \\ 0 \\ \vdots \\ 0 \\ y_{\sum_{i=g+1}^{n-r} v_i + n-r + \sum_{j=h+1}^{m-r} u_j + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma-1} q_j + 1,1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{z} \tag{2.7}$$

Remark a) The solution of system (1.1) has $n-r+\sigma \; (<n)$ arbitrarily chosen elements which are sufficiently smaller when r is close to n, i.e. we have full column rank, and the blocks depend on infinite elementary divisors, i.e. σ, tends to zero, as well.

b) If we are interested about *consistent* solution of systems (1.1), we should also consider the first part of Theorem 1, i.e. $\underline{d}_{i,1} = 0$, for every

$$i = 1, 2, \ldots, h, h + \sum_{i=g+1}^{n-r} v_i + u_{h+1} + 1, h + \sum_{i=g+1}^{n-r} v_i + u_{h+1} + u_{h+2} + 2, \ldots, h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j$$
$$+ m-h-r, h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r + \sum_{i=1}^{\nu} p_i + q_1, h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r$$
$$+ \sum_{i=1}^{\nu} p_i + q_1 + q_2, \ldots, h + \sum_{i=g+1}^{n-r} v_i + \sum_{j=h+1}^{m-r} u_j + m-h-r + \sum_{i=1}^{\nu} p_i + \sum_{j=1}^{\sigma} q_j .$$

3. Conclusions

The present paper discusses the classical problem of solving an ordinary system of linear equations with rectangular (or singular) coefficient matrices. Using the complex Kronecker canonical form, the solution is derived. The proposed methodology improves our knowledge of the problem into two main directions. Firstly, it decreases importantly the arbitrary - unknown elements of vector z. Secondly, in order to obtain consistent solutions, we know exactly the required and necessary mathematical conditions.

Finally, it should be stressed that the more our matrix tends to be full row/ column rank, and the less infinite elementary divisors it has, the better results are obtained, since the arbitrary elements are enormously diminished. That remark is very compatible to the known linear system theory, since our system tends to become regular.

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