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UNIFORM CONVERGENCE OF MONOTONE DIFFERENCE SCHEME FOR A SINGULARLY PERTURBED THIRD-ORDER CONVECTION-DIFFUSION EQUATION

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ABSTRACT. A singularly perturbed third-order convection-diffusion problem is considered. The third-order boundary value problem is transformed into a system of weakly coupled system of two differential equations. Then the problem is solved numerically using a monotone difference scheme on a Bakhvalov-Shishkin mesh. We use appropriate estimates of the discrete Green's function to obtain error estimates of the monotone difference scheme. Numerical experiments support our theoretical results.

Key Words. convection-diffusion problem, singularly perturbed, Bakhvalov-Shishkin mesh, finite difference scheme, Green's function.

1. INTRODUCTION

Singular perturbation problems appear in many branches of applied mathematics, and for more than two decades quite a good number of research works on the qualitative and quantitative analysis of these problems for both ordinary differential equations and partial differential equations have been reported in the literature. Most of the papers connected with computational aspects are confined to second order equation. But only few authors have developed numerical methods for singularly perturbed higher-order differential equations.

In this paper, we treat the following third-order singularly perturbed ordinary differential equations

$$-\varepsilon y'''(x) - a(x)y''(x) + b(x)y'(x) - c(x)y(x) = f(x), \quad x \in D, \tag{1.1}$$

$$y(0) = p, \; y'(0) = q, \; y'(1) = r, \tag{1.2}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, $a(x), b(x), c(x)$ and $f(x)$ are sufficiently smooth functions satisfying the following conditions:

$$a(x) \geq \alpha > 0, \tag{1.3}$$

$$b(x) \geq 0, \tag{1.4}$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{1.5}$$

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$$\alpha - \gamma(1+3\eta) \geq \eta' > 0 \quad \text{for some} \quad \eta \quad \text{and} \quad \eta', \tag{1.6}$$

with $D=(0,1), D_0=(0,1], \bar{D}=[0,1]$ and $y \in C^{(3)}(D) \cap C^{(1)}(\bar{D})$.

In [1], Niederdrenk and Yserentant considered higher order convection-diffusion type problems and derived conditions for the uniform stability of discrete and continuous problems. The works of Gartland [2] are connected with exponentially fitted higher order difference scheme with identity expansion method. Recently, Valarmathi and Ramanujam [3] proposed a method, 'boundary value technique', to find a numerical solution for the third-order problem (1.1)-(1.2). Also, Semper [4] and Roos et al. [5] considered fourth order equations and applied a standard finite element method.

The aim of this work is to illustrate an application of a priori estimates of the solutions of discrete problems, which are obtained using Green's function, to analyze the accuracy of finite difference schemes in the discrete maximum norm. The present work is an extension of L_1 norm stability inequality technique for second-order singularly perturbed boundary value problem to third-order singularly perturbed problem subject to a particular type of boundary conditions.

An outline of the paper is as follows.In the next section we derive bounds for the derivatives of the solution of (1.1). In section 3 we introduce a upwind difference scheme and a Bakhvalov-Shishkin mesh. We analyze the convergence properties of the scheme in section 4. Finally numerical results are presented in section 5.

Notation. Throughout the paper, C will denote a generic positive constant that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurence.

Assumption 1. Throughout the paper we shall also assume that $\varepsilon \leq CN^{-1}$ as is generally the case.

2. THE CONTINUOUS PROBLEM

In this section we will present a priori bounds on the solution of (1.1)-(1.2) and its derivatives.These bounds will be used in the error analysis in later sections.The singularly perturbed boundary value problem (1.1)-(1.2) can be transformed into an equivalent problem of the form

(2.1)

$$\mathbf{Ay} = \mathbf{F} \Longleftrightarrow \begin{cases} P_1\mathbf{y} \equiv y_1'(x) - y_2(x) = 0, \\ P_2\mathbf{y} \equiv -\varepsilon y_2''(x) - a(x)y_2'(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), x \in D, \\ y_1(0) = p, \ y_2(0) = q, \ y_2(1) = r, \end{cases}$$

where $\mathbf{y} = (y_1, y_2)$.

Lemma 1. (Maximum principle [3]) Consider the boundary value problem (2.1).Assume that $P_1\mathbf{u} \geq 0, P_2\mathbf{u} \geq 0$ in D, $u_1(0) \geq 0, u_2(0) \geq 0$, and $u_2(1) \geq 0$. Then $\mathbf{u}(x) \geq 0$ in $[0,1]$. Here $\mathbf{u}(x) = (u_1(x), u_2(x))$ for all $x \in \bar{D}$.

Lemma 2. (Stability result [3]) Consider the boundary value problem (2.1). If $\mathbf{y}$ is a smooth function, then

$$\|\mathbf{y}(x)\| \le C\max\{|y_1(0)|, |y_2(0)|, |y_2(1)|, \max_{x\in\bar{D}}|P_1\mathbf{y}|, \max_{x\in\bar{D}}|P_2\mathbf{y}|\} \quad \text{for all} \quad x \in \bar{D},$$

where $\|\mathbf{y}(x)\| = \max\{|y_1(x)|, |y_2(x)|\}$.

The construction of layer-adapted meshes and the analysis of numerical methods for singularly perturbed problems require precise knowledge about the behavior of the derivatives of the exact solution. The following lemma provides that information.

Lemma 3. If $a(x), b(x), c(x)$ and $f(x) \in C^{(j)}(\bar{D})$, then the solution $\mathbf{y}(x)$ of (1.1)-(1.2) has the representation $\mathbf{y} = \mathbf{v} + \mathbf{w}$ on $[0,1]$, where the smooth part $\mathbf{v}$ satisfies $P_1\mathbf{v}(x) = 0, P_2\mathbf{v}(x) = f(x)$ and

$$\|\mathbf{v}^{(k)}(x)\| \le C, \quad \text{for all} \quad k \le j \ , \ x \in \bar{D},$$

while the layer part $\mathbf{w}$ satisfies $P_1\mathbf{w}(x) = 0, P_2\mathbf{w}(x) = 0, \|\mathbf{w}(0)\| \le C, \|\mathbf{w}(1)\| \le C\exp(-\alpha/\varepsilon)$ and

$$|w_1^{(k)}(x)| \le C\varepsilon^{1-k}\exp(-\alpha x/\varepsilon) \ , \ |w_2^{(k)}(x)| \le C\varepsilon^{-k}\exp(-\alpha x/\varepsilon) \quad \text{for all} \quad k \le j \ , \ x \in \bar{D}.$$

Proof. The bounds can be obtained using results of reference [3]. □

3. MESH AND SCHEME

Our mesh is a modification of the Shishkin mesh:Bakhvalov-Shishkin mesh combined the Bakhvalov mesh in the boundary layer with the Shishkin-type transition point.

Let N, our discretization parameter, be an even positive integer. We choose the transition point σ as Shishkin does:

$$\sigma = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}. \tag{3.1}$$

Then our mesh is

$$x_i = \begin{cases} -\frac{2\varepsilon}{\alpha}\ln[1 - 2(1 - N^{-1})\frac{i}{N}] & \text{for } i = 0, 1, \cdots, N/2, \\ 1 - (1 - \frac{2\varepsilon}{\alpha}\ln N)^{\frac{2(N-i)}{N}} & \text{for } i = N/2+1, \cdots, N. \end{cases} \tag{3.2}$$

Let

$$D^N = \{x_i | i = 1, 2, \cdots, N-1\}, \quad D_0^N = \{x_i | i = 0, 1, \cdots, N-1\},$$
$$\bar{D}^N = \{x_i | i = 0, 1, \cdots, N\}.$$

The following lemma gives some estimates of the mesh sizes that will be used later.

Lemma 4. The step sizes of the mesh D^N satisfy

$$h_i \le \frac{8\varepsilon}{\alpha(N/2 - i + 1)} \le CN^{-1}, \quad h_{N/2+i} \le 2N^{-1}, \quad \text{for} \quad i = 1, 2, \cdots, N/2.$$

Proof. These can be easily calculated. See [7]. □

Now we consider the upwind difference scheme

$$P_1^N \mathbf{y}_i^N \equiv D y_{1,i}^N - y_{2,i}^N = 0, \quad i = 0, 1, \cdots, N-1, \tag{3.3}$$

$$P_2^N \mathbf{y}_i^N \equiv -\varepsilon D^+ D^- y_{2,i}^N - a_i D y_{2,i}^N + b_i y_{2,i}^N + c_i y_{1,i}^N = f_i, \ i = 1, 2, \cdots, N-1, \tag{3.4}$$

$$y_{1,0}^N = p, \ y_{2,0}^N = q, \ y_{2,N}^N = r, \tag{3.5}$$

where

$$D^+ v_i = \frac{v_{i+1} - v_i}{h_{i+1}}, \ D^- v_i = \frac{v_i - v_{i-1}}{h_i}, D v_i = \frac{v_{i+1} - v_i}{\hbar_i} \ \text{ and } \ \hbar_i = \frac{h_i + h_{i+1}}{2}, \hbar_0 = h_1.$$

4. ANALYSIS OF THE METHOD

Analogous to the continuous problem (2.1), we can get results for the discrete problem.

Lemma 5. (Discrete maximum principle [3]) Consider the discrete problem (3.3)-(3.5). If $y_{1,0} \geq 0, y_{2,0} \geq 0, y_{2,N} \geq 0, P_1^N \mathbf{y}_i \geq 0$ for $i = 0, 1, \cdots, N-1$, and $P_2^N \mathbf{y}_i \geq 0$ for $i = 1, 2, \cdots, N-1$, then $\mathbf{y}_i \geq 0$ for $i = 0, 1, \cdots, N$.

Lemma 6. (Stability result) If $\mathbf{y}_i$ is any mesh function, then

$$\begin{aligned} |y_{1,i}| &\leq C \max\{|y_{1,0}|, \max_{1 \leq i \leq N-1} |P_1^N \mathbf{y}_i|\}, \quad \text{for } \ i = 0, 1, \cdots, N, \\ |y_{2,i}| &\leq C \max\{|y_{2,0}|, |y_{2,N}|, \max_{1 \leq i \leq N-1} |P_2^N \mathbf{y}_i|\}, \quad \text{for } \ i = 1, \cdots, N. \end{aligned}$$

Proof. The proof is analogous with that of Lemma 2.1 in [3]. □

4.1 Discrete Green's function and its properties

For any mesh function w^N, we use $\|\cdot\|_\infty$ for the standard maximum norm, and we define a discrete L_1 norm by

$$\|w^N\|_1 = \sum_{i=1}^{N-1} \hbar_i |w_i^N|.$$

We also define the scalar product in $\mathbb{R}^{N+1}$ by

$$(v^N, w^N) = \sum_{j=1}^{N-1} v_j^N w_j^N \hbar_j, \quad \forall v^N, w^N \in \mathbb{R}^{N-1}.$$

Consider the Green's function $\mathbf{G}^N(x_i, \xi_j)$ of problem (3.3)-(3.4). As a function of x_i for fixed ξ_j this function is defined by the relations

$$P_1^N \mathbf{G}^N(x_i, \xi_j) = 0 \ , \quad x_i \in D_0^N, \xi_j \in D^N, \tag{4.1}$$

$$P_2^N \mathbf{G}^N(x_i, \xi_j) = \delta^N(x_i, \xi_j) \ , \quad x_i \in D^N, \xi_j \in D^N, \tag{4.2}$$

$$G_1^N(0, \xi_j) = G_2^N(0, \xi_j) = G_2^N(1, \xi_j) = 0 \ , \quad \xi_j \in D^N, \tag{4.3}$$

where

$$\delta^N(x_i,\xi_j)=\begin{cases}\hbar_i^{-1} & \text{for } \ x_i=\xi_j,\\ 0 & \text{for } \ x_i\neq\xi_j.\end{cases}$$

It is easy to see that using Green's function, we can give the following formula for the solution of problem (3.3)-(3.4)

$$y_{1,i}^N = \sum_{j=1}^{i-1} y_{2,i}^N + y_{1,0}^N, \tag{4.4}$$

$$y_{2,i}^N = \sum_{j=1}^{N-1} G_2^N(x_i,\xi_j)f_j\hbar_j\ ,\ x_i\in D^N. \tag{4.5}$$

Indeed, taking into account (3.6)-(3.7), we obtain

$$\begin{aligned}&(G_2^N(x_i,\xi_j),f_j)=(G_2^N(x_i,\xi_j),-\varepsilon D^+D^-y_{2,j}^N-a_jDy_{2,j}^N+b_jy_{2,j}^N+c_jy_{1,j}^N)\\&=(P_2^N\mathbf{G}^N,y_{2,j}^N)+(G_2^N(x_i,\xi_j),c_jy_{1,j}^N)-(G_1^N(x_i,\xi_j),c_jy_{2,j}^N)\\&=(\delta^N(x_i,\xi_j),y_{2,j}^N)+(D^+D^-G_1^N(x_i,\xi_j),c_jy_{1,j}^N)-(G_1^N(x_i,\xi_j),c_jy_{2,j}^N)\\&=y_{2,i}^N+(G_1^N(x_i,\xi_j),c_jD^+D^-y_{1,j}^N)-(G_1^N(x_i,\xi_j),c_jy_{2,j}^N)=y_{2,i}^N,\quad\text{for } \ x_i\in D^N.\end{aligned}$$

The Green's function $\mathbf{G}^N(x_i,\xi_j)$ as the function of a variable ξ_j for fixed x_i is the solution of the adjoint problem:

$$P_1^{N,*}\mathbf{G}^N(x_i,\xi_j)=0\ ,\ \xi_j\in D_0^N,x_i\in D^N, \tag{4.6}$$

$$P_2^{N,*}\mathbf{G}^N(x_i,\xi_j)=\delta^N(x_i,\xi_j)\ ,\ \xi_j\in D^N,x_i\in D^N, \tag{4.7}$$

$$G_1^N(x_i,0)=G_2^N(x_i,0)=G_2^N(x_i,1)=0\ ,\ x_i\in D^N. \tag{4.8}$$

This arises from the following arguments: using (4.5),(3.3) and (3.4), and the fact that $P_1^{N,*},P_2^{N,*}$ is adjoint to P_1^N,P_2^N respectively,we have

$$P_1^{N,*}\mathbf{G}^N(x_i,\xi_j)=0$$

and

$$\begin{aligned}y_{2,i}^N&=\sum_{j=1}^{N-1}G_2^N(x_i,\xi_j)f_j\hbar_j=\sum_{j=1}^{N-1}G_2^N(x_i,\xi_j)P_2^N\mathbf{y}_j^N\hbar_j\\&=\sum_{j=1}^{N-1}P_2^{N,*}\mathbf{G}^N(x_i,\xi_j)y_{2,j}^N\hbar_j\Longrightarrow P_2^{N,*}\mathbf{G}^N(x_i,\xi_j)=\delta^N(x_i,\xi_j),\end{aligned}$$

where we have used (4.6)-(4.7).

Lemma 7. The Green's function $\mathbf{G}^N(x_i,\xi_j)$ is nonnegative and bounded uniformly in ε:

$$0\leq\mathbf{G}^N(x_i,\xi_j)\leq\frac{1}{\alpha-\gamma}.$$

Proof. From Lemma 5, we can easily get the nonnegativity of the Green's function.

We now wish to prove the upper bound. Let the point $\xi_{j_0} \in D^N$ be such that

$$\max_{\xi_j \in D^N} G_2^N(x_i, \xi_j) = G_2^N(x_i, \xi_{j_0}) \ , \ x_i \in D^N.$$

Multiply (4.7) by $\hbar_j$ and sum with respect to j from 1 to j_0. Taking into account that $G_2^N(x_i, 0) = 0$, we obtain

$$\sum_{j=1}^{j_0} P_2^{N,*}\mathbf{G}^N(x_i, \xi_j)\hbar_j = -\varepsilon D_\xi^+ G_2^N(x_i, \xi_{j_0}) + \varepsilon D_\xi^- G_2^N(x_i, \xi_1)$$

$$+a_{j_0} G_2^N(x_i, \xi_{j_0}) + \sum_{j=1}^{j_0} (b_j G_2^N(x_i, \xi_j)\hbar_j + c_j G_1^N(x_i, \xi_j)\hbar_j). \tag{4.9}$$

Because of the choice of ξ_{j_0},

$$D_\xi^+ G_2^N(x_i, \xi_{j_0}) = (G_2^N(x_i, \xi_{j_0+1}) - G_2^N(x_i, \xi_{j_0}))h_{j_0+1} \leq 0, \tag{4.10}$$

and as $G_2^N(x_i, \xi_j)$ is nonnegative then

$$D_\xi^- G_2^N(x_i, \xi_1) = G_2^N(x_i, \xi_1)h_1^{-1} \geq 0. \tag{4.11}$$

On the other hand, from (4.6) we can get

$$\sum_{k=0}^{j-1} G_2^N(x_i, \xi_k)\hbar_k = G_1^N(x_i, \xi_j).$$

So

$$G_1^N(x_i, \xi_j) \leq G_2^N(x_i, \xi_{j_0}). \tag{4.12}$$

Combining (4.9)-(4.12), we obtain

$$(\alpha - \gamma) G_2^N(x_i, \xi_{j_0}) \leq \sum_{j=1}^{j_0} \delta^N(x_i, \xi_j)\hbar_j \leq 1. \tag{4.13}$$

Also, from (4.6),(4.9) we have

$$G_1^N(x_i, \xi_j) = \sum_{k=1}^{j-1} G_2(x_i, \xi_j)\hbar_k. \tag{4.14}$$

From (4.13) and (4.14) we can obtain the desired results. □

Lemma 8. The operator P_2^N satisfies

$$\|y_2^N\|_\infty \leq \frac{1}{\alpha - \gamma} \|P_2^N \mathbf{y}^N\|_1.$$

Proof. The proof follows directly from the representation of the solution in (4.5) and Lemma 7. □

4.2 Truncation error on a Bakhvalov-Shishkin mesh

Let $\mathbf{y}_i^N$ be the solution of the discrete problem (3.3)-(3.5) and $\mathbf{y}_i$ be the values of the solution of the original continuous problem at the nodes of mesh $\bar{D}^N$. Then $\mathbf{z}_i = \mathbf{y}_i^N - \mathbf{y}_i$ is the accuracy of the solution.Substituting $\mathbf{y}_i^N = \mathbf{z}_i + \mathbf{y}_i$ into (3.3)-(3.4). We see that $\mathbf{z}_i$ is the solution of the following problem

$$P_1^N \mathbf{z}_i = -P_1^N \mathbf{y}_i = -D^+ y_{1,i} + y_{2,i} \equiv \psi_{1,i}, \tag{4.15}$$

$$P_2^N \mathbf{z}_i = f_i - P_2^N \mathbf{y}_i = f_i + \varepsilon D^+ D^- y_{2,i} + a_i D^+ y_{2,i} - b_i y_{2,i} - c_i y_{1,i} \equiv \psi_{2,i}, \tag{4.16}$$

$$z_{1,0} = z_{2,0} = z_{2,N} = 0. \tag{4.17}$$

Using (2.1), we have one more representation

$$\begin{aligned} \psi_{1,i} &= -(D^+ y_{1,i} - y'_{1,i}), \\ \psi_{2,i} &= \varepsilon(D^+ D^- y_{2,i} - y''_{2,i}) + a_i (D^+ y_{2,i} - y'_{2,i}). \end{aligned}$$

We now estimate the truncation error ψ_i on the Bakhvalov-Shishkin mesh.

Lemma 9. The following estimates for the truncation error hold true:

$$\begin{aligned} |\psi_1(x_i)| &= C h_{i+1} \varepsilon^{-1} \exp(-\alpha x_i/\varepsilon) \le C N^{-1} \quad \text{for} \quad i = 0, 1, \cdots, N-1, \\ |\psi_2(x_i)| &= \begin{cases} C(h_{i+1} + N^{-1}\varepsilon^{-1} \exp(-\frac{\alpha x_i}{2\varepsilon}) & \text{for } i = 1, 2, \cdots, N/2-1, \\ C(h_{i+1} + \varepsilon^{-2}(h_i + h_{i+1}) \exp(-\frac{\alpha x_{i-1}}{\varepsilon})) & \text{for } i = N/2+1, \cdots, N-1, \\ C(h_{i+1} + h_i \exp(-\frac{\alpha x_i}{\varepsilon}) + 1) & \text{for } i = N/2, N/2+1. \end{cases} \end{aligned}$$

Proof. For $i = 0, 1, \cdots, N-1$ we use a Taylor expansion for $x = x_i$ to get

$$|\psi_{1,i}| = \frac{1}{2} h_{i+1} |y_1''(\xi_i)| \le C h_{i+1} \varepsilon^{-1} \exp(-\alpha x_i/\varepsilon) \le C N^{-1}, \ \xi_i \in (x_i, x_{i+1}), \tag{4.18}$$

where we have used

$$\frac{h_i}{\varepsilon} \exp(-\alpha x_i/\varepsilon) \le C N^{-1} \qquad \text{for} \quad i = 1, 2, \cdots, N/2.$$

Recalling the decomposition of Lemma 3, we have

$$|\psi_{2,i}| = |f_i - P_2^N \mathbf{y}_i| \le |P_2 \mathbf{v}_i - P_2^N \mathbf{v}_i| + |P_2 \mathbf{w}_i - P_2^N \mathbf{w}_i| \tag{4.19}$$

for $i = 1, 2, \cdots, N-1$. For the smooth part, we have

$$|P_2 \mathbf{v}_i - P_2^N \mathbf{v}_i| \le 2\varepsilon \int_{x_{i-1}}^{x_{i+1}} |v_2'''(t)| \mathrm{d}t + a_i \int_{x_i}^{x_{i+1}} |v_2''(t)| \mathrm{d}t \le C h_{i+1} \tag{4.20}$$

for $i = 1, 2, \cdots, N-1$. For the truncation error of the method with respect to the layer part $\mathbf{w}$ we have

$$\begin{aligned} |P_2 \mathbf{w}_i - P_2^N \mathbf{w}_i| &\le 2\varepsilon \int_{x_{i-1}}^{x_{i+1}} |w_2'''(t)| \mathrm{d}t + a_i \int_{x_i}^{x_{i+1}} |w_2''(t)| \mathrm{d}t \\ &\le C\varepsilon^{-2} \int_{x_{i-1}}^{x_{i+1}} \exp(-\alpha t/\varepsilon) \mathrm{d}t \quad \text{for} \quad i = 1, 2, \cdots, N-1. \end{aligned} \tag{4.21}$$

Let $x_i = \frac{2\varepsilon}{\alpha}\varphi(t) = -\frac{2\varepsilon}{\alpha}\ln[1-2(1-N^{-1})t]$ and $t_i = \varphi^{-1}(x_i)$ for $i = 1, 2, \cdots, N/2$. Then

$$\begin{aligned} |P_2^N(\mathbf{w}_i - \mathbf{w}_i^N)| &\leq C\varepsilon^{-1}\int_{t_{i-1}}^{t_{i+1}} \exp(-2\varphi(t))\varphi'(t)\mathrm{d}t \leq C\varepsilon^{-1}\int_{t_{i-1}}^{t_{i+1}} \exp(\varphi(t))\mathrm{d}t \\ &\leq C\varepsilon^{-1}\int_{t_{i-1}}^{t_{i+1}} \exp(-\frac{\alpha x_{i-1}}{2\varepsilon})\mathrm{d}t \leq CN^{-1}\varepsilon^{-1}\exp(-\frac{\alpha x_{i-1}}{2\varepsilon}) \\ &\leq CN^{-1}\varepsilon^{-1}\exp(-\frac{\alpha x_i}{2\varepsilon}) \ \text{ for } \ i = 1, 2, \cdots, N/2-1. \end{aligned} \tag{4.22}$$

and

$$|P_2\mathbf{w}_i - P_2^N\mathbf{w}_i| \leq C\varepsilon^{-2}(h_i + h_{i+1})\exp(-\alpha x_{i-1}/\varepsilon) \tag{4.23}$$

for $i = N/2+1, \cdots, N-1$.

Next we estimate $|P_2\mathbf{w}_{N/2} - P_2^N\mathbf{w}_{N/2}|$.

$$\begin{aligned} &|P_2\mathbf{w}_{N/2} - P_2^N\mathbf{w}_{N/2}| = |P_2^N\mathbf{w}_{N/2}| \\ &= |\varepsilon D^+D^- w_{2,N/2} + a_{N/2}D^+w_{2,N/2} - b_{N/2}w_{2,N/2} - c_{N/2}w_{1,N/2}| \\ &\leq \frac{1}{\hbar_{N/2}}|\varepsilon(D^+w_{2,N/2} - D^-w_{2,N/2}) + a_{N/2}(w_{2,N/2+1} - w_{2,N/2})| + C \\ &= \frac{1}{\hbar_{N/2}}[\varepsilon(w_2'(\xi_{N/2}) - w_2'(\xi_{N/2-1})) + a_{N/2}(w_{2,N/2+1} - w_{2,N/2})] + C \\ &\leq C(\hbar_{N/2}^{-1}\exp(-\frac{\alpha x_{N/2-1}}{\varepsilon}) + 1). \end{aligned} \tag{4.24}$$

Using similar reasoning, we obtain the following estimate

$$|P_2\mathbf{w}_{N/2+1} - P_2^N\mathbf{w}_{N/2+1}| \leq C(1 + h_{N/2+1}^{-1}\exp(-\frac{\alpha x_{N/2}}{\varepsilon})). \tag{4.25}$$

Combining (4.19)-(4.25) we can complete the local estimate of $\psi_{2,i}$. □

We can now derive our main result.

Theorem 1. The error of the difference scheme on the Bakhvalov-Shishkin mesh satisfies

$$\|\mathbf{y}_i - \mathbf{y}_i^N\| \leq CN^{-1} \ \text{ for } \ i = 0, 1, \cdots, N,$$

where $\|\mathbf{y}_i\| = \max\{|y_{1,i}|, |y_{2,i}|\}$ for $i = 0, 1, \cdots, N$.

Proof. By (4.5) and Lemma 8, we have the following a priori estimate for the accuracy $z_{2,i} = y_{2,i}^N - y_{2,i}$ of the solution in terms of the truncation error $\psi_{2,i}$

$$|y_{2,i} - y_{2,i}^N| \leq C\|\psi_{2,i}\|_1 \ \text{ for } \ i = 1, 2, \cdots, N-1. \tag{4.26}$$

Using Lemma 9, we obtain

$$\|\psi_{2,i}\|_1 = \sum_{i=1}^{N/2-1}|\psi_{2,i}|\hbar_i + |\psi_{2,N/2}|\hbar_{N/2} + |\psi_{2,N/2+1}|\hbar_{N/2+1} + \sum_{i=N/2+2}^{N-1}|\psi_{2,i}|\hbar_i$$

$$\begin{aligned}
&\leq C(\sum_{i=1}^{N/2-1} h_{i+1}\hbar_i + \hbar_{N/2} + \hbar_{N/2+1} + \sum_{i=N/2+2}^{N-1} h_{i+1}\hbar_i) \\
&+CN^{-1}\varepsilon^{-1}\sum_{i=1}^{N/2-1}\exp(-\frac{\alpha x_i}{2\varepsilon})\hbar_i \\
&+C(\exp(-\frac{\alpha x_{N/2-1}}{\varepsilon}) + \exp(-\frac{\alpha x_{N/2}}{\varepsilon})) \\
&+C\varepsilon^{-2}\sum_{i=N/2+2}^{N-1}(h_i + h_{i+1})\hbar_i \exp(-\frac{\alpha x_{i-1}}{\varepsilon}) \leq CN^{-1}. \quad (4.27)
\end{aligned}$$

Combining (4.26) and (4.27) we get

$$|y_{2,i} - y_{2,i}^N| \leq CN^{-1} \quad \text{for} \quad i = 0, 1, \cdots, N. \quad (4.28)$$

From Lemma 6 we have

$$|y_{1,i} - y_{1,i}^N| \leq C|\psi_{1,i}| \leq CN^{-1} \quad \text{for} \quad i = 0, 1, \cdots, N. \quad (4.29)$$

By (4.28) and (4.29) we get the desired results. □

5. NUMERICAL RESULTS

In this section we verify experimentally the theoretically results obtained in the preceding section. We present two examples to illustrate the method described in this paper.

Example 1. Consider the boundary value problem

$$-\varepsilon y'''(x) - 2y''(x) = 1 \ , \ \text{for} \ \ x \in (0,1) \quad (5.1)$$

$$y(0) = 1 \ , \ y'(0) = 1 \ , \ y'(1) = 1. \quad (5.2)$$

Example 2. Consider the boundary value problem

$$-\varepsilon y'''(x) - 2(1+x)y''(x) + 4xy'(x) - y(x) = f(x), \ \text{x} \in (0,1) \quad (5.3)$$

$$y(0) = 1 \ , \ y'(0) = 1 \ , y'(1) = 1, \quad (5.4)$$

where $f(x)$ is chosen such that

$$y(x) = \frac{x + \frac{\varepsilon}{2}\exp(-2x/\varepsilon)}{1 - \exp(-2/\varepsilon)} + x - x^2/2 + 1 - \frac{\varepsilon}{2(1 - \exp(-2/\varepsilon))}. \quad (5.5)$$

For our tests we take $\varepsilon = 10^{-8}$ which is a sufficient small choice to bring out the singularly perturbed nature of the problem. We measure the accuracy in the discrete maximum norm

$$e^N = \|\mathbf{y} - \mathbf{y}^N\|_\infty,$$

Table 1. Monotone difference scheme for Example 1

N	*error*	*rate*	*constant*
32	4.6480e-2	0.995	1.487
64	2.3328e-2	0.999	1.493
128	1.1690e-2	0.998	1.496
256	5.8520e-3	0.999	1.498
512	2.9278e-3	1.000	1.499
1024	1.4644e-3	-	1.500

Table 2. Monotone difference scheme for Example 2

N	*error*	*rate*	*constant*
32	6.5625e-2	0.959	2.100
64	3.3750e-2	0.978	2.160
128	1.7129e-2	0.989	2.193
256	8.6317e-3	0.996	2.210
512	4.3275e-3	0.995	2.216
1024	2.1717e-3	-	2.224

the convergence rate

$$r^{N,K} = \log_2\left(\frac{e^N}{e^{2N}}\right)$$

and the constants in the error estimate

$$C^N = e^N/N^{-1}.$$

From Tables 1 and 2 we see that e^N/e^{2N} is close to 2, which supports the convergence estimate of Theorem 1. They indicate that the theoretical results are fairly sharp.

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