Neural, Parallel, and Scientific Computations 17 (2009) 135-146

SOLVING TWO-POINT BOUNDARY VALUE PROBLEMS BY A FAMILY OF LINEAR MULTISTEP METHODS

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ABSTRACT. Linear Multistep Methods (LMMs) are developed and applied to solve two-point boundary value problems (BVPs). The derivation of the main methods lead to continuous approximations from which multiple finite difference methods (MFDMs) are obtained. The MFDMs are assembled into single block matrix equations which are used to solve BVPs. We obtain three specific methods with step numbers $k = 2, 3, 4$, which are used to illustrate the process. It is also shown that the methods have orders greater than one, zero-stable, and hence convergent. Numerical experiments are performed to show the efficiency of the methods.

AMS Subject Classification. 65L05, 65L06, 65L12

Key Words: Boundary Value Problems, Linear Multistep Methods, Multiple Finite Difference Method, Zero-stability

1. INTRODUCTION

In this paper, we consider the second-order ordinary differential equation of the form

$$y'' = f(x, y, y'),\ y(a) = y_0,\ y(b) = y_N \tag{1}$$

where a, b, y_0, y_N are real constants and $N > 0$ is the number of steps. We note that f is a continuous function and satisfies a Lipschitz condition. Keller [10] has given the theorem and the proof of the general conditions which ensure that the solution to (1) will exist and be unique. LMMs of the Adams-Moulton type for (1) are due to Onumanyi et al [14], and Brugnano and Trigiante [2]. However, these methods are applicable to (1) by first reducing it to a system of first-order differential equations, which involves more human effort and more computer time. We note that LMMS based on the Numerov's type method have been considered by Yusuph and Onumanyi [15], Lambert [12], and Henrici [6] for solving directly $y'' = f(x, y)$, $y(a) = y_0$, $y(b) = y_N$. Recently, Jator [7] and Jator and Li [8] proposed LMMs for the direct solution of the general second order IVPs, which were shown to be zero stable

Received June 1, 2009

and implemented without the need for either predictors or starting values from other methods. Therefore, an attempt has been made to use these LMMs and additional methods to solve BVPs directly.

In this paper, we discuss LMMs for $k = 2, 3, 4$ which are assembled and applied as single block matrix difference equations to provide the direct solution to (1) over on-overlapping intervals. It is worth noting that the simultaneous application of these MFDMs are more accurate than the standard finite difference methods (SFDMs) which are generally applied as single formulas over overlapping intervals as in Lambert [11] and Jennings [9]. In addition, higher order SFDMs are more tedious to derive and implement. Thus, the methods presented in this paper are more robust than the SFDMs. We also show that the MFDMs are zero-stable, consistent, and hence convergent. We emphasize that the main method is derived through interpolation and collocation, see Lie and Norsett [13], Atkinson [1], Onumanyi et al [14]. The approach facilitates the link between the finite difference methods and the k-step multistep collocation procedure, which are two important global methods which have been used with piecewise continuous approximate solution of ordinary differential equations (ODEs) Gladwell and Sayers [5].

The paper is organized as follows. In section two, we derive a continuous approximation $Y(x)$ for the exact solution $y(x)$. Section three is devoted to the specification of the methods and how the MFDMs are obtained. The analysis and implementation of the methods are discussed in section four. Numerical examples are given in section five to show the efficiency of the MFDMs. Finally, the conclusion of the paper is discussed in section six.

2. THE DERIVATION OF THE METHOD

In this section, we approximate the exact solution $y(x)$ by seeking the continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \lambda_j(x)\Upsilon_j(x) \tag{2}$$

where $x \in [a, b]$, $\lambda_j(x)$'s are unknown coefficients and $\Upsilon_j(x)$'s are polynomial basis functions of degree j. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \leq r \leq k$, and $0 < s \leq k+1$ respectively. The positive integer $k \geq 2$ denotes the step number of the method. We then construct a k-step multistep collocation method from (2) by imposing the following conditions.

$$Y(x_{n+j}) = y_{n+j}, \quad j = 0, 1, 2, \ldots, r-1 \tag{3}$$

$$Y''(x_{n+j}) = f_{n+j}, \quad j = 0, 1, 2, \ldots, s-1 \tag{4}$$

Equations (3) and (4) lead to a system of (r+s) equations, which is solved to obtain the continuous coefficients $\lambda_j(x)$'s. The k-step LMM is then, constructed by substituting the values of $\lambda_j(x)$'s into equation (2) and after some manipulation, our method is expressed in the form

$$Y(x) = \sum_{j=0}^{r-1} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} \tag{5}$$

which is used to generate MFDMs, which are applied as simultaneous numerical integrators to provide the discrete solution to (1). In this light, we seek a solution on the mesh

$$\pi_N : a = x_0 < x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots < x_N = b$$
$$h = x_{n+1} - x_n, \quad n = 0, 1, \ldots, N$$

where π_N is a partition of $[a, b]$ and h is the constant step-size of the partition of π_N.

3. SPECIFICATION OF THE METHODS

In this section, we use (5) and the formula for the derivative which is expressed as

$$Y'(x) = \frac{1}{h}\left(\sum_{j=0}^{r-1} \alpha_j'(x) y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j'(x) f_{n+j}\right) \tag{6}$$

which provides additional equations and derivatives obtained by imposing that

$$Y'(x) = \gamma(x) \tag{7}$$

to generate MFDMs for $k = 2, 3, 4$. In particular, we use (5) to obtain k-step LMMs with the following specifications: $r = 2$, $s = 3, 4, 5$; $k = 2, 3, 4$; $\Upsilon_j(x) = x^j$, $j = 0, 1, \ldots, s + r - 1$. We emphasize that we evaluate (5) at $x = x_{n+k}$ to generate the main methods. We also express $\alpha_j(x)$ and $\beta_j(x)$ as functions of t for convenience, where $t = (x - x_{n+k-1})/h$. The coefficients $\alpha_j'(x)$ and $\beta_j'(x)$ are easily obtained by differentiating $\alpha_j(x)$ and $\beta_j(x)$, which are displayed in Table 1. We discuss details of specific methods next.

Case $k = 2$

The following LMM (main method), which corresponds to the Numerov's method is obtained by evaluating (5) at $x = x_{n+2}$

$$y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{12}(f_n + 10f_{n+1} + f_{n+2}) \tag{8}$$

We assume that $\gamma(x)$ is continuous on the interval $[a, b]$. Thus, an additional method is obtained with a continuity equation imposed at $x = x_{n+2}$ as in Yusuph and Onumanyi

[15]. That is,

$$\lim_{x\to x_{n+2}^-}\gamma(x)=\lim_{x\to x_{n+2}^+}\gamma(x)$$

where

$$\gamma(x)=\begin{cases}\frac{1}{h}(-y_n+y_{n+1}+\frac{h^2}{24}(f_n+26f_{n+1}+9f_{n+2})) & \text{if } x_n\le x\le x_{n+2}\\ \frac{1}{h}(-y_{n+2}+y_{n+3}+\frac{h^2}{24}(-7f_{n+2}-6f_{n+3}+f_{n+4})) & \text{if } x_{n+2}\le x\le x_{n+4}\end{cases}$$

which gives

$$-y_n+\ y_{n+1}+\ y_{n+2}-y_{n+3}=\frac{h^2}{24}(-f_n-26f_{n+1}-16f_{n+2}-6f_{n+3}+f_{n+4}) \tag{9}$$

The derivatives are provided by $\gamma(x_{n+\tau})=\gamma_{n+\tau},\tau=0,\ldots,2$ as follows:

$$h\gamma_n=y_{n+1}-y_n+\frac{h^2}{24}(-7f_n-6f_{n+1}+f_{n+2})$$

$$h\gamma_{n+1}=y_{n+1}-y_n+\frac{h^2}{24}(3f_n+10f_{n+1}-f_{n+2})$$

$$h\gamma_{n+2}=y_{n+1}-y_n+\frac{h^2}{24}(f_n+26f_{n+1}+9f_{n+2})$$

Case $k=3$

The following LMMs (main method and an additional method) are obtained by evaluating (5) at $x=\{x_{n+3},x_{n+2}\}$

$$y_{n+3}=-2y_n+3y_{n+1}+\frac{h^2}{12}(2f_n+21f_{n+1}+12f_{n+2}+f_{n+3}) \tag{10}$$

$$y_{n+2}=-y_n+2y_{n+1}+\frac{h^2}{12}(f_n+10f_{n+1}+2f_{n+2}) \tag{11}$$

The following equation is obtained with a continuity equation imposed at $x=x_{n+3}$ as in Yusuph and Onumanyi [15]. That is,

$$\lim_{x\to x_{n+3}^-}\gamma(x)=\lim_{x\to x_{n+3}^+}\gamma(x)$$

where

$$\gamma(x)=\begin{cases}\frac{1}{h}(-y_n+y_{n+1}+\frac{h^2}{360}(38f_n+291f_{n+1}+444f_{n+2}+127f_{n+3}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_n\le x\le x_{n+3}\\ \frac{1}{h}(-y_{n+3}+y_{n+4}+\frac{h^2}{360}(-97f_{n+3}-114f_{n+4}+39f_{n+5}-8f_{n+6}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } x_{n+3}\le x\le x_{n+6}\end{cases}$$

which gives

$$\begin{aligned}-y_n+y_{n+1}+y_{n+3}-y_{n+4}=\frac{h^2}{360}(&-38f_n-291f_{n+1}-444f_{n+2}\\ &-224f_{n+3}-114f_{n+4}+39f_{n+5}-8f_{n+6})\end{aligned} \tag{12}$$

It is worth noting that the derivatives are provided by $\gamma(x_{n+\tau})=\gamma_{n+\tau},\tau=0,\ldots,3$ as follows:

$$h\gamma_n = -y_n + y_1 + \tfrac{h^2}{360}(-97f_n - 114f_{n+1} + 39f_{n+2} - 8f_{n+3})$$
$$h\gamma_{n+1} = -y_n + y_{n+1} + \tfrac{h^2}{360}(38f_n + 171f_{n+1} - 36f_{n+2} + 7f_{n+3})$$
$$h\gamma_{n+2} = -y_n + y_{n+1} + \tfrac{h^2}{360}(23f_n + 366f_{n+1} + 159f_{n+2} - 8f_{n+3})$$
$$h\gamma_{n+3} = -y_n + y_{n+1} + \tfrac{h^2}{360} + (38f_n + 291f_{n+1} + 444f_{n+2} + 127f_{n+3})$$

Case $k = 4$

The following LMMs (the main method and 2 additional methods) are obtained by evaluating (5) at $x = \{x_{n+4}, x_{n+3}, x_{n+2}\}$.

(13) $$y_{n+4} = -3y_n + 4y_{n+1} + \frac{h^2}{120}(27f_n + 332f_{n+1} + 222f_{n+2} + 132f_{n+3} + 7f_{n+4})$$

(14) $$y_{n+3} = -2y_n + 3y_{n+1} + \frac{h^2}{240}(37f_n + 432f_{n+1} + 222f_{n+2} + 32f_{n+3} - 3f_{n+4})$$

(15) $$y_{n+2} = -y_n + 2y_{n+1} + \frac{h^2}{240}(19f_n + 204f_{n+1} + 14f_{n+2} + 4f_{n+3} - f_{n+4})$$

The following equation is obtained with a continuity equation imposed at $x = x_{n+4}$. That is,

$$\lim_{x \to x_{n+4}^-} \gamma(x) = \lim_{x \to x_{n+4}^+} \gamma(x)$$

where

(16)
$$\gamma(x) = \begin{cases} \frac{1}{h}(-y_n + y_{n+1} + \frac{h^2}{1440}(81f_n + 1508f_{n+1} + 1050f_{n+2} + 1932f_{n+3} + 469f_{n+4})), \\ x_n \le x \le x_{n+4} \\ \frac{1}{h}(-y_{n+4} + y_{n+5} + \frac{h^2}{1440}(-367f_{n+4} - 540f_{n+5} + 282f_{n+6} - 116f_{n+7} + 21f_{n+8})), \\ x_{n+4} \le x \le x_{n+8} \end{cases}$$

which gives

(17) $$y_n - y_{n+1} - y_{n+4} + y_{n+5} = \frac{h^2}{1440}(81f_n + 1508f_{n+1} + 1050f_{n+2} + 1932f_{n+3} + 836f_{n+4} + 540f_{n+5} - 282f_{n+6} + 116f_{n+7} - 21f_{n+8})$$

It is worth noting that the derivatives are provided by $\gamma(x_{n+\tau}) = \gamma_{n+\tau}, \tau = 0, \dots, 4$ as follows:

$$h\gamma_n = -y_n + y_{n+1} - \frac{h^2}{1440}(367f_n + 540f_{n+1} - 282f_{n+2} + 116f_{n+3} - 21f_{n+4})$$
$$h\gamma_{n+1} = -y_n + y_{n+1} + \frac{h^2}{1440}(135f_n + 752f_{n+1} - 246f_{n+2} + 96f_{n+3} - 17f_{n+4})$$
$$h\gamma_{n+2} = -y_n + y_{n+1} + \frac{h^2}{1440}(97f_n + 1444f_{n+1} + 666f_{n+2} - 52f_{n+3} + 5f_{n+4})$$
$$h\gamma_{n+3} = -y_n + y_{n+1} + \frac{h^2}{1440}(119f_n + 1296f_{n+1} + 1578f_{n+2} + 640f_{n+3} - 33f_{n+4})$$
$$h\gamma_{n+4} = -y_n + y_{n+1} + \frac{h^2}{1440}(81f_n + 1508f_{n+1} + 1050f_{n+2} + 1932f_{n+3} + 469f_{n+4})$$

k	j	$\alpha_j(t)$	$\beta_j(t)$	$\alpha_j(1)$	$\beta_j(1)$
	0	$-t$	$\frac{1}{24}(t^4 - 2t^3 + 3t)$	-1	$\frac{1}{12}$
2	1	$(1+t)$	$\frac{1}{24}(-2t^4 + 12t^2 + 10t)$	2	$\frac{10}{12}$
	2	1	$\frac{1}{24}(t^4 + 2t^3 - t)$	1	$\frac{1}{12}$
	0	$-(1+t)$	$\frac{1}{360}(-3t^5 + 10t^3 + 23t + 30)$	-2	$\frac{2}{12}$
3	1	$(2+t)$	$\frac{1}{24}(-2t^4 + 12t^2 + 10t)$	3	$\frac{21}{12}$
	2	0	$\frac{1}{120}(3t^5 + 5t^4 - 20t^3 + 122t + 100)$	0	$\frac{12}{12}$
	3	1	$\frac{1}{360}(3t^5 + 15t^4 + 20t^3 - 8t)$	1	$\frac{1}{12}$
	0	$-(t+2)$	$\frac{1}{1440}(2t^6 + 6t^5 - 5t^4 - 20t^3 + 119t + 222)$	-3	$\frac{27}{120}$
	1	$(t+3)$	$\frac{1}{360}(-2t^6 - 9t^5 + 5t^4 + 30t^3 + 324t + 648)$	4	$\frac{332}{120}$
4	2	0	$\frac{1}{240}(2t^6 + 12t^5 + 5t^4 - 60t^3 + 263t + 222)$	0	$\frac{222}{120}$
	3	0	$\frac{1}{360}(-2t^6 - 15t^5 - 25t^4 + 50t^3 + 180t^2 + 160t + 48)$	0	$\frac{132}{120}$
	4	1	$\frac{1}{1440}(2t^6 + 18t^5 + 55t^4 + 60t^3 - 33t - 18)$	1	$\frac{7}{120}$

TABLE 1. Continuous coefficients $\alpha_j(t)$ and $\beta_j(t)$ as well as discrete coefficients $\alpha_j(1)$ and $\beta_j(1)$ of the main LMMs, for $k = 2, 3, 4$

4. ANALYSIS AND IMPLEMENTATION OF THE METHODS

The methods obtained in section three are specified members of the conventional LMM which can be represented as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} \tag{18}$$

where $\alpha_k \neq 0$, $\alpha_j' s$, $\beta_j' s$ are constants and α_0 and β_0 do not both vanish. We can also write (18) compactly in the form

$$\rho(E) y_n = h^2 \sigma(E) f_n \tag{19}$$

where $\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta^j$ and $\sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta^j$ are the characteristic polynomials, $\zeta \in \mathbb{C}$, and $E^j y_n = y_{n+j}$ is a shift operator.

Following Fatunla [4] and Lambert [11] we define the local truncation error associated with (18) to be the linear difference operator

$$L[y(x); h] = \sum_{j=0}^{k} \{\alpha_j y(x + jh) - h^2 \beta_j y''(x + jh)\} \tag{20}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (20) as a Taylor series about the point x to obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y' + \ldots, + C_q h^q y^q(x) + \ldots, \tag{21}$$

where the constant coefficients C_q , $q = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$\vdots C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} j^{q-2} \beta_j]$$

According to Henrici [6], we say that the method (18) has order p if

$$C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \qquad C_{p+2} \neq 0$$

therefore, C_{p+2} is the error constant and $C_{p+2} h^{p+2} y^{(p+2)}(x_n)$ is the principal local truncation error at the point x_n.

Theorem 4.1. *A necessary condition for the convergence of the LMM* (18) *is that the modulus of no root of the polynomial* $\rho(\zeta)$ *exceeds 1, and that the multiplicity of the roots of modulus* 1 *be at most* 2.

Proof. See Henrici [6, Page 301]. □

The methods specified in section three, which are of the form (18) can be analyzed by conveniently representing them by a matrix finite difference equation in block form. Thus, let the ν-vector (ν is the number of points within the block) Y_μ and F_μ, for $n = m\nu$, $m = 0, 1, \ldots$ be given as $Y_\mu = (y_{n+1}, \ldots, y_{n+\nu})^T$, $F_\mu = (f_{n+1}, \ldots, f_{n+\nu})^T$, then the k-block ν-point methods for (1) are given by

$$Y_\mu = \sum_{i=1}^{r-1} A^{(i)} Y_{\mu-i} + h^2 \sum_{j=0}^{s-1} B^{(j)} F_{\mu-i} \tag{22}$$

where $A^{(i)}$, $B^{(j)}$, $i = 0, \ldots, r-1$ and $j = 0, \ldots, s-1$ are ν by ν matrices (see Fatunla [4]).

Definition 4.2. In the sense of Fatunla [4], the block method (22) is zero stable provided the roots R_j, $j = 1, \ldots, k$ of the first characteristic polynomial $\rho(R)$ specified by

$$\rho(R) = \det[\sum_{j=0}^{r-1} A^{(j)} R^{r-(1+j)}] = 0, A^{(0)} = -I \tag{23}$$

satisfies $|R_j| \leq 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2.

Definition 4.3. The method (22) is said to be consistent if it has order at least one.

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (22) tends to the difference system

$$Y_\mu - \sum_{i=1}^{r-1} A^{(i)} Y_{\mu-i} = 0$$

whose first characteristic polynomial $\rho(R)$ is given by (23).

It is easily shown that for $k = 2$, (8) and (9) can be expressed in the form (22) with order $p = 4$ and error constant given by the vector $C_6 = (-\frac{1}{240}, -\frac{1}{30})^T$, where T denotes the transpose of the vector. It is also shown that for $\nu = k$, $r = 2$, $k = 2$, we obtained $\rho(R) = \det(RA^{(0)} - A^{(1)}) = (R-1)^2$ from (23) where $A^{(0)}$ is an identity matrix of dimension 2 and A^1 is a matrix of dimension 2 given by

$$A^1 = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$$

We note that for $k = 3$, (10), (11), and (12) can be expressed in the form (23) with order $p = 4$ and error constants given by the vector $C_6 = (\frac{-1}{80}, \frac{-1}{240}, \frac{3}{80})^T$. It is also shown that for $\nu = k$, $r = 2$, $k = 3$, we obtained the first characteristic polynomial

$\rho(R) = \det(RA^0 - A^{(1)}) = R(R-1)^2$ from (23) where $A^{(0)}$ is an identity matrix of dimension 3 and $A^{(1)}$ is a matrix of dimension 3 given by

$$A^1 = \begin{pmatrix} 0 & -1 & 2 \\ 0 & -2 & 3 \\ 0 & -3 & 4 \end{pmatrix}$$

For $k = 4$, (13) to (15), and (17) can be expressed in the form (23) with order $p = 5$ and error constants given by the vector $C_7 = (\frac{1}{120}, \frac{1}{120}, \frac{1}{240}, 0)^T$. It is also shown that for $\nu = k$, $r = 2$, $k = 4$, we obtained the first characteristic polynomial $\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^2(R-1)^2$ from (23), where $A^{(0)}$ is an identity matrix of dimension 4 and $A^{(1)}$ is a matrix of dimension 4 given by

$$A^1 = \begin{pmatrix} 0 & 0 & -1 & 2 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -4 & 5 \end{pmatrix}$$

According to definition 4.2, the methods given by (22) for $k = 2, 3, 4$ are zero-stable, since from (23), $\rho(R) = 0$ satisfy $|R_j| \leq 1$, $j = 1, \ldots, k$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2. Using definition 4.3, the methods are consistent as they have order $p > 1$. According to theorem 4.1, we can safely assert the convergence of the methods, since they are all zero-stable and consistent.

Our methods are implemented efficiently by combining the MFDMs in to a single matrix of finite difference equations which simultaneously provides the values generated by the sequence $\{y_n\}, n = 1, \ldots, N-1$ over sub-intervals $[x_0, x_k], \ldots, [x_{N-k}, x_N]$ which do not overlap (see [15]). In particular, for linear problems, we can solve (1) directly from the start with Gaussian elimination using partial pivoting, and for nonlinear problems, we can use a modified Newton-Raphson method.

5. NUMERICAL EXAMPLES

In this section, we have tested the performance our methods on 3 BVPs. For each example, we find absolute errors of the approximate solution in π_N, where N is chosen to be divisible by k. All computations were carried out using our written Mathematica code in Mathematica 5.2.

Example 5.1. Consider the non-homogeneous linear BVP which has also been solved by Burden and Faires [3].

$$y'' = -\frac{2}{x}y' + \frac{2}{x^2}y + \frac{\sin(\ln x)}{x^2}, \; 1 \leq x \leq 2, \; y(1) = 1, \; y(2) = 2$$

$$\text{Exact} : y(x) = c_1 x + \frac{c_2}{x^2} - \frac{3}{10}\sin(\ln x) - \frac{1}{10}\cos(\ln x)$$

$$c_2 = \frac{1}{70}[8 - 12\sin(\ln 2) - 4\cos(\ln 2)],\ c_1 = \frac{11}{10} - c_2$$

It is obvious that our method performs better than SFDM given in Burden and Faires [3]. Hence, for this example, our method is clearly superior. The details of the numerical results are given in Table 2.

x	Exact	SFDM	Our Method	SFDM-Error	Our Method-Error
1.0	1.0000000000	1.0000000000	1.0000000000	0.00×10^{-5}	0.00×10^{-6}
1.1	1.0926292985	1.0926005200	1.0926274983	2.88×10^{-5}	1.80×10^{-6}
1.2	1.1870848405	1.1870431300	1.1870814293	4.17×10^{-5}	3.41×10^{-6}
1.3	1.2833823641	1.2833368700	1.2833792604	4.55×10^{-5}	3.10×10^{-6}
1.4	1.3814459517	1.3814020500	1.3814430504	4.39×10^{-5}	2.90×10^{-6}
1.5	1.4811594170	1.4811202600	1.4811571004	3.92×10^{-5}	2.32×10^{-6}
1.6	1.5823924608	1.5823599000	1.5823906259	3.26×10^{-5}	1.83×10^{-6}
1.7	1.6850139617	1.6849890200	1.6850126615	2.49×10^{-5}	1.30×10^{-6}
1.8	1.7888985346	1.7888817500	1.7888976965	1.68×10^{-5}	8.38×10^{-7}
1.9	1.8939295092	1.8939211000	1.8939291145	8.41×10^{-6}	3.95×10^{-7}
2.0	2.0000000000	2.0000000000	2.0000000000	0.00×10^{-6}	0.00×10^{-7}

TABLE 2. Exact solution $y(x)$, approximate solution y, and absolute errors, $|y(x) - y|$, for Example 5.1 , for the method $k = 2$, $h = 01$.

Example 5.2. Consider BVP

$$y'' = y' - y + e^x - 3\sin x,\ y(1) = 1.09737491,\ y(2.2) = 10.79051685$$

$$\text{Exact} : y(x) = e^x - 3\cos x$$

Although the numerical results of this problem were not compared with another method, the results were compared with the theoretical solution as shown in Table 3.

Example 5.3. Consider the non-homogeneous linear BVP which has also been solved by Zill and Cullen [16] using the SFDM [16].

$$y'' + 3y' + 2y = 4x^2,\ y(1) = 1,\ y(2) = 6$$

$$\text{Exact} : y(x) = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2$$

$$c_1 = \frac{e^1(2 + 3e^2)}{(e^1 - 1)},\ c_2 = \frac{e^3(2 + 3e^1)}{(1 - e^1)}$$

x	$y(x)$	Our Method($k=2$) y	$\|y(x)-y\|$	Our Method($k=3$) y	$\|y(x)-y\|$	Our Method($k=4$) y	$\|y(x)-y\|$
1.0	1.0973749109	1.0973749109	0.00×10^{-7}	1.0973749109	0.00×10^{-7}	1.0973749109	0.00×10^{-9}
1.1	1.6433776597	1.6433781177	4.58×10^{-7}	1.6433773887	2.71×10^{-7}	1.6433776608	1.16×10^{-9}
1.2	2.2330436593	2.2330445641	9.05×10^{-7}	2.2330431148	5.45×10^{-7}	2.2330436625	3.15×10^{-9}
1.3	2.8668001817	2.8668014059	1.22×10^{-6}	2.8667993570	8.25×10^{-7}	2.8668001871	5.32×10^{-9}
1.4	3.5452985381	3.5453000358	1.50×10^{-6}	3.5452975503	9.88×10^{-7}	3.5452985461	7.91×10^{-9}
1.5	4.2694774653	4.2694790796	1.61×10^{-6}	4.2694763307	1.13×10^{-6}	4.2694774746	9.28×10^{-9}
1.6	5.0406309913	5.0406326398	1.65×10^{-6}	5.0406297238	1.27×10^{-6}	5.0406310011	9.84×10^{-9}
1.7	5.8604808746	5.8604823631	1.49×10^{-6}	5.8604796242	1.25×10^{-6}	5.8604808850	1.04×10^{-8}
1.8	6.7312537485	6.7312549557	1.21×10^{-6}	6.7312525559	1.19×10^{-6}	6.7312537602	1.17×10^{-8}
1.9	7.6557631429	7.6557638276	6.85×10^{-6}	7.6557620468	1.10×10^{-6}	7.6557631532	1.04×10^{-8}
2.0	8.6374966086	8.6374966086	0.00×10^{-7}	8.6374958081	8.00×10^{-7}	8.6374966152	6.67×10^{-9}

TABLE 3. Exact solution $y(x)$, approximate solution y, and absolute errors, $|y(x)-y|$, for Example 5.2 where $y(x) = e^x - 3\cos x$, $k = 2, 3, 4$ and $h = 01$.

It is obvious that our method performs better than SFDM given in Zill and Cullen [16]. Hence, for this example, our method is clearly superior. The details of the numerical results are given in Table 4.

x	Exact	SFDM	Our Method	SFDM-Error	Our Method-Error
1.0	1.0000000000	1.0000000000	1.0000000000	0.00×10^{-2}	0.00×10^{-5}
1.1	2.3936054033	2.4047000000	2.3935293502	1.11×10^{-2}	7.61×10^{-5}
1.2	3.4267162861	3.4432000000	3.4265955138	1.65×10^{-2}	1.21×10^{-4}
1.3	4.1828973100	4.2010000000	4.1827664601	1.81×10^{-2}	1.31×10^{-4}
1.4	4.7295217547	4.7469000000	4.7293925532	1.74×10^{-2}	1.29×10^{-4}
1.5	5.1208058367	5.1359000000	5.1206940913	1.51×10^{-2}	1.12×10^{-4}
1.6	5.4002835479	5.4124000000	5.4001912137	1.21×10^{-2}	9.23×10^{-5}
1.7	5.6028243302	5.6117000000	5.6027573176	8.88×10^{-3}	6.70×10^{-5}
1.8	5.7562772694	5.7620000000	5.7562331861	5.72×10^{-3}	4.41×10^{-5}
1.9	5.8828102454	5.8855000000	5.8827901425	2.69×10^{-3}	2.01×10^{-5}
2.0	6.0000000000	6.0000000000	6.0000000000	0.00×10^{-3}	0.00×10^{-6}

TABLE 4. Exact solution $y(x)$, approximate solution y, and absolute errors, $|y(x)-y|$, for Example 5.3 , for the method $k = 2$, $h = 0.1$

6. CONCLUSIONS

We have proposed a family of k-step LMMs with continuous coefficients from which MFDMs are obtained and applied as single block matrix equations to solve $y'' = f(x, y, y')$, subject to specified boundary conditions without first reducing it to an equivalent first order system of ODEs. We have discussed three specific LMMs with step numbers $k = 2, 3, 4$, which have orders greater than one, zero-stable, and hence convergent. The efficiency of the method is established numerically. Our future research will be focused on studying the stability properties of the methods.

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