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A New Coupled Approach High Accuracy Numerical Method for the Solution of 2D Non-Linear Biharmonic Equations

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Abstract: In this article, we derive a new fourth order finite difference approximation for the solution of two dimensional non-linear biharmonic partial differential equations on a 9-point compact stencil using coupled approach. The solutions of unknown variable and its Laplacian are obtained at each internal grid points. This discretization allows us to use the Dirichlet boundary conditions only and there is no need to discretize the derivative boundary conditions. We require only system of two equations to obtain the solution and its Laplacian. The main advantage of this work is that the proposed method is directly applicable to solve singular problems without any modifications. We compare the advantages and implementation of the proposed method with the standard central difference approximations in the context of basic iterative methods. Numerical examples are given to verify the fourth-order convergence rate of the method.

Keywords - finite differences; arithmetic average discretization; two dimensional non-linear biharmonic equations; Laplacian; high accuracy; compact approximation; maximum absolute errors.

1. INTRODUCTION

We are concerned with the numerical solution of two dimensional non-linear biharmonic partial differential equation of the form

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$$\varepsilon\nabla^4 u(x,y) \equiv \varepsilon\left(\frac{\partial^4 u}{\partial x^4}+2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}\right)=f(x,y,u,u_x,u_y,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y),0<x,y<1 \quad (1)$$

where $0<\varepsilon\le 1$, $(x,y)\in\Omega=\{(x,y)\,|\,0<x,y<1\}$ with boundary $\partial\Omega$ and $\nabla^2 u(x,y)\equiv \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$ represents the two dimensional Laplacian of the function $u(x,y)$. We assume that the solution $u(x,y)$ is smooth enough to maintain the order and accuracy of the scheme as high as possible under consideration.

Dirichlet boundary conditions of second kind are given by

$$u(x,0)=a_{10}(x),\quad u_{yy}(x,0)=a_{20}(x),\ 0\le x\le 1 \quad (2a)$$

$$u(x,1)=a_{11}(x),\quad u_{yy}(x,1)=a_{21}(x),\ 0\le x\le 1 \quad (2b)$$

$$u(0,y)=b_{01}(y),\quad u_{xx}(0,y)=b_{02}(y),\ 0\le y\le 1 \quad (2c)$$

$$u(1,y)=b_{11}(y),\quad u_{xx}(1,y)=b_{12}(y),\ 0\le y\le 1 \quad (2d)$$

The biharmonic equation is a fourth order elliptic partial differential equation which plays a very important role in areas of continuum mechanics, including linear elasticity theory and the solution of stokes flow. Different techniques for the numerical solution of the 2D biharmonic equations have been considered in the literature. (Smith, 1970; Ehrlich, 1971 & 1974) have solved 2D biharmonic equations using coupled second order accurate finite difference equations. (Bauer and Riess, 1972) have used block iterative method to solve the equation. Later, (Kwon *et al*, 1982; Stephenson, 1984; Mohanty and Pandey, 1996; Evans and Mohanty, 1998) have developed certain second- and fourth-order finite difference approximations for the second biharmonic problems using 9-point compact cell. Fourth order compact finite difference schemes have become quite popular as against the other lower order accurate schemes which require high mesh refinement and hence are computationally inefficient. On the other hand, the higher order accuracy of the fourth order compact methods combined with the compactness of the difference stencil yields highly accurate numerical solutions on relatively coarser grids with greater computational efficiency. A conventional approach for solving the 2D biharmonic equation is to discretize the differential equation (1) on a uniform grid using 25-point approximations with truncation error of order h^2. This approximation connect the values of central point in terms of 24 neighbouring values of u in 5×5 grid. We note that the central value of u is connected to grid points two grids away in each direction from the central point and the difference approximations needs to be modified at grid points near the boundaries. There are serious computational difficulties with solution of the linear and non-linear systems obtained through 25-point discretization of the 2D biharmonic equation. Approximations using compact cells avoid these difficulties. The compact approach involves discretizing the biharmonic equations using not just the grid values of the unknown solution u but also the values of the derivatives u_{xx} and u_{yy} at selected grid points (Mohanty and Pandey, 1996). Recently, (Mohanty, 2010) has proposed a new algorithm in coupled manner for the solution of two dimensional non-linear biharmonic problems of second kind, in which, a special technique is required to solve singular problems. Further, recently (Khattar *et al*, 2010) have also developed a new method based on arithmetic average discretization to solve three dimensional non-linear biharmonic problems of second kind using coupled approach, in which, no special technique is required to solve singular problems.

In this article, we split the differential equation into two coupled elliptic differential equations and introduce new ideas to handle boundary conditions without discretizing them in the coupled system of elliptic equations. We use only 9-point compact cell (see fig.1) for fourth order approximation of differential equation (1). The given Dirichlet boundary conditions are exactly satisfied and no approximations for derivatives need to be carried out at the boundaries. In next section, we discuss the finite difference approximation based on arithmetic average discretization for the differential equations (1). In section 3, we give the complete derivation of the method. In section 4, we discuss block iterative methods. In section 5, we have given the stability analysis, and illustrated the method and its fourth order convergence by solving four problems. Concluding remarks are given in section 6.

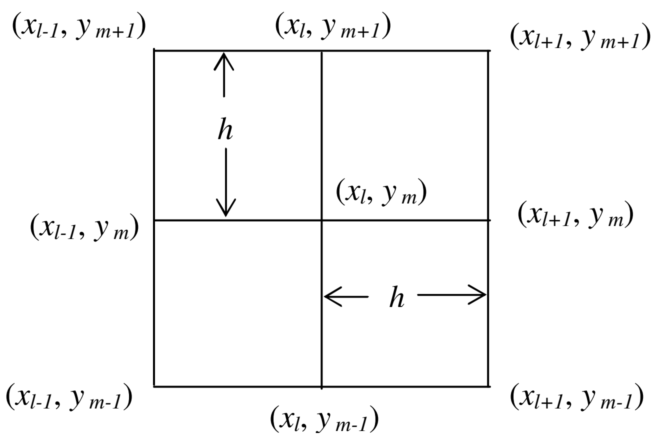

Fig.1: 9-point 2D single computational cell

2. DESCRIPTION OF THE DISCRETIZATION

Consider a two-dimensional uniform grid centered at the point (x_l, y_m), where $h > 0$ is the constant mesh length in x-, and y- directions and $x_l = lh$, $y_m = mh$; $l, m = 0,1,2,\dots,N$ with $(N+1)h=1$. Let $U_{l,m}$ and $u_{l,m}$ be the exact and approximate solution values of $u(x,y)$ at the grid point (x_l, y_m), respectively.

Note that, the Dirichlet boundary conditions are given by (2a)-(2d). Since the grid lines are parallel to coordinate axes and the values of u are exactly known on the boundary, this implies, the successive tangential partial derivatives of u are known exactly on the boundary. For example, on the line y=0, the values of $u(x, 0)$ and $u_{yy}(x,0)$ are known, i.e., the values of $u_x(x,0)$, $u_{xx}(x,0)$,… etc are known on the line y=0. This implies the values of $u(x, 0)$ and $\nabla^2 u(x,0) \equiv u_{xx}(x,0) + u_{yy}(x,0)$ are known on the line y=0. Similarly the values of u and $\nabla^2 u$ are known on all sides of the square region Ω.

Let us denote $\nabla^2 u = v$. Then we can re-write the boundary value problem (1) in a coupled manner as

$$\nabla^2 u(x,y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = v(x,y), \quad (x,y) \in \partial\Omega, \tag{3a}$$

$$\varepsilon \nabla^2 v(x,y) \equiv \varepsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = f(x,y,u,v,u_x,v_x,u_y,v_y), \quad (x,y) \in \partial\Omega. \tag{3b}$$

and the Dirichlet boundary conditions (2a)-(2d) may be replaced by

$$u(x,0) = a_{10}(x), \quad v(x,0) = \nabla^2 u(x,0) = a_{30}(x), 0 \le x \le 1 \tag{4a}$$

$$u(x,1) = a_{11}(x), \quad v(x,1) = \nabla^2 u(x,1) = a_{31}(x), \quad 0 \le x \le 1 \tag{4b}$$

$$u(0,y) = b_{01}(y), \quad v(0,y) = \nabla^2 u(0,y) = b_{03}(y), 0 \le y \le 1 \tag{4c}$$

$$u(1,y) = b_{11}(y), \quad v(1,y) = \nabla^2 u(1,y) = b_{13}(y), 0 \le y \le 1 \tag{4d}$$

Let at the grid points (x_l, y_m), the exact and approximate solution values of $v(x,y)$ be denoted as $V_{l,m}$ and $v_{l,m}$, respectively.

For fourth order approximation of the non-linear differential equation (1) on the 9-point compact cell, we need the following approximations:

$$\overline{U}_{l+\frac{1}{2},m} = \frac{1}{2}\left(U_{l+1,m} + U_{l,m}\right) \tag{5a}$$

$$\overline{V}_{l+\frac{1}{2},m} = \frac{1}{2}\left(V_{l+1,m} + V_{l,m}\right) \tag{5b}$$

$$\overline{U}_{l-\frac{1}{2},m} = \frac{1}{2}\left(U_{l-1,m} + U_{l,m}\right) \tag{6a}$$

$$\overline{V}_{l-\frac{1}{2},m} = \frac{1}{2}\left(V_{l-1,m} + V_{l,m}\right) \tag{6b}$$

$$\overline{U}_{l,m+\frac{1}{2}} = \frac{1}{2}\left(U_{l,m+1} + U_{l,m}\right) \tag{7a}$$

$$\overline{V}_{l,m+\frac{1}{2}} = \frac{1}{2}\left(V_{l,m+1} + V_{l,m}\right) \tag{7b}$$

$$\overline{U}_{l,m-\frac{1}{2}} = \frac{1}{2}\left(U_{l,m-1} + U_{l,m}\right) \tag{8a}$$

$$\overline{V}_{l,m-\frac{1}{2}} = \frac{1}{2}\left(V_{l,m-1} + V_{l,m}\right) \tag{8b}$$

$$\overline{U}_{xl,m} = \frac{1}{2h}\left(U_{l+1,m} - U_{l-1,m}\right) \tag{9a}$$

$$\overline{V}_{xl,m} = \frac{1}{2h}\left(V_{l+1,m} - V_{l-1,m}\right) \tag{9b}$$

$$\overline{U}_{xl+\frac{1}{2},m} = \frac{1}{h}\left(U_{l+1,m} - U_{l,m}\right) \tag{10a}$$

$$\overline{V}_{xl+\frac{1}{2},m} = \frac{1}{h}\left(V_{l+1,m} - V_{l,m}\right) \tag{10b}$$

$$\overline{U}_{xl-\frac{1}{2},m} = \frac{1}{h}\left(U_{l,m} - U_{l-1,m}\right) \tag{11a}$$

$$\overline{V}_{xl-\frac{1}{2},m} = \frac{1}{h}\left(V_{l,m} - V_{l-1,m}\right) \tag{11b}$$

$$\overline{U}_{xl,m+\frac{1}{2}} = \frac{1}{4h}\left(U_{l+1,m+1} - U_{l-1,m+1} + U_{l+1,m} - U_{l-1,m}\right) \tag{12a}$$

$$\overline{V}_{xl,m+\frac{1}{2}} = \frac{1}{4h}\left(V_{l+1,m+1} - V_{l-1,m+1} + V_{l+1,m} - V_{l-1,m}\right) \tag{12b}$$

$$\overline{U}_{xl,m-\frac{1}{2}} = \frac{1}{4h}\left(U_{l+1,m-1} - U_{l-1,m-1} + U_{l+1,m} - U_{l-1,m}\right) \tag{13a}$$

$$\overline{V}_{xl,m-\frac{1}{2}} = \frac{1}{4h}\left(V_{l+1,m-1} - V_{l-1,m-1} + V_{l+1,m} - V_{l-1,m}\right) \tag{13b}$$

$$\overline{U}_{yl,m} = \frac{1}{2h}\left(U_{l,m+1} - U_{l,m-1}\right) \tag{14a}$$

$$\overline{V}_{yl,m} = \frac{1}{2h}\left(V_{l,m+1} - V_{l,m-1}\right) \tag{14b}$$

$$\overline{U}_{yl+\frac{1}{2},m} = \frac{1}{4h}\left(U_{l+1,m+1} - U_{l+1,m-1} + U_{l,m+1} - U_{l,m-1}\right) \tag{15a}$$

$$\overline{V}_{yl+\frac{1}{2},m} = \frac{1}{4h}\left(V_{l+1,m+1} - V_{l+1,m-1} + V_{l,m+1} - V_{l,m-1}\right) \tag{15b}$$

$$\overline{U}_{yl-\frac{1}{2},m} = \frac{1}{4h}\left(U_{l-1,m+1} - U_{l-1,m-1} + U_{l,m+1} - U_{l,m-1}\right) \tag{16a}$$

$$\overline{V}_{yl-\frac{1}{2},m} = \frac{1}{4h}\left(V_{l-1,m+1} - V_{l-1,m-1} + V_{l,m+1} - V_{l,m-1}\right) \tag{l6b}$$

$$\overline{U}_{yl,m+\frac{1}{2}} = \frac{1}{h}\left(U_{l,m+1} - U_{l,m}\right) \tag{17a}$$

$$\overline{V}_{yl,m+\frac{1}{2}} = \frac{1}{h}\left(V_{l,m+1} - V_{l,m}\right) \tag{17b}$$

$$\overline{U}_{yl,m-\frac{1}{2}} = \frac{1}{h}\left(U_{l,m} - U_{l,m-1}\right) \tag{18a}$$

$$\overline{V}_{yl,m-\frac{1}{2}} = \frac{1}{h}\left(V_{l,m} - V_{l,m-1}\right) \tag{18b}$$

Then we evaluate

$$\overline{F}_{l,m} = f(x_l, y_m, U_{l,m}, V_{l,m}, \overline{U}_{xl,m}, \overline{V}_{xl,m}, \overline{U}_{yl,m}, \overline{V}_{yl,m}) \tag{19}$$

$$\overline{F}_{l\pm\frac{1}{2},m} = f(x_{l\pm\frac{1}{2}}, y_m, U_{l\pm\frac{1}{2},m}, V_{l\pm\frac{1}{2},m}, \overline{U}_{xl\pm\frac{1}{2},m}, \overline{V}_{xl\pm\frac{1}{2},m}, \overline{U}_{yl\pm\frac{1}{2},m}, \overline{V}_{yl\pm\frac{1}{2},m}) \qquad (20)$$

$$\overline{F}_{l,m\pm\frac{1}{2}} = f(x_l, y_{m\pm\frac{1}{2}}, U_{l,m\pm\frac{1}{2}}, V_{l,m\pm\frac{1}{2}}, \overline{U}_{xl,m\pm\frac{1}{2}}, \overline{V}_{xl,m\pm\frac{1}{2}}, \overline{U}_{yl,m\pm\frac{1}{2}}, \overline{V}_{yl,m\pm\frac{1}{2}}) \qquad (21)$$

Further, we define

$$\hat{U}_{l,m} = U_{l,m} + \frac{h^2}{4} V_{l,m} \qquad (22a)$$

$$\hat{V}_{l,m} = V_{l,m} + \frac{h^2}{4\varepsilon} \overline{F}_{l,m} \qquad (22b)$$

$$\hat{U}_{xl,m} = \overline{U}_{xl,m} + \frac{h}{8}(V_{l+1,m} - V_{l-1,m}) \qquad (23a)$$

$$\hat{V}_{xl,m} = \overline{V}_{xl,m} + \frac{h}{4\varepsilon}(\overline{F}_{l+\frac{1}{2},m} - \overline{F}_{l-\frac{1}{2},m}) \qquad (23b)$$

$$\hat{U}_{yl,m} = \overline{U}_{yl,m} + \frac{h}{8}(V_{l,m+1} - V_{l,m-1}) \qquad (24a)$$

$$\hat{V}_{yl,m} = \overline{V}_{yl,m} + \frac{h}{4\varepsilon}(\overline{F}_{l,m+\frac{1}{2}} - \overline{F}_{l,m-\frac{1}{2}}) \qquad (24b)$$

Finally, let

$$\hat{F}_{l,m} = f(x_l, y_m, \hat{U}_{l,m}, \hat{V}_{l,m}, \hat{U}_{xl,m}, \hat{V}_{xl,m}, \hat{U}_{yl,m}, \hat{V}_{yl,m}) \qquad (25)$$

Then at each internal grid point (x_l, y_m) of the solution region Ω, the given system of differential equations (3) are discretized by

$$L[U] \equiv U_{l-1,m-1} + 4U_{l,m-1} + U_{l+1,m-1} + 4U_{l-1,m} - 20U_{l,m} + 4U_{l+1,m} + U_{l-1,m+1} + 4U_{l,m+1} + U_{l+1,m+1}$$
$$= \frac{h^2}{2}[V_{l+1,m} + V_{l-1,m} + V_{l,m+1} + V_{l,m-1} + 8V_{l,m}] + O(h^6), \quad l,m=1(1)N \qquad (26a)$$

$$L[V] \equiv \varepsilon[V_{l-1,m-1} + 4V_{l,m-1} + V_{l+1,m-1} + 4V_{l-1,m} - 20V_{l,m} + 4V_{l+1,m} + V_{l-1,m+1} + 4V_{l,m+1} + V_{l+1,m+1}]$$
$$= 2h^2[\overline{F}_{l+\frac{1}{2},m} + \overline{F}_{l-\frac{1}{2},m} + \overline{F}_{l,m+\frac{1}{2}} + \overline{F}_{l,m-v} - \hat{F}_{l,m}] + \overline{T}_{l,m}, \quad l,m=1(1)N. \qquad (26b)$$

where $\overline{T}_{l,m} = O(h^6)$.

3. DERIVATION PROCEDURE

For the derivation of the method, we simply follow the ideas given by (Mohanty and Singh, 2006). At the grid point (x_l, y_m), let us denote

$$U_{ij}=\frac{\partial^{i+j}U}{\partial x_l^{\,i}\partial y_m^{\,j}},\ V_{ij}=\frac{\partial^{i+j}V}{\partial x_l^{\,i}\partial y_m^{\,j}},\ \alpha^{(1)}_{l,m}=\frac{\partial f}{\partial U_{l,m}},\ \alpha^{(2)}_{l,m}=\frac{\partial f}{\partial V_{l,m}},\ \beta^{(1)}_{l,m}=\frac{\partial f}{\partial U_{xl,m}},$$
$$\beta^{(2)}_{l,m}=\frac{\partial f}{\partial V_{xl,m}},\ \gamma^{(1)}_{l,m}=\frac{\partial f}{\partial U_{yl,m}},\ \gamma^{(2)}_{l,m}=\frac{\partial f}{\partial V_{yl,m}} \tag{27}$$

Further, at the grid point (x_l, y_m), we define

$$F_{l,m}=f(x_l,y_m,U_{l,m},V_{l,m},U_{xl,m},V_{xl,m},U_{yl,m},V_{yl,m}) \tag{28}$$

Then by the help of the notation (27), simplifying (5a)- (18b),we obtain

$$\overline{U}_{l\pm\frac{1}{2},m}=U_{l\pm\frac{1}{2},m}+\frac{h^2}{8}\mathrm{U}_{20}+O(h^3) \tag{29a}$$

$$\overline{V}_{l\pm\frac{1}{2},m}=V_{l\pm\frac{1}{2},m}+\frac{h^2}{8}\mathrm{V}_{20}+O(h^3) \tag{29b}$$

$$\overline{U}_{l,m\pm\frac{1}{2}}=U_{l,m\pm\frac{1}{2}}+\frac{h^2}{8}\mathrm{U}_{02}+O(h^3) \tag{30a}$$

$$\overline{V}_{l,m\pm\frac{1}{2}}=V_{l,m+\frac{1}{2}}+\frac{h^2}{8}\mathrm{V}_{02}+O(h^3) \tag{30b}$$

$$\overline{U}_{xl,m}=U_{xl,m}+\frac{h^2}{6}\mathrm{U}_{30}+O(h^4) \tag{31a}$$

$$\overline{V}_{xl,m}=V_{xl,m}+\frac{h^2}{6}\mathrm{V}_{30}+O(h^4) \tag{31b}$$

$$\overline{U}_{xl\pm\frac{1}{2},m}=U_{xl\pm\frac{1}{2},m}+\frac{h^2}{24}\mathrm{U}_{30}+O(h^4) \tag{32a}$$

$$\overline{V}_{xl\pm\frac{1}{2},m}=V_{xl\pm\frac{1}{2},m}+\frac{h^2}{24}\mathrm{V}_{30}+O(h^4) \tag{32b}$$

$$\overline{U}_{xl,m\pm\frac{1}{2}}=U_{xl,m\pm\frac{1}{2}}+\frac{h^2}{24}\left(4\mathrm{U}_{30}+3\mathrm{U}_{12}\right)+O(h^3) \tag{33a}$$

$$\overline{V}_{xl,m\pm\frac{1}{2}}=V_{xl,m\pm\frac{1}{2}}+\frac{h^2}{24}\left(4\mathrm{V}_{30}+3\mathrm{V}_{12}\right)+O(h^3) \tag{33b}$$

$$\overline{U}_{yl,m}=U_{yl,m}+\frac{h^2}{6}\mathrm{U}_{03}+O(h^4) \tag{34a}$$

$$\overline{V}_{yl,m}=V_{yl,m}+\frac{h^2}{6}\mathrm{V}_{03}+O(h^4) \tag{34b}$$

$$\overline{U}_{yl\pm\frac{1}{2},m}=U_{yl\pm\frac{1}{2},m}+\frac{h^2}{24}\left(3\mathrm{U}_{21}+4\mathrm{U}_{03}\right)+O(h^3) \tag{35a}$$

$$\overline{V}_{yl\pm\frac{1}{2},m} = V_{yl\pm\frac{1}{2},m} + \frac{h^2}{24}\left(3V_{21} + 4V_{03}\right) + O(h^3) \quad (35b)$$

$$\overline{U}_{yl,m\pm\frac{1}{2}} = U_{yl,m\pm\frac{1}{2}} + \frac{h^2}{24}U_{03} + O(h^3) \quad (36a)$$

$$\overline{V}_{yl,m\pm\frac{1}{2}} = V_{yl,m\pm\frac{1}{2}} + \frac{h^2}{24}V_{03} + O(h^3) \quad (36b)$$

At the grid point (x_l, y_m), we may write the difference equation (3b) as

$$\varepsilon\left(\frac{\partial^2 V_{l,m}}{\partial x^2} + \frac{\partial^2 V_{l,m}}{\partial y^2}\right) = f(x, y, U_{l,m}, V_{l,m}, U_{xl,m}, V_{xl,m}, U_{yl,m}, V_{yl,m}) \equiv F_{l,m} \quad (37)$$

By the help of Taylor expansion, we first obtain

$$\varepsilon\left[\delta_x^{\,2} + \delta_y^{\,2} + \frac{1}{6}\delta_x^{\,2}\delta_y^{\,2}\right]V_{l,m} = \frac{h^2}{3}\left[F_{l+\frac{1}{2},m} + F_{l-\frac{1}{2},m} + F_{l,m+\frac{1}{2}} + F_{l,m-\frac{1}{2}} - F_{l,m}\right] + O\left(h^6\right) \quad (38)$$

With the help of the approximation (29a)-(36b), from (19)-(21), we obtain

$$\overline{F}_{l,m} = F_{l,m} + O\left(h^2\right) \quad (39a)$$

$$\overline{F}_{l\pm\frac{1}{2},m} = F_{l\pm\frac{1}{2},m} + \frac{h^2}{24}T_1 \pm O\left(h^3\right) \quad (39b)$$

$$\overline{F}_{l,m\pm\frac{1}{2}} = F_{l,m\pm\frac{1}{2}} + \frac{h^2}{24}T_2 \pm O\left(h^3\right) \quad (39c)$$

where

$$T_1 = 3U_{20}\alpha^{(1)}_{l,m} + 3V_{20}\alpha^{(2)}_{l,m} + U_{30}\beta^{(1)}_{l,m} + V_{30}\beta^{(2)}_{l,m} + \left(3U_{21} + 4U_{03}\right)\gamma^{(1)}_{l,m} + \left(3V_{21} + 4V_{03}\right)\gamma^{(2)}_{l,m}$$

$$T_2 = 3U_{02}\alpha^{(1)}_{l,m} + 3V_{02}\alpha^{(2)}_{l,m} + \left(3U_{12} + 4U_{30}\right)\beta^{(1)}_{l,m} + \left(3V_{12} + 4V_{30}\right)\beta^{(2)}_{l,m} + U_{03}\gamma^{(1)}_{l,m} + V_{03}\gamma^{(2)}_{l,m}$$

Let

$$\hat{U}_{l,m} = U_{l,m} + ah^2 V_{l,m} \quad (40a)$$

$$\hat{V}_{l,m} = V_{l,m} + a'h^2\overline{F}_{l,m} \quad (40b)$$

$$\hat{U}_{xl,m} = \overline{U}_{xl,m} + bh\left(V_{l+1,m} - V_{l-1,m}\right) \quad (41a)$$

$$\hat{V}_{xl,m} = \overline{V}_{xl,m} + b'h\left(\overline{F}_{l+\frac{1}{2},m} - \overline{F}_{l-\frac{1}{2},m}\right) \quad (41b)$$

$$\hat{U}_{yl,m} = \overline{U}_{yl,m} + ch\left(V_{l,m+1} - V_{l,m-1}\right) \quad (42a)$$

$$\hat{V}_{yl,m} = \overline{V}_{yl,m} + c'h\left(\overline{F}_{l,m+\frac{1}{2}} - \overline{F}_{l,m-\frac{1}{2}}\right) \quad (42b)$$

where a, a', b, b', c, c' are parameters to be determined.

With the help of the approximations (39a)-(39c) and simplifying (40a)-(42b), we get

$$\hat{U}_{l,m} = U_{l,m} + \frac{h^2}{6} T_3 + O\left(h^4\right) \tag{43a}$$

$$\hat{V}_{l,m} = V_{l,m} + \frac{h^2}{6} T_3' + O\left(h^4\right) \tag{43b}$$

$$\hat{U}_{xl,m} = U_{xl,m} + \frac{h^2}{6} T_4 + O\left(h^4\right) \tag{44a}$$

$$\hat{V}_{xl,m} = V_{xl,m} + \frac{h^2}{6} T_4' + O\left(h^4\right) \tag{44b}$$

$$\hat{U}_{yl,m} = U_{yl,m} + \frac{h^2}{6} T_5 + O\left(h^4\right) \tag{45a}$$

$$\hat{V}_{yl,m} = V_{yl,m} + \frac{h^2}{6} T_5' + O\left(h^4\right) \tag{45b}$$

where

$$T_3 = 6a\left(U_{20} + U_{02}\right)$$

$$T_3' = 6\varepsilon a'\left(V_{20} + V_{02}\right)$$

$$T_4 = U_{30} + 12b\left(U_{30} + U_{12}\right) = \left(1 + 12b\right)U_{30} + 12bU_{12}$$

$$T_4' = V_{30} + 6\varepsilon b'\left(V_{30} + V_{12}\right) = \left(1 + 6\varepsilon b'\right)V_{30} + 6\varepsilon b' V_{12}$$

$$T_5 = U_{03} + 6c\left(U_{03} + U_{21}\right) = \left(1 + 12c\right)U_{03} + 12cU_{21}$$

$$T_5' = V_{03} + 6\varepsilon c'\left(V_{03} + V_{21}\right) = \left(1 + 6\varepsilon c'\right)V_{03} + 6\varepsilon c' V_{21}$$

Now,

$$\hat{F}_{l,m} = F_{l,m} + \frac{h^2}{6} T_6 + O(h^4) \tag{46}$$

where

$$T_6 = T_3\alpha_{l,m}^{(1)} + T_3'\alpha_{l,m}^{(2)} + T_4\beta_{l,m}^{(1)} + T_4'\beta_{l,m}^{(2)} + T_5\gamma_{l,m}^{(1)} + T_5'\gamma_{l,m}^{(2)}$$

Substituting the approximations (39a)-(39c) and (46) into (25b) and by the help of (38), we obtain the local truncation error

$$\overline{T}_{l,m} = -\frac{h^4}{6}\left[T_1 + T_2 - 2T_6\right] + O(h^6) \tag{47}$$

The proposed difference method to be of fourth order, the coefficient of h^4 in (47) must be zero and we obtain

$$T_1 + T_2 - 2T_6 = 0 \tag{48}$$

Substituting the values of T_1, T_2 and T_6 in (47), we obtain the values of parameters

$$a = \frac{1}{4}, a' = \frac{1}{4\varepsilon}, \quad b = \frac{1}{8}, b' = \frac{1}{4\varepsilon}, c = \frac{1}{8}, c' = \frac{1}{4\varepsilon},$$

and the local truncation error (47) reduces to $\overline{T}_{l,m} = O(h^6)$.

4. BLOCK ITERATIVE METHODS

By combining the difference equations at each internal grid points, we obtain a large sparse system of matrix to solve. At each interior mesh point, we have two unknowns u and $\nabla^2 u \equiv v$, that is, the number of bands with non-zero entries is increased, and so is the size of the final matrix for the same mesh size. However, by this new method, the value of the Laplacian, which is often of interest, is also computed.

Whenever $f(x, y, u, v, u_x, v_x, u_y, v_y)$ is linear in u, v, u_x, v_x, u_y and v_y, the difference equations (26a) and (26b) form a linear block system. To solve such a system or indeed to demonstrate the existence of a solution, one can use block successive over relaxation (BSOR) iterative method (Hageman and Young, 2004; Kelly, 1995; Meurant, 1999; Parter, 1981; Saad, 1996; Varga, 2000).

To define BSOR method, we first write (26a) and (26b) in the form

$$\boldsymbol{A}_1\boldsymbol{u} + \boldsymbol{B}_1\boldsymbol{v} = \boldsymbol{0} \tag{49a}$$

$$\boldsymbol{A}_2\boldsymbol{u} + \boldsymbol{B}_2\boldsymbol{v} = \boldsymbol{d} \tag{49b}$$

where $\boldsymbol{A}_{1\mathrm{L}}$ = [1, 4, 1], $\boldsymbol{A}_{1\mathrm{D}}$ = [4, -20, 4], $\boldsymbol{A}_{1\mathrm{U}}$ =[1, 4, 1] represent lower, main and upper diagonal tri-diagonal matrices of the tri-block diagonal matrix $\boldsymbol{A}_1 = [\boldsymbol{A}_{1\mathrm{L}}, \boldsymbol{A}_{1\mathrm{D}}, \boldsymbol{A}_{1\mathrm{U}}]$ and $\boldsymbol{B}_{1\mathrm{L}}$ = [0, 1, 0], $\boldsymbol{B}_{1\mathrm{D}}$ = [1, 8, 1], $\boldsymbol{B}_{1\mathrm{U}}$ =[0, 1, 0] are tri-diagonal matrices of tri-block diagonal matrix $\boldsymbol{B}_1 = \frac{-h^2}{2} [\boldsymbol{B}_{1\mathrm{L}}, \boldsymbol{B}_{1\mathrm{D}}, \boldsymbol{B}_{1\mathrm{U}}]$. The structure of block tri-diagonal matrices $\boldsymbol{A}_2$ and $\boldsymbol{B}_2$ depends upon the linear form of the function $f(x, y, u, v, u_x, v_x, u_y, v_y)$ and $\boldsymbol{u}$, $\boldsymbol{v}$ are solution vectors, and $\boldsymbol{d}$ is the vector consisting of right hand side functions and associated boundary conditions.

Let $\boldsymbol{B}_2 = [\boldsymbol{B}_{2\mathrm{L}}, \boldsymbol{B}_{2\mathrm{D}}, \boldsymbol{B}_{2\mathrm{U}}]$ be the block tri-diagonal matrix associated with (49b). Relative to the partitioning (49a) and (49b), the BSOR method is defined by

$$\boldsymbol{A}_{1\mathrm{D}}\, \boldsymbol{u}^{(k+1)} = \omega[-(\boldsymbol{A}_{1\mathrm{L}}+\boldsymbol{A}_{1\mathrm{U}})\, \boldsymbol{u}^{(k)} - \boldsymbol{B}_1 \boldsymbol{v}^{(k)}] + (1-\omega)\, \boldsymbol{A}_{1\mathrm{D}} \boldsymbol{u}^{(k)} \tag{50a}$$

$$\boldsymbol{B}_{2\mathrm{D}}\, \boldsymbol{v}^{(k+1)} = \omega[-(\boldsymbol{B}_{2\mathrm{L}}+\boldsymbol{B}_{2\mathrm{U}})\, \boldsymbol{v}^{(k)} - \boldsymbol{A}_2 \boldsymbol{u}^{(k+1)} + \boldsymbol{d}] + (1-\omega)\, \boldsymbol{B}_{2\mathrm{D}} \boldsymbol{v}^{(k)} \tag{50b}$$

where $0<\omega<2$ is a relaxation parameter. The above system of equations can be solved by using a line solver. For ω=1, the BSOR method reduces to block-Gauss Seidel iterative method.

Whenever $f(x, y, u, v, u_x, v_x, u_y, v_y)$ is non-linear in u, v, u_x, v_x, u_y and v_y, the difference equations (26a) and (26b) form a non-linear block system. To solve such a system, one can apply Newton's non-linear block successive over relaxation (NBSOR) iterative method (Hageman and Young, 2004; Kelly, 1995; Meurant, 1999; Parter, 1981; Saad, 1996; Varga, 2000).

To define NBSOR method, we first write (26a) and (26b) in the form

$$\boldsymbol{Au} + \boldsymbol{Bv} = \boldsymbol{0} \tag{51a}$$
$$H(\boldsymbol{u}, \boldsymbol{v}) = \boldsymbol{0} \tag{51b}$$

where $\boldsymbol{A} = [\boldsymbol{A}_L, \boldsymbol{A}_D, \boldsymbol{A}_U]$ and $\boldsymbol{B} = [\boldsymbol{B}_L, \boldsymbol{B}_D, \boldsymbol{B}_U]$ are tri-block diagonal matrices already defined earlier, and $\boldsymbol{u}$, $\boldsymbol{v}$ are solution vectors of the linear system (51a) and non-linear system (51b).

Now we compute the value of $\boldsymbol{u}$ from (51a) using linear iterative method and value of $\boldsymbol{v}$ from (51b) using non-linear iterative method.

The Jacobian $\boldsymbol{J}$ of $\boldsymbol{H}$ is easily found to be the block tri-diagonal matrix $\boldsymbol{J} = [\boldsymbol{J}_L, \boldsymbol{J}_D, \boldsymbol{J}_U]$, where

$$\boldsymbol{J}_L = \left[\frac{\partial H}{\partial v_{l-1,m-1}}, \frac{\partial H}{\partial v_{l,m-1}}, \frac{\partial H}{\partial v_{l+1,m-1}} \right],$$

$$\boldsymbol{J}_D = \left[\frac{\partial H}{\partial v_{l-1,m}}, \frac{\partial H}{\partial v_{l,m}}, \frac{\partial H}{\partial v_{l+1,m}} \right],$$

and

$$\boldsymbol{J}_U = \left[\frac{\partial H}{\partial v_{l-1,m+1}}, \frac{\partial H}{\partial v_{l,m+1}}, \frac{\partial H}{\partial v_{l+1,m+1}} \right]$$

are N^{th} order tri-diagonal matrices.

Now matrix equation for Newton NBSOR method is given by

$$\boldsymbol{J}\ \Delta \boldsymbol{v}^{(k)} = -H(\boldsymbol{u}^{(k+1)}, \boldsymbol{v}^{(k)}) \tag{52}$$

where $(\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)})$ is the initial approximation of $(\boldsymbol{u}, \boldsymbol{v})$ and $\Delta \boldsymbol{v}^{(k)}$ is any intermediate vector and the values of $\boldsymbol{u}^{(k+1)}$ are known from the previous step. We define

$$\boldsymbol{v}^{(k+1)} = \boldsymbol{v}^{(k)} + \Delta \boldsymbol{v}^{(k)},\ k = 0,1,2,\ldots \tag{53}$$

We can solve (51a) and (51b) by using BSOR and Newton NBSOR method as follows:

$$\boldsymbol{A}_D\ \boldsymbol{u}^{(k+1)} = \omega[-(\boldsymbol{A}_L + \boldsymbol{A}_U)\ \boldsymbol{u}^{(k)} - \boldsymbol{B}\,\boldsymbol{v}^{(k)}] + (1-\omega)\,\boldsymbol{A}_D \boldsymbol{u}^{(k)},\ k = 0,1,2,\ldots \tag{54a}$$
$$\boldsymbol{J}_D\ \Delta \boldsymbol{v}^{(k+1)} = \omega[-H(\boldsymbol{u}^{(k+1)}, \boldsymbol{v}^{(k)}) - (\boldsymbol{J}_L + \boldsymbol{J}_U)\,\Delta \boldsymbol{v}^{(k)}] + (1-\omega)\,\boldsymbol{J}_D \Delta \boldsymbol{v}^{(k)},\ k = 0,1,2,\ldots \tag{54b}$$

where $0<\omega<2$ is a relaxation parameter. For $\omega=1$, the above system reduces to Newton block-Gauss Seidel iterative method. Above system can be solved by using a tri-diagonal solver. Then by using the outer iterative method (53), we can evaluate $\boldsymbol{v}^{(k+1)}$, $k = 0,1,2,\ldots$.

In order for this method to converge it is sufficient that the initial iterate $(\boldsymbol{u}^{(0)}, \boldsymbol{v}^{(0)})$ be close to the solution.

The second order approximations for the coupled system of differential equations (3a) and (3b) are straightforward and can be written in a coupled manner

$$U_{l,m-1}+U_{l-1,m}-4U_{l,m}+U_{l+1,m}+U_{l,m+1}=h^2V_{l,m}+O(h^4)\,,\ \ l,m=1(1)N \tag{55a}$$

$$V_{l,m-1}+V_{l-1,m}-4V_{l,m}+V_{l+1,m}+V_{l,m+1}$$
$$=h^2 f(x_l,y_m,U_{l,m},V_{l,m},\overline{U}_{xl,m},\overline{V}_{xl,m},\overline{U}_{yl,m},\overline{V}_{yl,m})+O(h^4)\,,l,m=1(1)N \tag{55b}$$

Note that, the second order approximations (55a) and (55b) require only 5-grid points on a single computational cell (see Fig.1) applicable to linear biharmonic problems with singular coefficients. In a similar manner, we can discuss the block iterative methods for the systems (55a) and (55b).

5. STABILITY ANALYSIS AND EXPERIMENTAL RESULTS

Let us consider the test equation

$$\nabla^4 u=g(x,y)\,,\ 0<x,y<1 \tag{56}$$

Applying the proposed method (26a)-(26b) to the above equation, we obtain

$$U_{l-1,m-1}+4U_{l,m-1}+U_{l+1,m-1}+4U_{l-1,m}-20U_{l,m}+4U_{l+1,m}+U_{l-1,m+1}+4U_{l,m+1}+U_{l+1,m+1}$$
$$=\frac{h^2}{2}[V_{l+1,m}+V_{l-1,m}+V_{l,m+1}+V_{l,m-1}+8V_{l,m}],\quad l,m=1(1)N \tag{57a}$$

$$V_{l-1,m-1}+4V_{l,m-1}+V_{l+1,m-1}+4V_{l-1,m}-20V_{l,m}+4V_{l+1,m}+V_{l-1,m+1}+4V_{l,m+1}+V_{l+1,m+1}$$
$$=2h^2[g_{l+\frac{1}{2},m}+g_{l-\frac{1}{2},m}+g_{l,m+\frac{1}{2}}+g_{l,m-\frac{1}{2}}-g_{l,m}],\ l,m=1(1)N \tag{57b}$$

where $g_{l,m}=g(x_l,y_m), g_{l\pm\frac{1}{2},m}=g(x_l\pm\frac{1}{2},y_m)$etc.

An iterative method for (57a)-(57b) can be written as

$$20\boldsymbol{Iu}^{(k+1)}=\boldsymbol{Au}^{(k)}-\frac{h^2}{2}\boldsymbol{Bv}^{(k)}+\boldsymbol{RHU} \tag{58a}$$

$$20\boldsymbol{Iv}^{(k+1)}=\boldsymbol{0u}^{(k)}+\boldsymbol{Av}^{(k)}+\boldsymbol{RHV} \tag{58b}$$

where $\boldsymbol{u}^{(k)},\boldsymbol{v}^{(k)}$ are solution vectors and ***RHU, RHV*** are right hand side vectors consists of boundary and homogenous function values.

Above system in matrix form can be written as

$$\begin{bmatrix} \boldsymbol{U}^{(k+1)} \\ \boldsymbol{V}^{(k+1)} \end{bmatrix} = \boldsymbol{G} \begin{bmatrix} \boldsymbol{U}^{(k)} \\ \boldsymbol{V}^{(1)} \end{bmatrix} + \boldsymbol{RH} \tag{59}$$

where

$$\boldsymbol{G} = \frac{1}{20}\begin{bmatrix} \boldsymbol{A} & \dfrac{-h^2}{2}\boldsymbol{B} \\ \boldsymbol{0} & \boldsymbol{A} \end{bmatrix}, \quad \boldsymbol{RH} = \begin{bmatrix} \boldsymbol{RHU} \\ \boldsymbol{RHV} \end{bmatrix},$$

$\boldsymbol{A}$ = [$\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{P}$] , $\boldsymbol{B}$=[$\boldsymbol{T}$, $\boldsymbol{S}$, $\boldsymbol{T}$] , $\boldsymbol{P}$ = [1, 4, 1], $\boldsymbol{Q}$ = [4, 0, 4], $\boldsymbol{T}$=[0, 1, 0], $\boldsymbol{S}$=[1, 8, 1]

where we denote

$$[a, b, c] = \begin{bmatrix} b & c & & & \boldsymbol{0} \\ a & b & c & & \\ & & \ddots & & \\ & & a & b & c \\ \boldsymbol{0} & & & a & b \end{bmatrix}_{N\times N}$$

as Nth order tri-diagonal matrix and its eigen values are given by

$$\lambda_j = b + 2\sqrt{ac}\cos\left(\frac{\pi j}{N+1}\right), \quad j = 1, 2, ...N. \tag{60}$$

Above iterative method is stable as long as $\rho(\boldsymbol{G}) \le 1$, where $\rho(\boldsymbol{G})$ is spectral radius of $\boldsymbol{G}$**.**

Now eigenvalues of $\boldsymbol{Q}$ are given by

$$\lambda_k = 8\cos\frac{k\pi}{N+1} \equiv 8\cos(k\pi h), k = 1(1)N \tag{61}$$

and eigenvalues of $\boldsymbol{P}$ are given by

$$\mu_k = 4 + 2\cos\frac{k\pi}{N+1} \equiv 4 + 2\cos(k\pi h), k = 1(1)N\,. \tag{62}$$

and hence, the eigenvalues of $\boldsymbol{A}$ are given by

$$\nu_{jk} = \lambda_k + 2\mu_k\cos(j\pi h) \equiv 8[\cos(k\pi h) + \cos(j\pi h)] + 4\cos(k\pi h)\cos(j\pi h),$$
$$j = 1(1)N, k = 1(1)N \tag{63}$$

Thus the eigenvalues of $\boldsymbol{G}$ are given by

$$\xi_{jk} = \frac{1}{20}\nu_{jk} = \frac{1}{20}[8(\cos(k\pi h) + \cos(j\pi h)) + 4\cos(k\pi h)\cos(j\pi h)],$$
$$j = 1(1)N, k = 1(1)N \tag{64}$$

The maximum eigenvalue of $\boldsymbol{G}$ occurs at $j = k = 1$.

Hence $\rho(\boldsymbol{G}\ ) = \max.\ \left|\xi_{jk}\right| = \dfrac{\cos(\pi h)}{5}[4+\cos(\pi h)] \le 1$, (65)

which is satisfied for all variable angles πh. Hence the iterative method (58a)-(58b) is stable.

In order to validate the proposed fourth order method and test its robustness, we solve the following four test problems in the region 0<*x,y*<1, whose exact solutions are known. The Dirichlet boundary conditions and right hand side homogeneous functions are obtained by using the exact solutions. We have solved the linear systems by using block Gauss-Seidel iterative method, and the non-linear system of equations by using Newton block Gauss-Seidel iterative method. We have also compared the numerical results obtained by proposed fourth order approximations (26a) and (26b) with the numerical results obtained by corresponding second order approximations (55a) and (55b). In all cases, we have considered $\boldsymbol{u}^{(0)} = \boldsymbol{0}$ as the initial approximation and the iterations were stopped when the absolute error tolerance $\left|\boldsymbol{u}^{(k+1)} - \boldsymbol{u}^{(k)}\right| \le 10^{-12}$ was achieved. In all cases, we have calculated maximum absolute errors (l_∞-norm) for different grid sizes. All computations were performed using double precision arithmetic.

Example 1: (Variable coefficient problems)

(a) $$\nabla^4 u = (1+x^2)(u_{xxx}+u_{xyy}) + (1+y^2)(u_{xxy}+u_{yyy}) + (1+\sin^2 x)u_x + (1+\sin^2 y)u_y + G(x,y),\ 0<x,y<1 \tag{66a}$$

The exact solution is $u(x,y) = \sin(\pi x).\sin(\pi y)$.

(b) $$\nabla^4 u = (1+\cos^2 x)(u_{xxx}+u_{xyy}) + (1+\cos^2 y)(u_{xxy}+u_{yyy}) + (1+x^4)u_x + (1+y^4)u_y + G(x,y),\ 0<x,y<1 \tag{66b}$$

The exact solution is $u(x,y) = e^{x+y}$.

The maximum absolute errors (MAEs) are tabulated in Table 1.

Example 2: (Singular problem)

$$\nabla^4 u + \frac{\alpha}{x}(u_{xxx}+u_{xyy}) = G(x,y), \quad \alpha \ne 0\ ,\ 0<x,y<1 \tag{67}$$

The exact solution is $u(x,y) = x^4 \sinh y$.

The MAEs are tabulated in Table 2 for α=1 and 2.

Example 3: (Navier-Stokes model equation in terms of stream function ψ), (Spotz and Carey, 1995).

$$\frac{1}{R_e}\nabla^4\psi = \psi_y(\psi_{xxx}+\psi_{xyy}) - \psi_x(\psi_{xxy}+\psi_{yyy}) + G(x,y)\,,\ 0<x,y<1 \tag{68}$$

The exact solution is $\psi(x,y) = e^x \sin(\pi y)$.

The MAEs are tabulated in Table 3 for various values of Reynolds number R_e.

Table 1: The MAE errors

h		Example 1(a)		Example 1(b)	
		$O(h^4)-$Method	$O(h^2)-$Method	$O(h^4)-$Method	$O(h^2)-$Method
1/8	u	0.3717(-03)	0.2579(-01)	0.1348(-06)	0.7276(-03)
	$\nabla^2 u$	0.6260(-02)	0.2539(+00)	0.7649(-05)	0.3746(-02)
1/16	u	0.2306(-04)	0.6380(-02)	0.8309(-08)	0.1886(-03)
	$\nabla^2 u$	0.4032(-03)	0.6333(-01)	0.4873(-06)	0.9415(-03)
1/32	u	0.1449(-05)	0.1590(-02)	0.5243(-09)	0.4725(-04)
	$\nabla^2 u$	0.2513(-04)	0.1594(-01)	0.3041(-07)	0.2357(-03)
1/64	u	0.9055(-07)	0.3433(-03)	0.3262(-10)	0.1181(-04)
	$\nabla^2 u$	0.1569(-05)	0.3773(-02)	0.1900(-08)	0.5896(-04)

Table 2: The MAE errors

h		$\alpha = 1$		$\alpha = 2$	
		$O(h^4)-$Method	$O(h^2)-$Method	$O(h^4)-$Method	$O(h^2)-$Method
1/8	u	0.1119(-05)	0.1073(-02)	0.8111(-06)	0.9318(-03)
	$\nabla^2 u$	0.1755(-04)	0.4461(-02)	0.2925(-04)	0.7921(-02)
1/16	u	0.7073(-07)	0.2704(-03)	0.5099(-07)	0.2375(-03)
	$\nabla^2 u$	0.1113(-05)	0.1151(-02)	0.1840(-05)	0.2011(-02)
1/32	u	0.4397(-08)	0.6780(-04)	0.3189(-08)	0.5983(-04)
	$\nabla^2 u$	0.7049(-07)	0.2934(-03)	0.1152(-06)	0.5047(-03)
1/64	u	0.2741(-09)	0.1693(-04)	0.1995(-09)	0.1514(-04)
	$\nabla^2 u$	0.4451(-08)	0.7439(-04)	0.7210(-08)	0.1169(-03)

Table 3: The MAE errors

h		$O(h^4)-$Method		$O(h^2)-$Method
		$R_e = 10^2$	$R_e = 10^4, 10^6, 10^8$	$R_e = 10^2, 10^4, 10^6, 10^8$
1/4	ψ	0.3455(-02)	0.3496(-02)	Over Flow
	$\nabla^2\psi$	0.3033(-01)	0.3101(-01)	
1/8	ψ	0.1923(-03)	0.2154(-03)	Over Flow
	$\nabla^2\psi$	0.1708(-02)	0.1911(-02)	
1/16	ψ	0.8174(-05)	0.1350(-04)	Over Flow
	$\nabla^2\psi$	0.8992(-04)	0.1198(-03)	
1/32	ψ	0.3426(-06)	0.8437(-06)	Over Flow
	$\nabla^2\psi$	0.7437(-05)	0.7483(-05)	

6. CONCLUSIONS

In this article, we presented a new fourth order compact finite difference discretization using coupled approach based on arithmetic average discretization for the solution of 2D non-linear biharmonic partial differential equations. The proposed methods are directly applicable to singular problems without any modification.The method is derived on a 9-point compact stencil using the values of u and its Laplacian as the unknowns. We have obtained the numerical solution of Laplacian of u as a by-product, which is quite often of interest in many applied mathematics problems. Our method is used to solve several problems including Navier Stokes model equation and enables us to obtain high accuracy solutions with great efficiency. While solving Navier Stokes equations of motion, numerical experiments confirms that the proposed fourth order discretization method produces oscillation free solution for high Reynolds number, whereas the second order method is unstable. We are currently working to extend this technique to solve multi-dimensional non-linear time dependent biharmonic partial differential equations using coupled approach.

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