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NUMERICAL VERIFICATION OF CERTAIN OSCILLATION RESULT ON TIME SCALES

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ABSTRACT. We first investigate several examples of second order nonlinear dynamic equation

$$\left(a\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)+q(t)x^{\beta}(t)=0$$

which can also be rewritten in the form of two-dimensional dynamic system

$$x^{\Delta}(t)=b(t)g[y^{\sigma}(t)] \quad \text{and} \quad y^{\Delta}(t)=-c(t)f[x(t)]$$

where α and β are ratios of positive odd integers, a and q are real–valued, positive and rd–continuous functions on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\ \mathbb{T}=\infty$. Under oscillation criteria, some equations are then selected. Exploring the numerical solution of corresponding dynamic system individually on different time scales not only visualizes the oscillating motion as theoretically expected but also reveals other interesting behavior patterns. This study finally suggests that a time domain also plays an important role on the boundedness of oscillatory solution.

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1. INTRODUCTION

The concept of dynamic equations and dynamic systems on time scales is one of the most practical approach to the real world applications such as chemotherapy regimen. At the present time, cancer becomes one of the leading causes of human death. A cancer patient is traditionally received chemotherapy treatment involving various drugs. Different drugs normally active at different times. Therefore, each patient may take drugs not only continuously but also on a single day or several consecutive days which are discrete. The drug schedule obviously becomes a combination of several time scales which should not be modeled in the form of difference or differential equations.

Fortunately, Stefan Hilger initiated the theory of calculus on time scales in 1988. Since then, several authors have continuously contributed theoretical results on various aspects of time scales. Recently, many research activities concerning dynamic equations (see [1–10]) and dynamic systems (see [11, 12]) on time scales have been

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extensively investigated. Among them, the oscillation problem of second-order nonlinear dynamic equations has received most of attention. The motivation for this study comes from the following papers.

In [7], the authors investigated and provided oscillation criteria for the second order nonlinear dynamic equation

$$\left(a\left(x^{\Delta}\right)^{\alpha}\right)^{\Delta}(t)+q(t)x^{\beta}(t)=0 \tag{1.1}$$

where α and β are ratios of positive odd integers, a and q are real-valued, positive and rd-continuous functions on a time scale $\mathbb{T}\subset\mathbb{R}$ with $\sup\mathbb{T}=\infty$.

In [12], the authors considered the two-dimensional dynamic system

$$\begin{aligned} x^{\Delta}(t)&=b(t)g[y^{\sigma}(t)]\\ y^{\Delta}(t)&=-c(t)f[x(t)] \end{aligned} \tag{1.2}$$

and established sufficient conditions for oscillation on a time scale.

This paper aims to demonstrate the benefit of using numerical method to support theoretical prediction and to discover new interesting pattern beyond theory. It is organized as follows. In the next section, we recall the basic concepts and notations of calculus on time scales. Section 3 presents oscillation criteria for dynamic equations and dynamic systems that are needed in the remainder of this paper. In Section 4, various examples are investigated and illustrated in order to show applicability of theoretical results. The solution patterns of each dynamic system on different time scales are also revealed. We make some conclusions and discussions in Section 5. Finally, Appendix contains an algorithm for computing the numerical solution of dynamic system on time scales.

2. PRELIMINARIES

A time scale is an arbitrary nonempty closed subset of the real numbers. The examples of time scales are the real numbers ($\mathbb{R}$), the integers ($\mathbb{Z}$), the natural numbers ($\mathbb{N}$) and the nonnegative integers ($\mathbb{N}_0$). However, the rational numbers ($\mathbb{Q}$), the irrational numbers ($\mathbb{R}\backslash\mathbb{Q}$), the complex numbers ($\mathbb{C}$) and the open interval between 0 and 1 ((0,1)) are not time scales.

For a point $t\in\mathbb{T}$, the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ are defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$, respectively. In this definition, $\inf\emptyset=\sup\mathbb{T}$ and $\sup\emptyset=\inf\mathbb{T}$. In addition, t is called right-scattered if $\sigma(t)>t$, right-dense if $\sigma(t)=t$, left-scattered if $\rho(t)<t$ and left-dense if $\rho(t)=t$. The graininess function $\mu:\mathbb{T}\to[0,\infty)$ is defined by $\mu(t)=\sigma(t)-t$.

For a function $f:\mathbb{T}\to\mathbb{R}$, the notation f^{σ} denotes $f\circ\sigma$. The set $\mathbb{T}^{\kappa}$ is defined by $\mathbb{T}\backslash\{m\}$ if $\mathbb{T}$ has a left-scattered maximum m. Otherwise, $\mathbb{T}^{\kappa}=\mathbb{T}$. Moreover, f

is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The (delta) derivative $f^{\Delta}(t)$ is defined to be the number (if it exists) with the property that for all $\epsilon > 0$, there is a neighborhood U of $t \in \mathbb{T}^{\kappa}$ such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

Another useful relationship concerning the (delta) derivative is defined by

$$f^{\Delta}(t) = \begin{cases} \lim_{s\to t} \frac{f(t)-f(s)}{t-s} & \text{if} \quad \mu(t) = 0 \\ \frac{f(\sigma(t))-f(t)}{\sigma(t)-t} & \text{if} \quad \mu(t) \neq 0. \end{cases}$$

If $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$, then we say that f is (delta) differentiable on $\mathbb{T}^{\kappa}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of a rd-continuous function f if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^{\kappa}$. If $t_0 \in \mathbb{T}$ then $F(t) = \int_{t_0}^{t} f(s)\Delta s$ for $t \in \mathbb{T}$. For all $m, n \in \mathbb{T}$, the Cauchy integral and the infinite integral are defined by

$$\int_m^n f(t)\Delta t = F(n) - F(m) \quad \text{and} \quad \int_m^{\infty} f(t)\Delta t = \lim_{n\to\infty} \int_m^n f(t)\Delta t,$$

respectively. Note that if $m < n$, in the case $\mathbb{T} = \mathbb{R}$, we have

$$\sigma(t) = \rho(t) = t, \quad \mu(t) = 0, \quad f^{\Delta}(t) = f'(t), \quad \int_m^n f(t)\Delta t = \int_m^n f(t)dt.$$

In the case $\mathbb{T} = \mathbb{Z}$, we have

$$\sigma(t) = t+1, \rho(t) = t-1, \mu(t) = 1, f^{\Delta}(t) = \Delta f(t) = f(t+1)-f(t), \int_m^n f(t)\Delta t = \sum_{t=m}^{n-1} f(t).$$

In the case $\mathbb{T} = \mathbb{P}_{l,h} := \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l]$ where $l, h > 0$, we have

$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l) \\ t+h & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{k(l+h)+l\}, \end{cases}$$

$$\rho(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty}(k(l+h), k(l+h)+l] \\ t-h & \text{if} \quad t \in \bigcup_{k=1}^{\infty}\{k(l+h)\}, \end{cases}$$

$$\mu(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l) \\ h & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{k(l+h)+l\}, \end{cases}$$

$$f^{\Delta}(t) = \begin{cases} f'(t) & \text{if} \quad t \in \bigcup_{k=0}^{\infty}[k(l+h), k(l+h)+l) \\ \frac{f(\sigma(t))-f(t)}{h} & \text{if} \quad t \in \bigcup_{k=0}^{\infty}\{k(l+h)+l\}, \end{cases}$$

$$\int_m^n f(t)\Delta t = \begin{cases} \int_m^n f(t)dt & \text{if} \quad m, n \in [k(l+h), k(l+h)+l), \ k \in \mathbb{N}_0 \\ hf(m) & \text{if} \quad m \in \{k(l+h)+l\} \text{ and } n = \sigma(m), \ k \in \mathbb{N}_0. \end{cases}$$

In the case $\mathbb{T} = q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}, q > 1\}$, we have

$$\sigma(t) = qt, \quad \rho(t) = \frac{t}{q}, \quad \mu(t) = (q-1)t,$$

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \quad \int_{q^m}^{q^n} f(t)\Delta t = \sum_{i=m}^{n-1} \mu(q^i) f(q^i).$$

Assume $t_0 \in \mathbb{T}$ and sometimes it is convenient to let $t_0 > 0$. The time scale interval $[t_0, \infty)_{\mathbb{T}}$ is defined by $[t_0, \infty) \bigcap \mathbb{T}$. The nontrivial function, $z(t)$, is called the solution of dynamic system

$$z^{\triangle}(t) = f(t, z(t)), \ z \in \mathbb{R}^n, \ t \in \mathbb{T} \tag{2.1}$$

when $z(t) \in \mathbf{C}^1_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^n)$ and satisfies system (2.1). If $z(t)$ also satisfies the initial condition

$$z(t_0) = z_0 \tag{2.2}$$

then $z(t)$ is called the solution of initial value problem (2.1) and (2.2).

Normally, the solution of dynamic equation or dynamic system is said to be oscillatory if it is neither eventually positive nor negative. Otherwise, it is said to be nonoscillatory. Dynamic equation or dynamic system is oscillatory in case all solutions are oscillatory. The solution, $z(t)$, is bounded if there exists a constant $C(t_0, z_0)$ (that may depend on starting time, t_0, and initial condition, z_0) such that $\|z(t)\| \leq C(t_0, z_0)$. The solution is uniformly bounded if C is independent of t_0.

For more details on time scales, we refer the reader to [13–15].

3. OSCILLATION CRITERIA

S. R. Grace et al. [7] investigated the second order nonlinear dynamic equation (1.1) associating with the two following conditions, i.e.,

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s < \infty \tag{3.1}$$

and

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \infty. \tag{3.2}$$

They also established some oscillation criteria. However, we choose only one for each condition and provide some examples.

Theorem 3.1. *Let condition* (3.1) *hold. If there exists a positive nondecreasing delta differentiable function ξ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$,*

$$\limsup_{t \to \infty} \int_{t_1}^{t} \left[\xi(s)q(s) - \delta_1(s)\eta^{\alpha}(s)\xi^{\Delta}(s)\right] \Delta s = \infty \tag{3.3}$$

and

$$\int_{t_1}^{\infty}\left(\frac{1}{a(s)}\int_{t_1}^{s}\theta^{\beta}(u)q(u)\Delta u\right)^{1/\alpha}\Delta s=\infty \tag{3.4}$$

where

$$\delta_1(t)=\begin{cases} c_1, & c_1 \text{ is any positive constant, if } \beta>\alpha\\ 1, & \text{if } \beta=\alpha\\ c_2\eta^{\beta-\alpha}(t), & c_2 \text{ is any positive constant, if } \beta<\alpha,\end{cases}$$

$$\eta(t)=\left(\int_{t_1}^{t}a^{-1/\alpha}(s)\Delta s\right)^{-1} \quad \text{and} \quad \theta(t)=\int_{t}^{\infty}a^{-1/\alpha}(s)\Delta s$$

then equation (1.1) *is oscillatory.*

For example,

1. $\left(t^2x^{\Delta}\right)^{\Delta}(t)+100t^{1/3}x^{1/3}(t)=0$
2. $(t^2x^{\Delta})^{\Delta}(t)+t^4x(t)=0$
3. $(t^2x^{\Delta})^{\Delta}(t)+t^3x^3(t)=0$

Theorem 3.2. *Let condition* (3.2) *hold. If there exists a positive nondecreasing delta differentiable function* ξ *such that for every* $t_1\in[t_0,\infty)_{\mathbb{T}}$ *condition* (3.3) *holds, then equation* (1.1) *is oscillatory.*

For example,

4. $x^{\Delta\Delta}(t)+t^{1/3}x^{1/3}(t)=0$
5. $x^{\Delta\Delta}(t)+|\sin t|x(t)=0$
6. $x^{\Delta\Delta}(t)+t^2x^3(t)=0$
7. $(\frac{1}{t}x^{\Delta})^{\Delta}(t)+\frac{1}{t^2}x(t)=0$
8. $x^{\Delta\Delta}(t)+4x(t)=0$

Not only are these examples represented in the form of dynamic equation (1.1) but they can also be rearranged in the form of dynamic system (1.2) which were considered by Y. Xu and Z. Xu [12]. They established some oscillation criteria where

$$B(t)=\int_{t_0}^{t}b(s)\Delta s,\quad t>t_0 \quad \text{and} \quad C(t)=\int_{\sigma(t)}^{\infty}c(s)\Delta s,\quad t\geq t_0.$$

All examples appeared in this paper can be verified by the following theorem as well.

Theorem 3.3. *Let* $f\in\mathbf{C}^1(\mathbb{R},\mathbb{R})$, $b(t)\geq 0$ *and* $b(t)$ *does not vanish identically in* $[t_0,\infty)_{\mathbb{T}}$. *Suppose that* $C(\tau)=\infty$ *for all* $\tau\in[t_0,\infty)_{\mathbb{T}}$. *Then system* (1.2) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

4. VERIFICATION OF OSCILLATION

Inspired by previous section, the oscillation of each example is theoretically guaranteed in both stated forms. The supportive illustrations are finally demonstrated.

Before exploring the numerical results, it will be more convenient to write dynamic equation (1.1) as the equivalent system

$$x^{\triangle}(t) = \left(\frac{1}{a(t)}y(t)\right)^{1/\alpha} \qquad y^{\triangle}(t) = -q(t)x^{\beta}(t). \tag{4.1}$$

Obviously, we can separate into two cases:

Case 1 If $\mu(t) = 0$, system (4.1) becomes

$$\frac{dx}{dt} = \left(\frac{1}{a(t)}y(t)\right)^{1/\alpha} \qquad \frac{dy}{dt} = -q(t)x^{\beta}(t).$$

Case 2 If $\mu(t) \neq 0$, system (4.1) becomes

$$x(\sigma(t)) = x(t) + \mu(t)\left(\frac{1}{a(t)}y(t)\right)^{1/\alpha}$$
$$y(\sigma(t)) = y(t) - \mu(t)q(t)x^{\beta}(t).$$

The following examples on $\mathbb{R}$, $\mathbb{P}$, $\mathbb{Z}$ and $q^{\mathbb{Z}}$ satisfy condition (3.1).

Example 4.1. Consider the dynamic equation

$$\left(t^2 x^{\Delta}\right)^{\Delta}(t) + 100t^{1/3}x^{1/3}(t) = 0. \tag{4.2}$$

Comparing with equation (1.1), $a(t) = t^2$, $q(t) = \ 100t^{1/3}$ and $\alpha = 1 > \beta = 1/3$. The equivalent system is

$$x^{\triangle}(t) = \frac{y(t)}{t^2} \quad y^{\triangle}(t) = -100(tx)^{1/3}. \tag{4.3}$$

Comparing with system (1.2), $b(t) = 1/t^2$, $c(t) = 100t^{1/3}$, $f(x(t)) = x^{1/3}(t)$ and

$$g(y(t)) = \begin{cases} y(t) & \text{if } \mu = 0 \\ y(t) - \mu y^{\Delta}(t) & \text{if } \mu \neq 0. \end{cases} \tag{4.4}$$

For convenience, all given examples in this paper share the same $g(y(t))$. Next, the oscillation of dynamic equation (4.2) on different time scales is investigated by using S .R. Grace et al. oscillation criteria.

On $\mathbb{R}$ time scale,

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \int_{t_0}^{\infty} s^{-2}ds = -\frac{1}{s}\bigg|_{t_0}^{\infty} = \frac{1}{t_0} < \infty.$$

Obviously, condition (3.1) holds. Next, consider

$$\theta(t) = \int_{t}^{\infty} a^{-1/\alpha}(s)\Delta s = \int_{t}^{\infty} s^{-2}ds = -\frac{1}{s}\bigg|_{t}^{\infty} = \frac{1}{t}.$$

$$\eta(t) = \left(\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s\right)^{-1} = \left(\int_{t_1}^{t} s^{-2} ds\right)^{-1} = -\frac{t_1 t}{t_1 - t}.$$

Then we can choose $\xi(t) = 1$ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$, equation (3.3) is

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[100 s^{1/3} - 0\right] ds = 100 \limsup_{t\to\infty} \left.\frac{3s^{4/3}}{4}\right|_{t_1}^{t} = \infty.$$

Equation (3.4) becomes

$$\int_{t_1}^{\infty} \left(\frac{1}{s^2}\int_{t_1}^{s} \frac{1}{u^{1/3}} 100 u^{1/3} du\right) ds = 100 \int_{t_1}^{\infty} \left(\frac{s - t_1}{s^2}\right) ds = 100 \left.\left(\ln s + \frac{t_1}{s}\right)\right|_{t_1}^{\infty} = \infty.$$

Thus, equation (4.2) is oscillatory by applying Theorem 3.1.

On $\mathbb{P}_{l,h}$ time scale, suppose C_1 is a constant. Equation (3.1) becomes

$$\begin{aligned}
&\int_{t_0}^{k_1(l+h)+l} \frac{1}{s^2} ds + \frac{h}{(k_1(l+h)+l)^2} + \sum_{k=k_1+1}^{\infty} \int_{k(l+h)}^{k(l+h)+l} \frac{1}{s^2} ds + \sum_{k=k_1+1}^{\infty} \frac{h}{(k(l+h)+l)^2} \\
&= -\left.\frac{1}{s}\right|_{t_0}^{k_1(l+h)+l} + \frac{h}{(k_1(l+h)+l)^2} + \sum_{k_1=k+1}^{\infty} -\left.\frac{1}{s}\right|_{k(l+h)}^{k(l+h)+l} + \sum_{k=k_1+1}^{\infty} \frac{h}{(k(l+h)+l)^2} \\
&= C_1 + \sum_{k=k_1+1}^{\infty} \left(-\frac{1}{k(l+h)+l} + \frac{1}{k(l+h)}\right) + \sum_{k=k_1+1}^{\infty} h(k(l+h)+l)^{-2}.
\end{aligned}$$

Obviously, the convergence of two infinite series must be investigated. Firstly, let $f(x) = -\frac{1}{x(l+h)+l} + \frac{1}{x(l+h)}$. By using the integral test, the former series converges as described below,

$$\begin{aligned}
\int_{1}^{\infty} f(x) dx &= \lim_{b\to\infty} \int_{1}^{b} \left[\frac{-1}{(l+h)x+l} + \frac{1}{(l+h)x}\right] dx \\
&= \lim_{b\to\infty} \frac{1}{l+h}\left[\ln\left|\frac{(l+h)b}{(l+h)b+h}\right| - \ln\left|\frac{l+h}{l+2h}\right|\right] \\
&= -\frac{1}{l+h} \ln\left|\frac{l+h}{l+2h}\right| < \infty.
\end{aligned}$$

Next, under algebraic operations, we know that

$$\begin{gathered}
k(l+h) < k(l+h) + l \\
\frac{1}{k(l+h)} > \frac{1}{k(l+h)+l} \\
\sum_{k=k_1+1}^{\infty} \frac{1}{(k(l+h))^2} > \sum_{k=k_1+1}^{\infty} \frac{1}{(k(l+h)+l)^2}.
\end{gathered}$$

As a conclusion, the latter series converges by using the comparison test together with the fact that $\sum_{k=k_1+1}^{\infty} \frac{1}{k^2}$ is P series where $p = 2 > 1$. Thus, condition (3.1) holds. Let C_2 be a positive constant. Then $\theta(t)$ is

$$\int_t^{k_1(l+h)+l} \frac{1}{s^2}ds + \sum_{k=k_1}^{\infty} \frac{h}{(k(l+h)+l)^2} + \sum_{k=k_1+1}^{\infty} \int_{k(l+h)}^{k(l+h)+l} \frac{1}{s^2}ds$$

$$= \frac{-1}{k_1(l+h)+l} + \frac{1}{t} + \sum_{k=k_1}^{\infty} \frac{h}{(k(l+h)+l)^2} + \sum_{k=k_1+1}^{\infty} \left[\frac{-1}{k(l+h)+l} + \frac{1}{k(l+h)}\right]$$

$$= \frac{1}{t} + \sum_{k=k_1}^{\infty} \frac{h}{(k(l+h)+l)^2} - \sum_{k=k_1}^{\infty} \frac{h}{[k(l+h)+l][(k+1)(l+h)]} = \frac{1}{t} + C_2$$

and $\eta(t)$ is

$$\left[\int_{t_1}^{k_1(l+h)+l} \frac{1}{s^2}ds + \sum_{k=k_1}^{k_n-1} \frac{h}{(k(l+h)+l)^2} + \sum_{k=k_1+1}^{k_n-1} \int_{k(l+h)}^{k(l+h)+l} \frac{1}{s^2}ds + \int_{k_n(l+h)}^{t} \frac{1}{s^2}ds\right]^{-1}.$$

In the same manner, $\xi(t) = 1$ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$, equation (3.3) is

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(100s^{1/3} - 0\right)\Delta s$$

$$= 100\left[\int_{t_1}^{k_1(l+h)+l} s^{1/3}ds + \sum_{k=k_1}^{\infty} s^{1/3} + \sum_{k=k_1+1}^{\infty} \int_{k(l+h)}^{k(l+h)+l} s^{1/3}ds\right] = \infty.$$

Suppose C_3 is a constant. Then equation (3.4) is

$$\int_{t_1}^{\infty} \left(\frac{1}{s^2}\int_{t_1}^{s} \left(\frac{1}{u} + C_1\right)^{1/3} 100u^{1/3}\Delta u\right)\Delta s$$

$$\geqslant \int_{t_1}^{\infty} \left(\frac{1}{s^2}\int_{t_1}^{s} 100\Delta u\right)\Delta s$$

$$= 100\int_{t_1}^{\infty} \frac{1}{s^2}\left[\int_{t_1}^{k_1(l+h)+l} du + \sum_{k=k_1}^{k_n-1} h + \sum_{k=k_1+1}^{k_n-1} \int_{k(l+h)}^{k(l+h)+l} du + \int_{k_n(l+h)}^{s} du\right]\Delta s$$

$$= 100\int_{t_1}^{\infty} \frac{s-t_1}{s^2}\Delta s$$

$$= 100\left[\int_{t_1}^{k_1(l+h)+l} \frac{s-t_1}{s^2}ds + \sum_{k=k_1}^{\infty} \frac{s-t_1}{s^2} + \sum_{k=k_1+1}^{\infty} \int_{k(l+h)}^{k(l+h)+l} \frac{s-t_1}{s^2}ds\right]$$

$$= 100\left[C_3 + \sum_{k=k_1}^{\infty} h\left(\frac{-t_1}{(k(l+h)+l)^2} + \frac{1}{k(l+h)+l}\right)\right].$$

In addition, $\sum_{k=k_1}^{\infty} \frac{h}{k(l+h)+l}$ is greater than $\sum_{k=k_1}^{\infty} \frac{h}{(k+1)(l+h)}$ which diverges. By using the comparison test, $\sum_{k=k_1}^{\infty} \frac{h}{k(l+h)+l}$ diverges. Then, $\int_{t_1}^{\infty} \left(\frac{1}{a(s)}\int_{t_1}^{s} \theta^{\beta}(u)q(u)\Delta u\right)^{1/\alpha}\Delta s$ $= \infty$. Consequently, equation (4.2) is oscillatory.

On $\mathbb{Z}$ time scale, suppose C, C_1, C_2 are constant. Then

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \sum_{s=t_0}^{\infty} s^{-2} < \infty$$

as same reasoning as on $\mathbb{P}$. Hence condition (3.1) holds. Furthermore,

$$\theta(t) = \int_t^\infty a^{-1/\alpha}(s)\Delta s = \sum_{s=t}^{\infty} s^{-2} = C_1.$$

$$\eta(t) = \left(\int_{t_1}^{t} a^{-1/\alpha}(s)\Delta s\right)^{-1} = \left(\sum_{s=t_1}^{t-1} s^{-2}\right)^{-1} = C_2.$$

Then we choose $\xi(t) = 1$ such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$, equation (3.3) becomes

$$\limsup_{t\to\infty} \sum_{s=t_1}^{t} \left(100s^{1/3} - 0\right) = 100 \sum_{s=t_1}^{\infty} \left(\frac{1}{s^{-1/3}}\right) = \infty$$

and equation (3.4) is

$$\int_{t_1}^{\infty} \left(\frac{1}{s^2} \sum_{u=t_1}^{s-1} \left(\sum_{s=u}^{\infty} s^{-2}\right)^{1/3} 100u^{1/3} \right) \Delta s = \int_{t_1}^{\infty} \left(\frac{1}{s^2} \sum_{u=t_1}^{s-1} C^{1/3} 100u^{1/3} \right) \Delta s$$
$$= 100C^{1/3} \int_{t_1}^{\infty} \left(\frac{1}{s^2} \sum_{u=t_1}^{s-1} u^{1/3} \right) \Delta s.$$

Let $s = t_1 + n$. Then we obtain

$$100C^{1/3} \int_{t_1}^{\infty} \left(\frac{1}{s^2} \sum_{u=t_1}^{t_1+n-1} u^{1/3} \right) \Delta s \le 100C^{1/3} \int_{t_1}^{\infty} \left(\frac{1}{s^2} n t_1^{1/3} \right) \Delta s$$
$$= 100C^{1/3} \sum_{n=0}^{\infty} \left(\frac{n t_1^{1/3}}{(t_1+n)^2} \right).$$

According to the integral test, we must evaluate

$$\int_1^\infty \frac{n}{(t_1+n)^2} dn.$$

After letting $u = t_1 + n$ and $du = dn$, we obtain

$$\int_1^\infty \frac{u - t_1}{u^2} du = \int_1^\infty \left(\frac{1}{u} - \frac{t_1}{u^2}\right) du = \infty.$$

Therefore, equation (4.2) is oscillatory.

On $2^{\mathbb{Z}}$ time scale, given $t_0 = 2^0$, $s = 2^i$ and $f(s) = s^{-2}$ where $i \in \mathbb{N}_0$,

$$\int_{t_0}^{\infty} a^{-1/\alpha}(s)\Delta s = \sum_{s=t_0}^{\infty} \mu(s) s^{-2} = \sum_{i=0}^{\infty} (2-1)2^i 2^{-2i} = \sum_{i=0}^{\infty} \frac{1}{2^i}.$$

This geometric series converges to 2. Hence condition (3.1) holds. Let $t = 2^n$ where $n \in \mathbb{N}_0$. We obtain

$$\theta(t) = \sum_{i=n}^{\infty} \frac{1}{2^i} = \frac{2}{2^n} = \frac{2}{t}.$$

Let $t_1 = 2^m$ and $t = 2^n$ where $m, n \in \mathbb{N}_0$ and $n > m$. We get

$$\eta(t) = \left(\sum_{s=t_1}^{t-1} \mu(s)s^{-2}\right)^{-1} = \left(\sum_{i=m}^{n-1} \frac{1}{2^i}\right)^{-1} = \frac{2^m(2^{n-1})}{2^n - 1} = \frac{t_1 t}{2(t-1)}.$$

Again, $\xi(t) = 1$ is chosen such that for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$, equation (3.3) is

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left(100s^{1/3} - 0\right) \Delta s = 100 \limsup_{t\to\infty} \sum_{s=m}^{t} 2^s (2^s)^{1/3} = \infty$$

and by setting $t_1 = 2^m$ and $s = 2^{(m+r)}$ where $m, r \in \mathbb{N}_0$, equation (3.4) is

$$\int_{t_1}^{\infty} \left(\frac{1}{s^2} \int_{t_1}^{s} \left(\frac{2}{u}\right)^{1/3} 100u^{1/3} \Delta u\right) \Delta s = 2^{1/3} 100 \int_{t_1}^{\infty} \left(\frac{1}{s^2} \sum_{u=m}^{m+r-1} 2^u\right) \Delta s$$

$$= 2^{1/3} 100 \int_{t_1}^{\infty} \frac{1}{s^2}(2^{m+r} - 2^m) \Delta s = 2^{1/3} 100 \sum_{r=0}^{\infty} \frac{2^s}{2^{2s}} (2^{m+r} - 2^m)$$

$$= 2^{1/3} 100 \sum_{n=0}^{\infty} \left(1 - \frac{1}{2^n}\right) = \infty.$$

Therefore, equation (4.2) is oscillatory.

An alternative approach for theoretical verification of oscillation begins by converting equation (4.2) to system (4.3) and applying Theorem 3.3. Here, $f(x) = x^{1/3}$. Clearly, $f \in \mathbf{C}^1(\mathbb{R} \to \mathbb{R})$ and its first derivative is continuous everywhere except at $x = 0$. Moreover, $b(t) = \frac{1}{t^2} > 0$ and $c(t) = q(t)$. Using the same technique as before, $C(t)$ can be evaluated on involving time scales.

For all numerical solutions of equation (4.2), the initial condition is $(x_0, y_0) = (1, 1)$ and the starting time is $t_0 = 1$. In Figure 1 (top-left), we choose $\mathbb{T} = \mathbb{R}$, step size $H = 0.01$ and running step $M = 5,000$. In Figure 1 (top-right), we choose $\mathbb{T} = \mathbb{P}_{1,0.5}$, length of continuous interval $l = 1$, step size $H = 0.01$, running step $M = 100$, forward jump distance $h = 0.5$ and number of continuous interval $P = 100$. In Figure 1 (bottom-left), we choose $\mathbb{T} = \mathbb{Z}$, step size $H = 1$ and running step $M = 500$ while Figure 1 (bottom-right) shows the solution behavior on $\mathbb{T} = \{2^k : k \in [0, 10]\}$. Excluding Figure 1 (bottom-right), the pattern of illustrations are oscillatory and bounded.

Later on, the theoretical verifications of oscillation are omitted due to similar technique whereas, the numerical verifications of oscillation together with all parameter values are finally provided.

Example 4.2. Consider the dynamic equation

$$(t^2 x^{\Delta})^{\Delta}(t) + t^4 x(t) = 0. \tag{4.5}$$

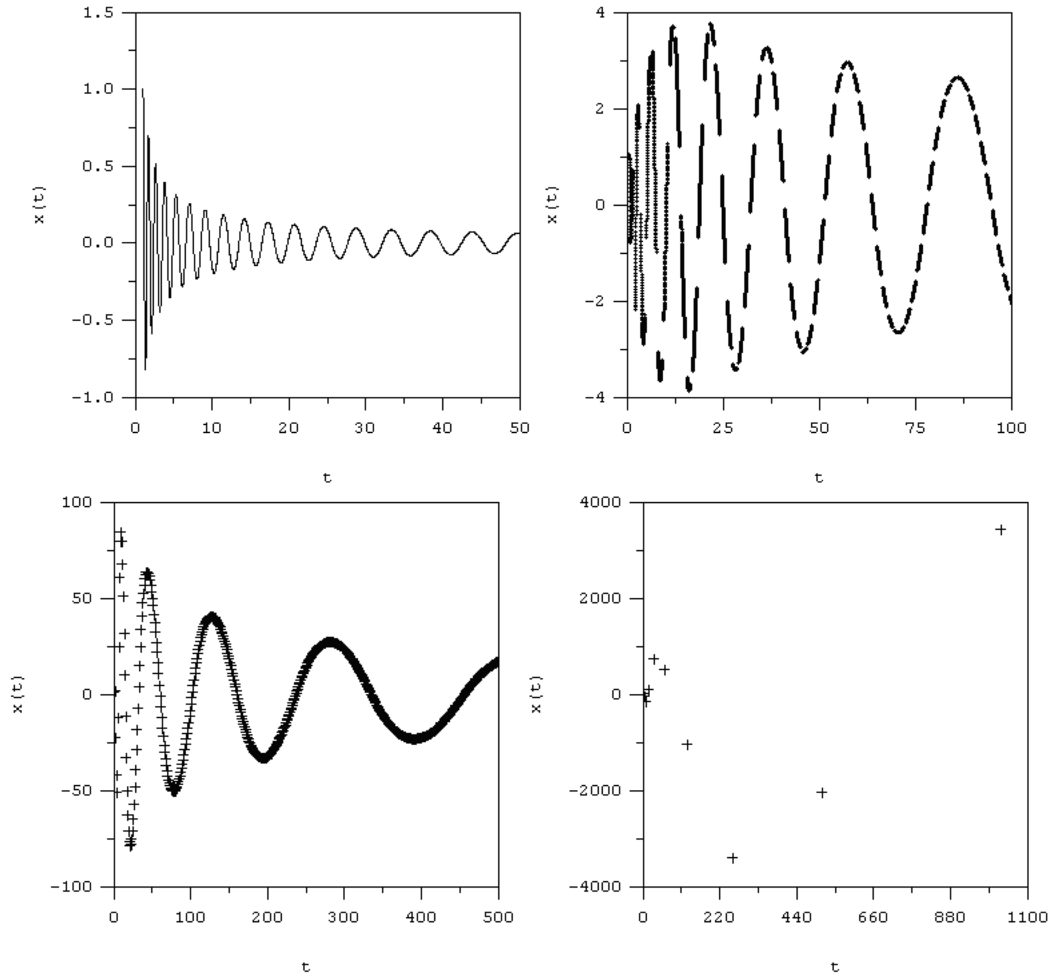

FIGURE 1. $\left(t^2x^{\Delta}\right)^{\Delta}(t) + 100t^{1/3}x^{1/3}(t) = 0.$

Comparing with equation (1.1), $a(t) = t^2$, $q(t) = t^4$ and $\alpha = \beta = 1$. The equivalent system is

$$x^{\triangle}(t) = \frac{1}{t^2}y(t) \quad y^{\triangle}(t) = -t^4x(t). \tag{4.6}$$

Comparing with system (1.2), $b(t) = \frac{1}{t^2}$, $c(t) = t^4$ and $f(x(t)) = x(t)$.

For all numerical solutions of equation (4.5), the initial condition is $(x_0, y_0) = (1, 1)$ and the starting time is $t_0 = 1$. In Figure 2 (left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$ and $M = 1,000$. In Figure 2 (right), we choose $\mathbb{T} = \mathbb{P}_{2,0.5}$, $l = 2$, $H = 0.01$, $M = 200$, $h = 0.5$ and $P = 5$. Obviously, both figures reveal different solution behaviors. The bounded and unbounded oscillatory solutions are appeared in the left and right figures, respectively.

Example 4.3. Consider the dynamic equation

$$(t^2x^{\Delta})^{\Delta}(t) + t^3x^3(t) = 0. \tag{4.7}$$

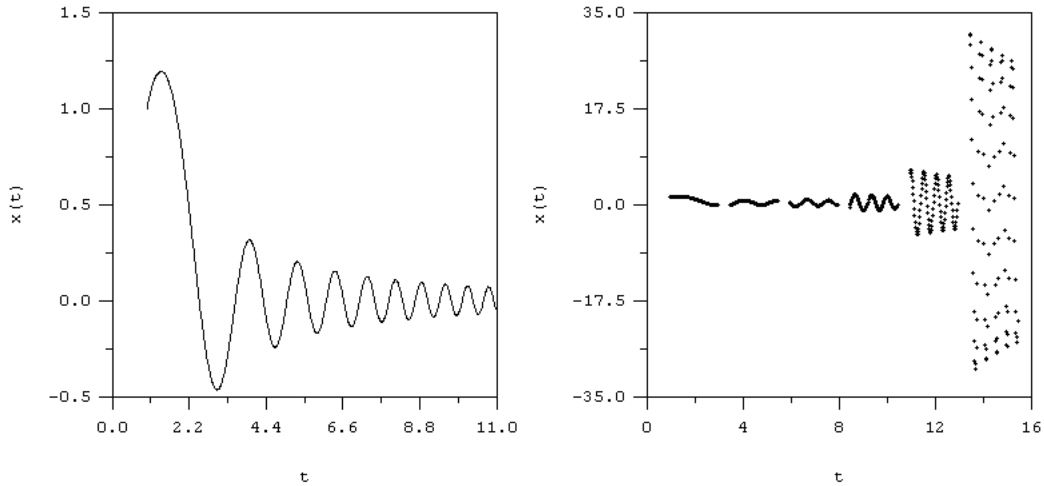

FIGURE 2. $(t^2x^{\Delta})^{\Delta}(t) + t^4x(t) = 0$.

Comparing with equation (1.1), $a(t) = t^2$, $q(t) = t^3$ and $\alpha = 1 < \beta = 3$. The equivalent system is

$$x^{\triangle}(t) = \frac{y(t)}{t^2} \quad y^{\triangle}(t) = -t^3x^3(t). \tag{4.8}$$

Comparing with system (1.2), $b(t) = \frac{1}{t^2}$, $c(t) = t^3$ and $f(x(t)) = x^3(t)$.

For all numerical solutions of (4.7), the initial condition is $(x_0, y_0) = (2, 0.5)$ and the starting time is $t_0 = 1$. The necessary values in case $\mathbb{T} = \mathbb{R}$ are $H = 0.01$ and $M = 10,000$ and in case $\mathbb{T} = \mathbb{P}_{1,0.5}$ are $l = 1$, $H = 0.01$, $M = 100$, $h = 0.5$ and $P = 10$. Here, both graphs are omitted because the solution behaviors of previous and present examples are alike.

Next, the following equations on $\mathbb{R}, \mathbb{P}$ and $\mathbb{Z}$ satisfy condition (3.2).

Example 4.4. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + t^{1/3}x^{1/3}(t) = 0. \tag{4.9}$$

Comparing with equation (1.1), $a(t) = 1$, $q(t) = t^{1/3}$ and $\alpha = 1 > \beta = 1/3$. The equivalent system is

$$x^{\triangle}(t) = y(t) \quad y^{\triangle}(t) = -(tx)^{1/3}. \tag{4.10}$$

Comparing with system (1.2), $b(t) = 1$, $c(t) = t^{1/3}$, and $f(x(t)) = x^{1/3}(t)$.

For all numerical solutions of equation (4.9), the initial condition is $(x_0, y_0) = (1, 1)$ and the starting time is $t_0 = 1$. In Figure 3 (top-left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$ and $M = 5,000$. In Figure 3 (top-right), we choose $\mathbb{T} = \mathbb{P}_{0.9,0.1}$, $l = 0.9$, $H = 0.01$, $M = 90$, $h = 0.1$ and $P = 50$. In Figure 3 (bottom-left), we choose $\mathbb{T} = \mathbb{P}_{0.1,0.2}$, $l = 0.1$, $H = 0.01$, $M = 10$, $h = 0.2$ and $P = 100$. In Figure 3 (bottom-right), we choose $\mathbb{T} = \mathbb{Z}$, $H = 1$ and $M = 30$. The oscillatory solutions gradually

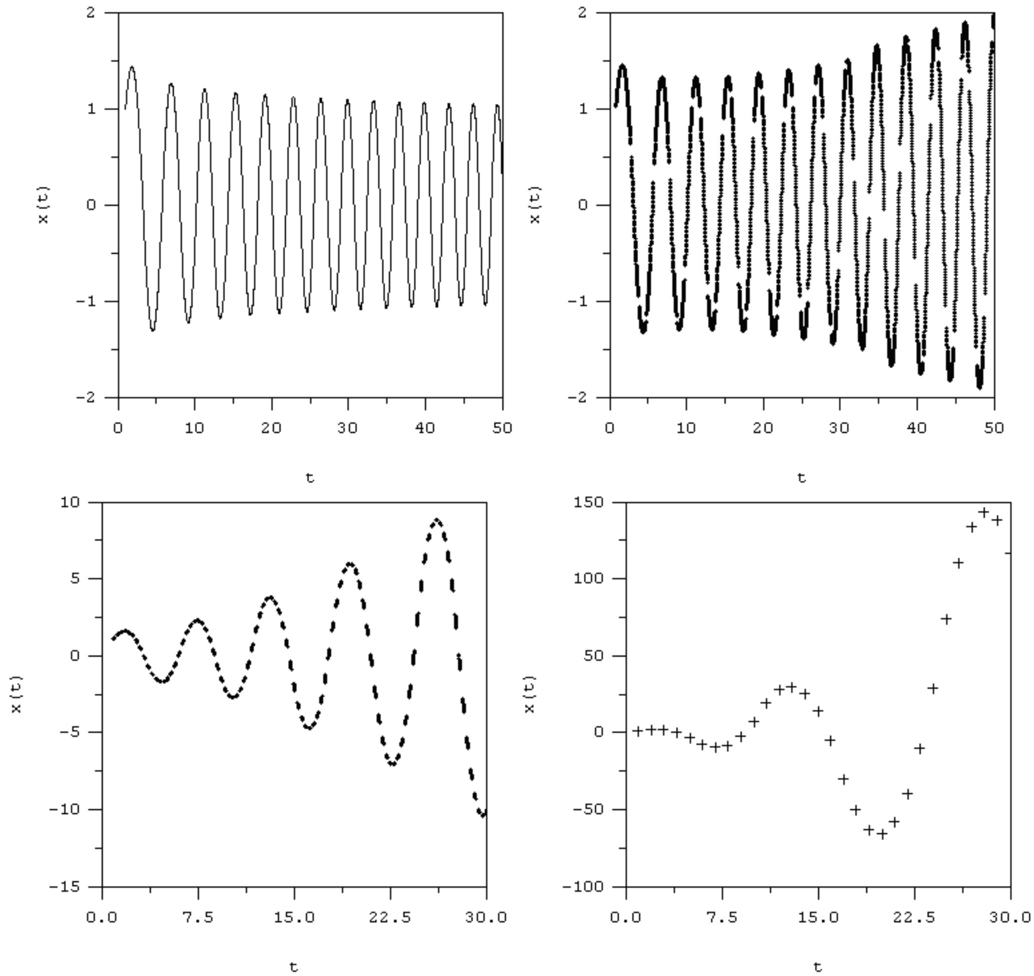

FIGURE 3. $x^{\Delta\Delta}(t) + t^{1/3}x^{1/3}(t) = 0$

change from bounded to unbound when the discrete jump points occur and the jump distance increases.

Example 4.5. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + |\sin t|x(t) = 0. \tag{4.11}$$

Comparing with equation (1.1), $a(t) = 1$, $q(t) = \ |\sin t|$, and $\alpha = 1 = \beta$. The equivalent system is

$$x^{\triangle}(t) = y(t) \quad y^{\triangle}(t) = -|\sin t|x(t). \tag{4.12}$$

Comparing with system (1.2), $b(t) = 1$, $c(t) = |\sin t|$ and $f(x(t)) = x(t)$.

All numerical solutions of equation (4.11), the initial condition is $(x_0, y_0) = (1, 0.4)$ and the starting time is $t_0 = 0$. In Figure 4 (top-left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$ and $M = 6,000$. In Figure 4 (top-right), we choose $\mathbb{T} = \mathbb{P}_{0.9,0.1}$, $l = 0.9$, $H = 0.01$, $M = 90$, $h = 0.1$ and $P = 60$. In Figure 4 (bottom-left), we choose

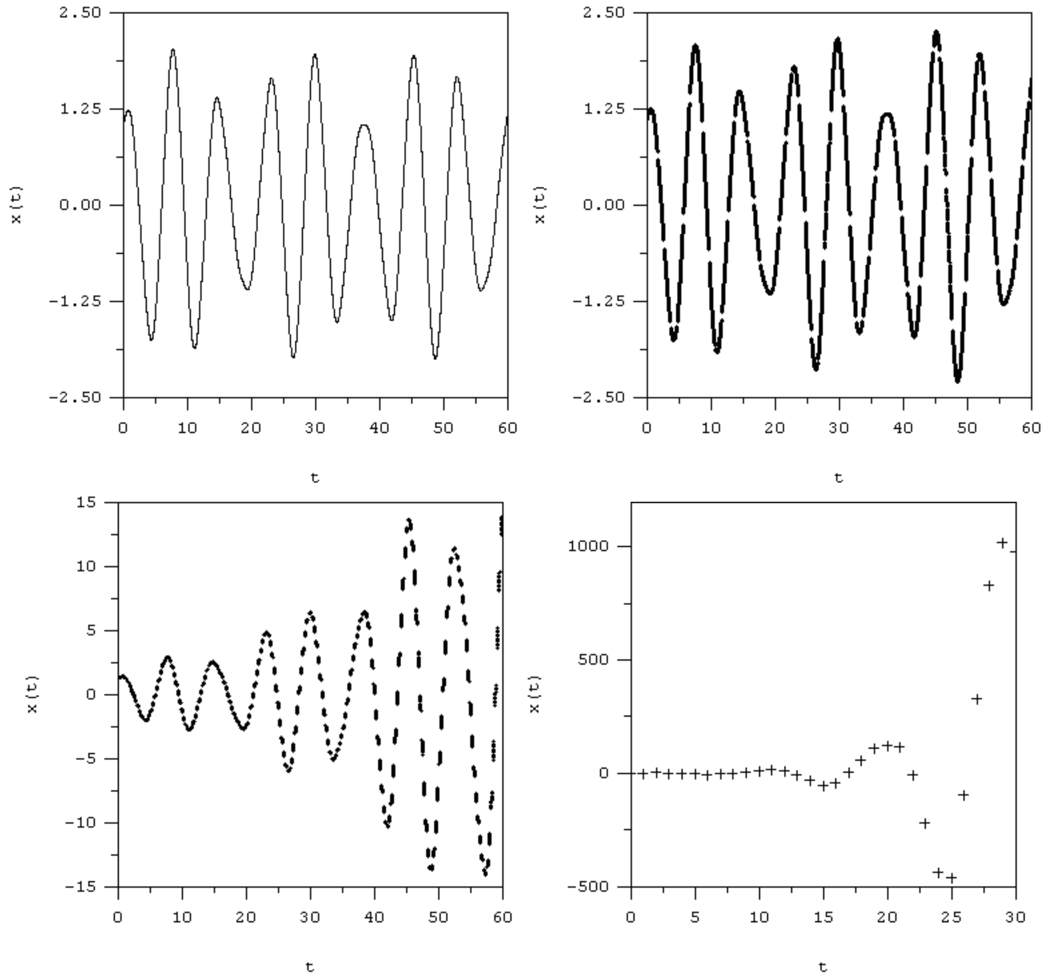

FIGURE 4. $x^{\Delta\Delta}(t) + |\sin t|x(t) = 0$.

$\mathbb{T} = \mathbb{P}_{0.1,0.2}$, $l = 0.1$, $H = 0.01$, $M = 10$, $h = 0.2$ and $P = 200$. In Figure 4 (bottom-right), we choose $\mathbb{T} = \mathbb{Z}$, $H = 1$ and $M = 30$. More interesting patterns of oscillation are visualized.

Example 4.6. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + t^2 x^3(t) = 0. \tag{4.13}$$

Comparing with equation (1.1), $a(t) = 1$, $q(t) = t^2$ and $\alpha = 1 < \beta = 3$. The equivalent system is

$$x^{\triangle}(t) = y(t) \quad y^{\triangle}(t) = -t^2 x^3(t). \tag{4.14}$$

Comparing with system (1.2), $b(t) = 1$, $c(t) = t^2$ and $f(x(t)) = x^3(t)$.

For all numerical solutions of equation (4.13), the initial condition is $(x_0, y_0) = (1, 1)$ and the starting time is $t_0 = 1$. In Figure 5 (top-left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$ and $M = 2,000$. In Figure 5 (top-right), we choose $\mathbb{T} = \mathbb{P}_{1,0.01}$, $l = 1$, $H = 0.01$, $M = 100$, $h = 0.01$ and $P = 20$. In Figure 5 (bottom-left), we choose $\mathbb{T} = \mathbb{P}_{0.9,0.1}$,

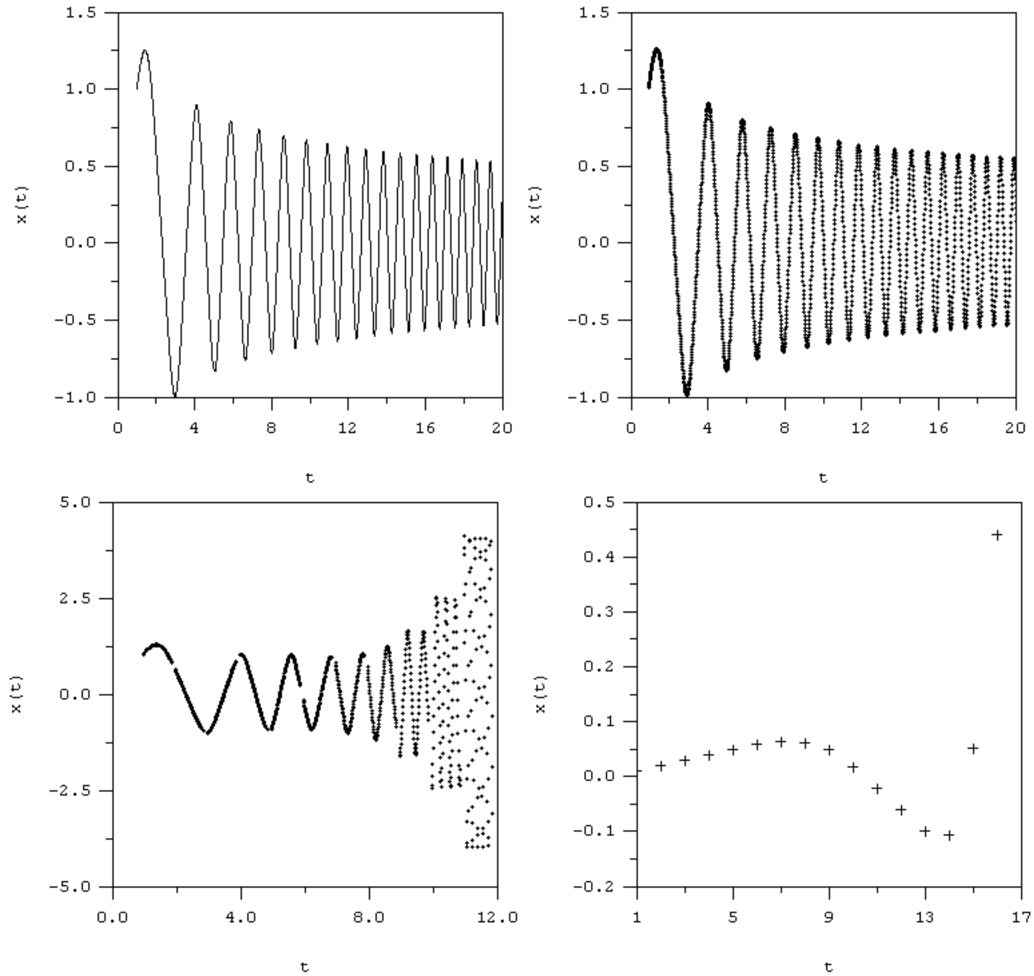

FIGURE 5. $x^{\Delta\Delta}(t) + t^2 x^3(t) = 0$.

$l = 0.9$, $H = 0.01$, $M = 90$, $h = 0.1$ and $P = 10$. In Figure 5 (bottom-right), we choose $\mathbb{T} = \mathbb{Z}$, initial condition $(x_0, y_0) = (0.01, 0.01)$ and $t \in [1, 16]$. Obviously, the solution behavior is bounded when the jump distance is small. However, the oscillatory solution become unbounded when the jump distance is large enough.

Example 4.7. Consider the dynamic equation

$$\left(\frac{1}{t}x^{\Delta}\right)^{\Delta}(t) + \frac{1}{t^2}x(t) = 0. \tag{4.15}$$

Comparing with equation (1.1), $a(t) = \frac{1}{t}$, $q(t) = \frac{1}{t^2}$ and $\alpha = 1 = \beta$. The equivalent system is

$$x^{\triangle}(t) = ty(t) \quad y^{\triangle}(t) = -\frac{1}{t^2}x(t). \tag{4.16}$$

Comparing with the system (1.2), $b(t) = t$, $c(t) = \frac{1}{t^2}$ and $f(x(t)) = x(t)$.

For all numerical solutions of equation (4.15), the initial condition is $(x_0, y_0) = (1, 1)$ and the starting time is $t_0 = 1$. In Figure 6 (left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$

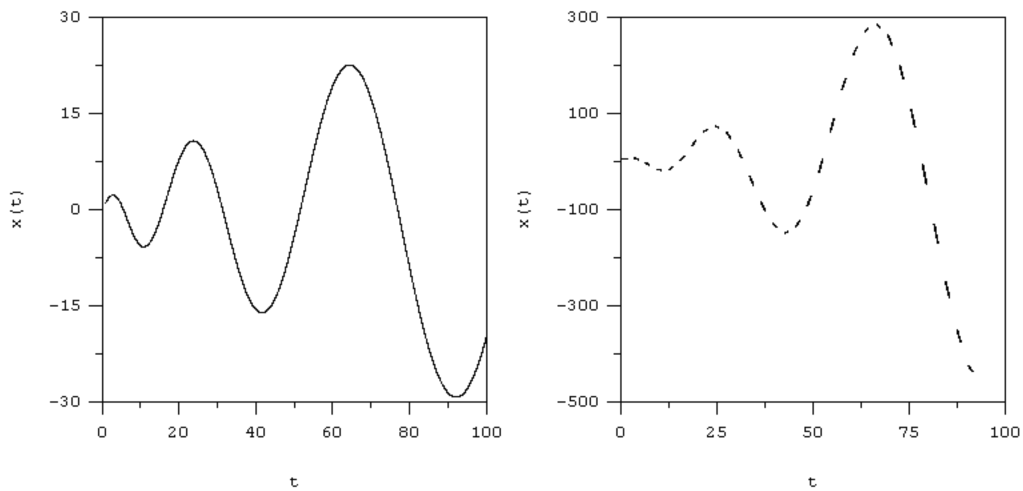

FIGURE 6. $(\frac{1}{t}x^{\Delta})^{\Delta}(t) + \frac{1}{t^2}x(t) = 0$.

and $M = 10,000$. In Figure 6 (right), we choose $\mathbb{T} = \mathbb{P}_{1,2}$, $l = 1$, $H = 0.01$, $M = 100$, $h = 2$ and $P = 30$. Both oscillatory solutions are unbounded.

Obviously, it is difficult to find closed-form solutions of the earlier examples. However, these examples do have solutions which are graphically shown in previous figures. In order to understand solution behavior better, we propose Example 4.8 which can be solved analytically.

Example 4.8. Consider the dynamic equation

$$x^{\Delta\Delta}(t) + 4x(t) = 0. \tag{4.17}$$

Comparing with equation (1.1), $a(t) = 1$, $q(t) = 4$ and $\alpha = 1 = \beta$. The equivalent system is

$$x^{\triangle}(t) = y(t) \quad y^{\triangle}(t) = -4x(t). \tag{4.18}$$

Comparing with system (1.2), $b(t) = 1$, $c(t) = 4$ and $f(x(t)) = x(t)$.

In fact, the general solution of equation (4.17) on $\mathbb{T} = \mathbb{R}$ is given by $x(t) = C_1 \cos 2t + C_2 \sin 2t$ where C_1, C_2 are constant. $x(t)$ is uniformly bounded because

$$\|x(t)\| = \|C_1 \cos 2t + C_2 \sin 2t\| \le |C_1| + |C_2|.$$

Next, the general solution of equation (4.17) on $\mathbb{T} = \mathbb{Z}$ is given by $x(t) = (C_1 + C_2 t)x_0$ where C_1, C_2 are constant. For $t \in [t_0, \infty)$,

$$\|x(t)\| = \|(C_1 + C_2 t)x_0\| = |C_1 + C_2 t||x_0| = \infty.$$

Therefore, $x(t)$ is unbounded.

Now we turn our attention to $\mathbb{P}_{l,h}$ time scale which combines the continuous time intervals $[k(l+h), k(l+h)+l)$ and the discrete jump points $k(l+h)+l$ where $k \in \mathbb{N}_0$. In general, the solutions of equation (4.17) on $\mathbb{R}$ and $\mathbb{P}_{l,h}$ are the same but the initial

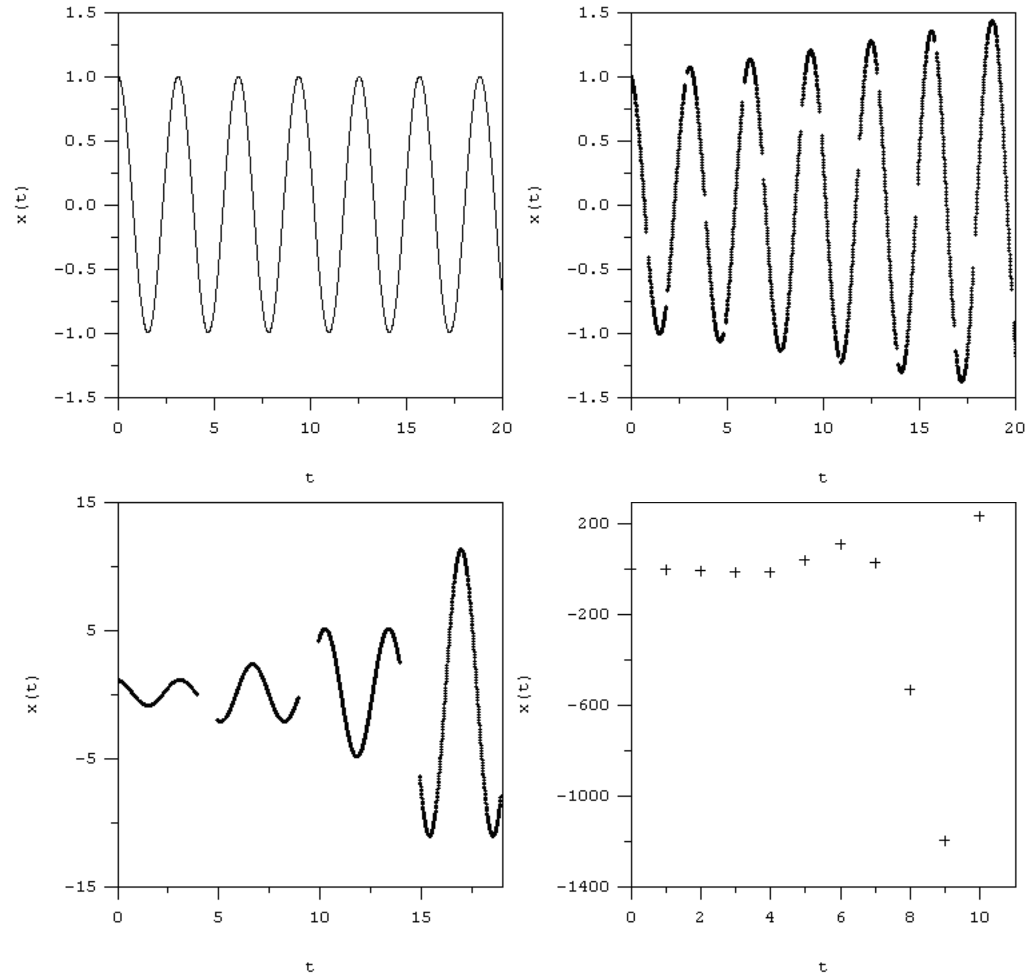

FIGURE 7. $x^{\Delta\Delta}(t) + 4x(t) = 0$.

condition is assigned differently. For more detail, the solutions of equation (4.17) on $\mathbb{R}$ and $\mathbb{P}_{l,h}$ coincide in the first interval if we start with the same initial condition. Then the discrete jump point occurs at the end of interval. Consequently,

$$x(\sigma(t)) = x(t) + x^{\Delta}(t)\mu(t)$$

and the initial conditions for the next interval need to be adjusted.

For all numerical solutions of equation (4.17), the initial condition is $(x_0, y_0) = (1, 0)$ and the starting time is $t_0 = 0$. In Figure 7 (top-left), we choose $\mathbb{T} = \mathbb{R}$, $H = 0.01$ and $M = 2,000$. In Figure 7 (top-right), we choose $\mathbb{T} = \mathbb{P}_{0.9,0.1}$, $l = 0.9$, $H = 0.01$, $M = 90$, $h = 0.1$ and $P = 20$. In Figure 7 (bottom-left), we choose $\mathbb{T} = \mathbb{P}_{4,1}$, $l = 4$, $H = 0.01$, $M = 400$, $h = 1$ and $P = 3$. In Figure 7 (bottom-right), we choose $\mathbb{T} = \mathbb{Z}$, $H = 1$ and $M = 10$. The oscillatory solution is bounded on $\mathbb{T} = \mathbb{R}$ only.

5. CONCLUSION

Obviously, all pictures share the same property (i.e., oscillation). It is amazing that the same dynamic equation can perform the different patterns in the numerical solutions after a time scale is changed. Throughout all examples, we can finally conclude that for each dynamic equation, if the bounded oscillation appears on a continuous time scale then the bounded oscillation probably appears on other discrete time scales too as shown in Example 4.1 and Example 4.6. However, if the unbounded oscillation exists on a continuous time scale then the bounded oscillation can no longer exists on any time scales as revealed in Example 4.7.

In addition, the forward jump operator plays an important role. When the jump distance is large enough, the oscillatory solution is growing as shown in Figure (2-4). Actually, equation (4.17) on $\mathbb{P}$ time scale is discovered that the solution behavior in each continuous time interval is locally bounded because norm of solution is less than a constant. It can be easily noticed when the length of interval is extended. According to the discrete jump point, the initial condition is always modified when the jump occurs. Consequently, norm of solution in the next interval is less than that the present interval. This is the reason why the global solution behavior looks unbounded.

All of our examples are sensitive to time. Truly speaking, there are a lot of papers mentioning about oscillation criteria but not many of them stating about the boundedness of oscillation. Thus, theoretical prediction on the boundedness of oscillation is await for further study.

6. ACKNOWLEDGEMENT

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7. **APPENDIX**

An algorithm for computing the numerical solution of dynamic system on four time scales.

INPUT: A: starting time X,Y: Initial value

M: Number of step size (Case 2, M=l/H)

H: Step size for continuous interval

Case1: Real number

```
    FOR j=0,1,2,...,M DO
        T=A+H*(j)
        SAVE T,X,Y in the OUTPUT LIST
        RUNGE-KUTTA-FEHLBERG METHOD
    END FOR
```

case 2: $P_{l,h}$ P: Numbers of period

l: Length of continuous interval h: Length of jump

```
        DUM3=A
        FOR j=0,1,2,...,M DO
            T=DUM3+H*(j)
            SAVE T,X,Y in the OUTPUT LIST
            DUM1=X
```

```
            DUM2=Y
            RUNGE-KUTTA-FEHLBERG METHOD
        END FOR
        X=DUM1+h*F(T,DUM1,DUM2)
        Y=DUM2+h*G(T,DUM1,DUM2)
    FOR k=1,2,3,..,P DO
        DUM3=A+k*(l+h)
        FOR j=0,1,2,...,M DO
            T=DUM3+H*(j)
            SAVE T,X,Y in the OUTPUT LIST
            DUM1=X
            DUM2=Y
            RUNGE-KUTTA-FEHLBERG METHOD
        END FOR
        X=DUM1+h*F(T,DUM1,DUM2);
        Y=DUM2+h*G(T,DUM1,DUM2);
    END FOR
case 3: Equal space
    DUM3=A
    FOR j=0,1,2,...,M DO
        T=DUM3+H*(j)
        SAVE T,X,Y in the OUTPUT LIST
        DUM1=X
        DUM2=Y
        X=DUM1+H*F(T,DUM1,DUM2)
        Y=DUM2+H*G(T,DUM1,DUM2)
    END FOR
case 4: b^k b: Value of base
    FOR k=0,1,2,...,M DO
        DUM3=pow(b,k);
        T=DUM3;
        SAVE T,X,Y in the OUTPUT LIST
        DUM1=X
        DUM2=Y
        X=DUM1+(b-1.)*DUM3*F(DUM3,DUM1,DUM2)
        Y=DUM2+(b-1.)*DUM3*G(DUM3,DUM1,DUM2)
    END FOR
```

F:= $X^{\Delta}(T, X, Y)$
G:= $Y^{\Delta}(T, X, Y)$