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TWO-ORDER UPWIND FINITE VOLUME METHOD FOR 1D SHALLOW WATER EQUATIONS

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ABSTRACT. In this paper, we apply two-order upwind finite volume method to analyze 1D shallow water model numerically. we present a full discrete generalized upwind finite volume scheme and its L^2 error estimates for the 1D shallow water equations and we show the error estimate is $O(\Delta t + h^{3/2})$. At last, we give numerical experiments that demonstrate the efficiency of our method .

Keywords: Shallow water equations; Finite volume method; Error estimate

1. INTRODUCTION

Numerical models based on the nonlinear shallow water equations (SWE) are used to model predominantly horizontal, free surface flows such as in shallow lake, wide rivers, estuaries and the coastal zone. In recent years, numerical methods for the SWE have attracted many attentions.

Computational techniques using finite difference, finite element and finite volume methods have been extensively reported in the literature. Although widely applied to shallow water equations, the finite difference technique has the major drawback that it does not guarantee strict conservation of mass and momentum. Furthermore, the necessity of including process across a range of spatial scales means that techniques capable of operating on unstructured meshes will be more appropriate than those such as the finite difference methods which rely on structured and often regular meshes. The finite element method has been used with irregular meshes of triangular or quadrilateral elements to model shallow water flows (Heniche et al., 2000; González, 2005). However, the finite element method can experience difficulty when both subcritical and supercritical flows are encountered (Akanbi and Katopodes, 1988), and may produce solutions with local mass conservation errors in some implementations (Horrit, 2000). The finite volume method is therefore adopted in the present work. For a

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comprehensive review of recent developments in finite volume methods for shallow water equations we refer to (Toro, 2001).

Various numerical methods developed for general systems of hyperbolic conservation laws have been applied to the shallow water equations. For instance, most shock-capturing finite volume schemes for shallow water equations are based on approximate Riemann solvers which have been originally designed for hyperbolic systems without accounting for source terms such as bed slopes and friction losses. Therefore, most of these schemes suffer from numerical instability and may produce nonphysical oscillations mainly because discrete of the flux and source terms are not well-balanced in their reconstruction. Alcrudo and Garcia-Navarro (Alcrudo and Garcia-Navarro, 1993) have presented a Godunov-type scheme for numerical solution of shallow water equations. Alcrudo and Benkhaldoun (Alcrudo and Benkhaldoun, 2001) have developed exact solutions for the Riemann problem at the interface with a sudden variation in the topography. LeVeque (LeVeque, 1998) proposed a Riemann solver inside a cell for balancing the source terms and the flux gradients. However, the extension of this scheme for unstructured meshes is not trivial. Numerical methods based on surface gradient techniques have also been applied to shallow water equations by Zhou et al. (Zhou et al., 2001). The TVD-MacCormak scheme has been used by Ming-Heng (Ming-Heng, 2003) to solve water flows in variable bed topography. A different approach based on local hydrostatic reconstructions have been studied by Audusse et al. (Audusse et al., 2004) for open channel flows with topography. The performance of discontinuous Galerkin methods has been examined by Xing and Shu (Xing and Shu, 2006) for some test examples on shallow water flows. However, most of these methods present results with an order of accuracy smaller than the expected in the solutions for unstructured grids. Besides this fact, it is well known that TVD schemes have their order of accuracy reduced to first order in the presence of shocks due to the effects of limiters. A central-upwind scheme using the surface elevation instead of the water depth has been used by Kurganov and Levy (Kurganov and Levy, 2002). Vukovic and Sopta (Vukovic and Sopta, 2002) extended the ENO and WENO schemes to one-dimensional shallow water equations. Unfortunately, most ENO and WENO schemes that solves real flows correctly are still very computationally expensive. On the other hand, numerical methods based on kinetic reconstructions have been studied by Perthame and Simeoni (Perthame and Simeoni, 2001) for one-dimensional problems. In the framework of kinetic schemes, Seaïd (Seaïd, 2004) proposed a class of relaxation methods for solving shallow water equations. The principal drawback of kinetic methods is that they are very difficult to implement on unstructured grids.

In this paper, we describe the development of a generalized upwind finite volume method for shallow water equations. The original generalized upwind difference scheme has been recently proposed by Ronghua Li (Ronghua Li, 1999). In 1978, Ronghua Li utilized finite element spaces and generalized characteristic functions on dual elements, i.e., the common terms of the local Taylor expansions, to rewrite integral interpolation methods in a form of generalized Galerkin methods, and thus obtained a generalization of difference methods on irregular networks, that is, the so- called generalized difference methods (GDM). Both the theoretical observations and the computational experiments show that GDM enjoy not only the simplicity of difference methods but also the accuracy of finite element methods.

This paper is organized as follows. In part 2, the governing equations are given; In Part 3, we give a full discrete upwind finite volume schemes; In Part 4, error estimate for the 1D shallow water equations are given. At last, we give some numerical experiments that demonstrate our method is efficient.

2. GOVERNING EQUATIONS

Consider the following equations (Xing and Shu, 2006; Kurganov and Levy, 2002):

$$\frac{\partial H}{\partial t}+\frac{\partial (Hu)}{\partial x}=s, \tag{1.1a}$$

$$\frac{\partial (Hu)}{\partial t}+\frac{\partial (Hu^2+gH^2/2)}{\partial x}=-gHB'(x). \tag{1.1b}$$

where $x\in[a,b]$, $t\in[0,T]$. H is the depth of water, u is the velocity of the fluid, s is a fluid mass source term, the function B represents the bottom topography (B $\equiv$ const corresponds to the flat bottom case) and g is the gravitational constant.

Let $p=u+2c$, $q=u-2c$, $a_1=u+c$, $a_2=u-c$. So $u=\frac{1}{2}(p+q)$, $H=\frac{1}{16g}(p-q)^2$, the characteristic equations of equations (1.1) are:

$$\frac{\partial p}{\partial t}+a_1\frac{\partial p}{\partial x}=s_1, \tag{1.2a}$$

$$\frac{\partial q}{\partial t}+a_2\frac{\partial q}{\partial x}=s_2. \tag{1.2b}$$

where $c=\sqrt{gH}$.

3. UPWIND FINITE VOLUME SCHEME

Let us decompose the interval $I=[a,b]$ into a grid T_h with nodes

$$a=x_0<x_{\frac{1}{2}}<x_1<x_{\frac{3}{2}}<\cdots<x_{n-\frac{1}{2}}<x_n=b.$$

where $x_{i-\frac{1}{2}}=\frac{(x_i+x_{i-1})}{2}$. Denote the length of the element I_i by $h_i=x_i-x_{i-1}$ and write $h=\max\limits_{1\le i\le n}h_i$. We assume the grid satisfies the quasi-uniform condition $h_i\ge\mu h$ $(i=1,\ldots,n)$ for some positive constant μ.

Next we place a dual grid T_h^* with nodes:

$$a=x_0<x_{\frac{1}{4}}<x_{\frac{3}{4}}<\cdots<x_{n-\frac{3}{4}}<x_{n-\frac{1}{4}}<x_n=b.$$

$T_h^* = \{I_i^*, I_{i-\frac{1}{2}}^* : I_i^* = [x_{i-\frac{1}{4}}, x_{i+\frac{1}{4}}]; I_{i-\frac{1}{2}}^* = [x_{i-\frac{3}{4}}, x_{i-\frac{1}{4}}], i = 1, 2, \ldots, n-1, I_0^* = [x_0, x_{\frac{1}{4}}],$ $I_n^* = [x_{n-\frac{1}{4}}, x_n]\}$, where $x_{i-\frac{k}{4}} = x_i - \frac{k}{4}h_i (k = 1, 3, i = 1, 2, \ldots, n)$.

Choose the trial function space $U_h = \{\phi_i, \phi_{i-\frac{1}{2}}, i = 1, 2, \ldots, n\}$, where

$$\phi_i(x) = \begin{cases} (\frac{2|x-x_i|}{h_i} - 1)(\frac{|x-x_i|}{h_i} - 1), & x_{i-1} \le x \le x_i, \\ (\frac{2|x-x_i|}{h_{i+1}} - 1)(\frac{|x-x_i|}{h_{i+1}} - 1), & x_i \le x \le x_{i+1}, \\ 0, & \text{eleswhere.} \end{cases}$$

$$\phi_{i-\frac{1}{2}}(x) = \begin{cases} 4(1 - \frac{x-x_{i-1}}{h_i})(\frac{x-x_{i-1}}{h_i}), & x_{i-1} \le x \le x_i, \\ 0, & \text{eleswhere.} \end{cases}$$

Accordingly we choose the test function space $V_h = \{\psi_i(x), \psi_{i-\frac{1}{2}}(x), i = 1, 2, \ldots, n\}$ as the piecewise constant function space, where:

$$\psi_i(x) = \begin{cases} 1, & x_{i-\frac{1}{4}} \le x \le x_{i+\frac{1}{4}}, \\ 0, & \text{elsewhere.} \end{cases}$$

$$\psi_{i-\frac{1}{2}}(x) = \begin{cases} 1, & x_{i-\frac{3}{4}} \le x \le x_{i-\frac{1}{4}}, \\ 0, & \text{eleswhere.} \end{cases}$$

Due to the discontinuity of V_h on the boundaries of the elements, one can not apply the Galerkin finite element method on the entire region I. But it is feasible to apply it on a single dual element $I_j^*, I_{j-\frac{1}{2}}^*$. So we seek $p_h \in U_h$ satisfying:

$$\text{(3.1)} \qquad \begin{cases} \int_{I_j^*} [\frac{\partial p_h}{\partial t} + a_1 \frac{\partial p_h}{\partial x}] v_h dx = \int_{I_j^*} s_1 v_h dx, \\ \int_{I_{j-\frac{1}{2}}^*} [\frac{\partial p_h}{\partial t} + a_1 \frac{\partial p_h}{\partial x}] v_h dx = \int_{I_{j-\frac{1}{2}}^*} s_1 v_h dx, \end{cases} \quad \forall v_h \in V_h.$$

Employ Green's formula

$$\text{(3.2)} \qquad \begin{cases} \int_{I_j^*} a_1 \frac{\partial p_h}{\partial x} v_h dx = (a_1 p_h v_h)_{\partial I_j^*} - \int_{I_j^*} p_h \frac{\partial (a_1 v_h)}{\partial x} dx, \\ \int_{I_{j-\frac{1}{2}}^*} a_1 \frac{\partial p_h}{\partial x} v_h dx = (a_1 p_h v_h)_{\partial I_{j-\frac{1}{2}}^*} - \int_{I_{j-\frac{1}{2}}^*} p_h \frac{\partial (a_1 v_h)}{\partial x} dx, \end{cases}$$

Then we can rewrite (3.1) as

$$\text{(3.3)} \qquad \begin{cases} \int_{I_j^*} \frac{\partial p_h}{\partial t} v_h dx - \int_{I_j^*} p_h \frac{\partial (a_1 v_h)}{\partial x} dx + (a_1 p_h v_h)_{\partial I_j^*} = \int_{I_j^*} s_1 v_h dx, \\ \int_{I_{j-\frac{1}{2}}^*} \frac{\partial p_h}{\partial t} v_h dx - \int_{I_{j-\frac{1}{2}}^*} p_h \frac{\partial (a_1 v_h)}{\partial x} dx + (a_1 p_h v_h)_{\partial I_{j-\frac{1}{2}}^*} = \int_{I_{j-\frac{1}{2}}^*} s_1 v_h dx, \end{cases}$$

Denote the boundary of I by ∂I, define:

$$(\partial I)_- = a, \ (\partial I)_+ = b, \ a_i > 0;$$

$$(\partial I)_+ = a, \ (\partial I)_- = b, \ a_i \le 0; i = 1, 2.$$

Similarly, we can define $(\partial I_i^*)_\pm$. For $\forall x \in \partial I_i^*$, set

$$p_h^+(x) = \begin{cases} \lim\limits_{x' \to x} p_h(x'), & x \in (\partial I_i^*)_-, x' \bar{\in} I_i^*, \\ \lim\limits_{x' \to x} p_h(x'), & x \in (\partial I_i^*)_+, x' \in I_i^*. \end{cases}$$

$$p_h^-(x)=\begin{cases}\lim\limits_{x'\to x} p_h(x'), & x\in(\partial I_i^*)_-, x'\in I_i^*,\\ \lim\limits_{x'\to x} p_h(x'), & x\in(\partial I_i^*)_+, x'\bar{\in} I_i^*.\end{cases}$$

Where $[p_h]=p_h^+-p_h^-$ is the jump of p_h across the boundary of dual element. Similarly for $[q_h]$.

They are referred to the upwind and the downwind values of $p_h(x)$ at $x\in\partial I_i^*$, respectively. On the analogy of the classical upwind scheme, we replace p_h in the line integral of the left-hand side of (3.3) by p_h^+ to obtain:

(3.4)

$$\begin{cases}\int_{I_j^*}\frac{\partial p_h}{\partial t}v_h dx-\int_{I_j^*}p_h\frac{\partial(a_1v_h)}{\partial x}dx+(a_1p_h^+v_h)_{\partial I_j^*}=\int_{I_j^*}s_1v_h dx,\\ \int_{I_{j-\frac{1}{2}}^*}\frac{\partial p_h}{\partial t}v_h dx-\int_{I_{j-\frac{1}{2}}^*}p_h\frac{\partial(a_1v_h)}{\partial x}dx+(a_1p_h^+v_h)_{\partial I_{j-\frac{1}{2}}^*}=\int_{I_{j-\frac{1}{2}}^*}s_1v_h dx.\end{cases}\quad\forall v_h\in V_h.$$

It follows from (3.2) that

$$-\int_{I_j^*}p_h\frac{\partial(a_1v_h)}{\partial x}dx=\int_{I_j^*}a_1\frac{\partial p_h}{\partial x}v_h dx-(a_1p_hv_h)_{\partial I_j^*},$$

$$-\int_{I_{j-\frac{1}{2}}^*}p_h\frac{\partial(a_1v_h)}{\partial x}dx=\int_{I_{j-\frac{1}{2}}^*}a_1\frac{\partial p_h}{\partial x}v_h dx-(a_1p_hv_h)_{\partial I_{j-\frac{1}{2}}^*}.$$

Let

$$\bar{p}=pe^{-c_1t},\text{where } c_1=c_0+\sup_{x\in\Omega}|diva_1(x)|, c_0>0.$$

$$\bar{q}=qe^{-c_2t},\text{where } c_2=c_0+\sup_{x\in\Omega}|diva_2(x)|, c_0>0.$$

Define bilinear form

$$\begin{cases}A_1(\bar{p},\psi_j)=\sum_{j=1}^n\{\int_{I_j^*}a_1\frac{\partial\bar{p}}{\partial x}dx+(a_1[\bar{p}])_{(\partial I_j^*)_-}+\int_{I_j^*}c_1\bar{p}dx\},\\ A_1(\bar{p},\psi_{j-\frac{1}{2}})=\sum_{j=1}^n\{\int_{I_{j-\frac{1}{2}}^*}a_1\frac{\partial\bar{p}}{\partial x}dx+(a_1[\bar{p}])_{(\partial I_{j-\frac{1}{2}}^*)_-}+\int_{I_{j-\frac{1}{2}}^*}c_1\bar{p}dx\}.\end{cases}\tag{3.5}$$

$$\begin{cases}A_2(\bar{q},\psi_j)=\sum_{j=1}^n\{\int_{I_j^*}a_2\frac{\partial\bar{q}}{\partial x}dx+(a_2[\bar{q}])_{(\partial I_j^*)_-}+\int_{I_j^*}c_2\bar{q}dx\},\\ A_2(\bar{q},\psi_{j-\frac{1}{2}})=\sum_{j=1}^n\{\int_{I_{j-\frac{1}{2}}^*}a_2\frac{\partial\bar{q}}{\partial x}dx+(a_2[\bar{q}])_{(\partial I_{j-\frac{1}{2}}^*)_-}+\int_{I_{j-\frac{1}{2}}^*}c_2\bar{q}dx\}\end{cases}\tag{3.6}$$

Therefore, from (3.4) yields a semi-discrete upwind scheme for (2.1a):

$$\begin{cases}(\bar{p}_{h,t},\psi_j)+A_1(\bar{p}_h,\psi_j)=(\bar{s}_1,\psi_j),\\ (\bar{p}_{h,t},\psi_{j-\frac{1}{2}})+A_1(\bar{p}_h,\psi_{j-\frac{1}{2}})=(\bar{s}_1,\psi_{j-\frac{1}{2}}).\end{cases}\quad\forall\psi_j,\psi_{j-\frac{1}{2}}\in V_h.\tag{3.7}$$

Similarly, we have a semi-discrete upwind scheme for (2.1b):

$$\begin{cases}(\bar{q}_{h,t},\psi_j)+A_2(\bar{q}_h,\psi_j)=(\bar{s}_2,\psi_j),\\ (\bar{q}_{h,t},\psi_{j-\frac{1}{2}})+A_2(\bar{q}_h,\psi_{j-\frac{1}{2}})=(\bar{s}_2,\psi_{j-\frac{1}{2}}).\end{cases}\quad\forall\psi_j,\psi_{j-\frac{1}{2}}\in V_h.\tag{3.8}$$

Various kinds of finite quotients can be used to further discretize the time derivative, such as forward difference, back-ward difference, or Crank-Nicolson difference.

Denote $\bar{p}_h^n = \bar{p}_h(x,t_n), \bar{s}_1^n = \bar{s}_1(x,t_n), t_n = n\triangle t$. Using the back-ward differencing on the time direction yields the full discrete scheme:

$$(\bar{p}_h^n, v_h) + \triangle t A_1(\bar{p}_h^n, v_h) = (\bar{p}_h^{n-1} + \triangle t \bar{s}_1^n, v_h), \quad \forall v_h \in V_h. \tag{3.9}$$

$$(\bar{q}_h^n, v_h) + \triangle t A_2(\bar{q}_h^n, v_h) = (\bar{q}_h^{n-1} + \triangle t \bar{s}_2^n, v_h), \quad \forall v_h \in V_h. \tag{3.10}$$

where $\triangle t = t_n - t_{n-1}$ is the time step.

4. ERROR ESTIMATES

Lemma 1.1: For $A_1(\cdot,\cdot)$ which defined in above, there have:

$$A_1(\bar{p}_h, \bar{p}_h) \geq \gamma_0(\|\bar{p}_h\|_{\partial I}^2 + \|\bar{p}_h\|_0^2).$$

where: $\gamma_0 = min(\sigma_1, \frac{1}{2}), \sigma_1 = c_1 - \frac{1}{2}diva_{1h},\ \|\bar{p}_h\|_0^2 = (\bar{p}_h, \bar{p}_h)$,

$$\|\bar{p}_h\|_{\partial I}^2 = \sum_{j=0}^{n}\{(|a_{1h}|[\bar{p}_h]^2)_{(\partial I_j^*)_-} + (|a_{1h}|[\bar{p}_h]^2)_{(\partial I^*_{j+\frac{1}{2}})_-}\} + \frac{1}{2}(|a_{1h}|(\bar{p}_h^+)^2)_{(\partial I_0^*)_+}.$$

Proof: Because $div(av_h) = v_h diva + a\cdot\nabla v_h$, so we have

$$(av_h^2)_{\partial I_j^*} = 2\int_{I_j^*} v_h(a\cdot\nabla v_h)dx + \int_{I_j^*} v_h^2 divadx.$$

Therefore

$$\begin{aligned}
& A_1(\bar{p}_h, \bar{p}_h) \\
= & \sum_{j=1}^{n}\{\int_{I_j^*} a_{1h}\frac{\partial \bar{p}_h}{\partial x}\bar{p}_h dx + (a_{1h}[\bar{p}_h]\bar{p}_h)_{(\partial I_j^*)_-} + \int_{I_j^*} c_1\bar{p}_h^2 dx\} \\
+ & \sum_{j=1}^{n}\{\int_{I^*_{j-\frac{1}{2}}} a_{1h}\frac{\partial \bar{p}_h}{\partial x}\bar{p}_h dx + (a_{1h}[\bar{p}_h]\bar{p}_h)_{(\partial I^*_{j-\frac{1}{2}})_-} + \int_{I^*_{j-\frac{1}{2}}} c_1\bar{p}_h^2 dx\} \\
= & \sum_{j=1}^{n}\{\frac{1}{2}(a_{1h}\bar{p}_h^2)_{\partial I_j^*} - \frac{1}{2}\int_{I_j^*}\bar{p}_h^2 diva_{1h}dx + (a_{1h}[\bar{p}_h]\bar{p}_h)_{(\partial I_j^*)_-} + \int_{I_j^*} c_1\bar{p}_h^2 dx\} \\
+ & \sum_{j=1}^{n}\{\frac{1}{2}(a_{1h}\bar{p}_h^2)_{\partial I^*_{j-\frac{1}{2}}} - \frac{1}{2}\int_{I^*_{j-\frac{1}{2}}}\bar{p}_h^2 diva_{1h}dx + (a_{1h}[\bar{p}_h]\bar{p}_h)_{(\partial I^*_{j-\frac{1}{2}})_-} + \int_{I^*_{j-\frac{1}{2}}} c_1\bar{p}_h^2 dx\} \\
= & \sum_{j=1}^{n}\{\frac{1}{2}(a_{1h}\bar{p}_h^2)_{(\partial I_j^*)_+} + \frac{1}{2}(a_{1h}\bar{p}_h^2)_{(\partial I_j^*)_-} + [a_{1h}(\bar{p}_h^+ - \bar{p}_h^-)\bar{p}_h]_{(\partial I_j^*)_-} \\
+ & \frac{1}{2}[a_{1h}((\bar{p}_h^+)^2 - 2\bar{p}_h^+\bar{p}_h^- + (\bar{p}_h^-)^2)]_{(\partial I_j^*)_-} - \frac{1}{2}(a_{1h}[\bar{p}_h]^2)_{(\partial I_j^*)_-} + \int_{I_j^*}\sigma_1\bar{p}_h^2 dx\} \\
+ & \sum_{j=1}^{n}\{\frac{1}{2}(a_{1h}\bar{p}_h^2)_{(\partial I^*_{j-\frac{1}{2}})_+} + \frac{1}{2}(a_{1h}\bar{p}_h^2)_{(\partial I^*_{j-\frac{1}{2}})_-} + [a_{1h}(\bar{p}_h^+ - \bar{p}_h^-)\bar{p}_h]_{(\partial I^*_{j-\frac{1}{2}})_-} \\
+ & \frac{1}{2}[a_{1h}((\bar{p}_h^+)^2 - 2\bar{p}_h^+\bar{p}_h^- + (\bar{p}_h^-)^2)]_{(\partial I^*_{j-\frac{1}{2}})_-} - \frac{1}{2}(a_{1h}[\bar{p}_h]^2)_{(\partial I^*_{j-\frac{1}{2}})_-} + \int_{I^*_{j-\frac{1}{2}}}\sigma_1\bar{p}_h^2 dx\}
\end{aligned}$$

$$
\begin{aligned}
= & \frac{1}{2}\sum_{j=1}^{n}\{(a_{1h}\bar{p}_h^2)_{(\partial I_j^*)+} - (a_{1h}[\bar{p}_h]^2)_{(\partial I_j^*)-} + (a_{1h}(\bar{p}_h^+)^2)_{(\partial I_j^*)-} + 2\int_{I_j^*}\sigma_1\bar{p}_h^2dx\} \\
+ & \frac{1}{2}\sum_{j=1}^{n}\{(a_{1h}\bar{p}_h^2)_{(\partial I_{j-\frac{1}{2}}^*)+} - (a_{1h}[\bar{p}_h]^2)_{(\partial I_{j-\frac{1}{2}}^*)-} + (a_{1h}(\bar{p}_h^+)^2)_{(\partial I_{j-\frac{1}{2}}^*)-} + 2\int_{I_{j-\frac{1}{2}}^*}\sigma_1\bar{p}_h^2dx\}.
\end{aligned}
$$

Hence

$$(\partial I_{j-\frac{1}{2}}^*)_+ = (\partial I_j^*)_-, (\partial I_j^*)_+ = (\partial I_{j+\frac{1}{2}}^*)_-,$$

$$
\begin{aligned}
& A_1(\bar{p}_h,\bar{p}_h) \\
= & \frac{1}{2}\sum_{j=0}^{n}\{(|a_{1h}|[\bar{p}_h]^2)_{(\partial I_j^*)-} + (|a_{1h}|[\bar{p}_h]^2)_{(\partial I_{j+\frac{1}{2}}^*)-}\} + \int_{I_j^*}\sigma_1\bar{p}_h^2dx + \int_{I_{j-\frac{1}{2}}^*}\sigma_1\bar{p}_h^2dx \\
+ & \frac{1}{2}(|a_{1h}|(\bar{p}_h^+)^2)_{(\partial I_0^*)+} - (|a_{1h}|(\bar{p}_h^+)^2)_{(\partial I_n^*)-},
\end{aligned}
$$

Notice $\bar{p}_h^+ \mid_{(\partial I_n^*)-} = 0$.

So

$$A_1(\bar{p}_h,\bar{p}_h) \geq \gamma_0(\|\bar{p}_h\|_{\partial I}^2 + \|\bar{p}_h\|_0^2).$$

Therefore, for $\bar{p} \in H^2(I)$, the Ritz project $R_h\bar{p} \in V_h$ exists and unique, moreover,

$$A_1(R_h\bar{p}, v_h) = A_1(\bar{p}, v_h), \quad \forall v_h \in V_h, \tag{4.1}$$

Similarly,

$$A_2(\bar{q}_h, \prod_h^* \bar{q}_h) \geq \|\bar{q}_h\|_{\partial I}^2$$

where: $\|\bar{q}_h\|_{\partial I}^2 = \sum_{j=0}^{n}\{(\bar{q}_{j-\frac{1}{2}} - \bar{q}_j)(|a_2|\bar{q}_h^+)_{(\partial I_j^*)-} + (\bar{q}_j - \bar{q}_{j+\frac{1}{2}})(|a_2|\bar{q}_h^+)_{(\partial I_{j+\frac{1}{2}}^*)-}\} + \bar{q}_{\frac{1}{2}}(|a_2|\bar{q}_h^+)_{(\partial I_0^*)+}$.

So $A_2(\bar{q}_h, \prod_h^* \bar{q}_h)$ is positive definite. Therefore, for $\bar{q} \in H^2(I)$, the Ritz project $R_h\bar{q} \in V_h$ exists and is unique, moreover,

$$A_2(R_h\bar{q}, v_h) = A_2(\bar{q}, v_h), \quad \forall v_h \in V_h, \tag{4.2}$$

Lemma 1.2: For $\bar{p}, \bar{q} \in H^2(I)$, there holds the following estimate:

$$|||\bar{p} - R_h\bar{p}||| \leq Ch^{\frac{3}{2}}\|\bar{p}\|_2, |||\bar{q} - R_h\bar{q}||| \leq Ch^{\frac{3}{2}}\|\bar{q}\|_2, \tag{4.3}$$

where $|||v|||^2 = \|v\|_{\partial I}^2 + h\sum_{x_i}(a_1\frac{\partial v}{\partial x})^2dx$, so $|||v|||$ and $\|v\|$ is equivalent in V_h.

For the full-discrete upwind finite volume scheme, there holds the following error estimate:

Theorem 1.3: let p, q, p_h and q_h be the solutions to $(2.1), (3.9)$ and (3.10), respectively, satisfying $p_t, q_t \in H^2(I), p_{tt}, q_{tt} \in L^2(I)$. Then there holds the following error estimate:

$$\|p(t_n) - p_h^n\|_0 \leq \|p^0 - p_h^0\|_0 + Ch^{\frac{3}{2}}\|p^0\|_2 + \triangle t\int_0^{t_n}\|p_{tt}\|_0dt + Ch^{\frac{3}{2}}\int_0^{t_n}\|p_t(t)\|_2dt \tag{4.4}$$

and

$$\|q(t_n)-q_h^n\|_0 \le \|q^0-q_h^0\|_0 + Ch^{\frac{3}{2}}\|q^0\|_2 + \triangle t\int_0^{t_n}\|q_{tt}\|_0 dt + Ch^{\frac{3}{2}}\int_0^{t_n}\|q_t(t)\|_2 dt \tag{4.5}$$

Proof : The first we proof (4.4).

Note $p_h^n - p(t_n) = \xi_1^n + \eta_1^n$, where $\xi_1^n = R_h p(t_n) - p(t_n), \eta_1^n = p_h^n - R_h p(t_n)$. It follows from (4.3) that

$$\|\xi_1^n\| \le Ch^{\frac{3}{2}}\|p(t_n)\|_2.$$

Also observe that

$$p(t_n) = p(0) + \int_0^{t_n} p_t(t)dt$$

$$\|p(t_n)\|_2 \le \|p(0)\|_2 + \int_0^{t_n}\|p_t(t)\|_2 dt.$$

Thus

$$\|\xi_1^n\| \le Ch^{\frac{3}{2}}(\|p^0\|_2 + \int_0^{t_n}\|p_t(t)\|_2 dt). \tag{4.6}$$

Next, we turn to deal with η_1^n. Write $\overline{\partial}_t p_h^n = (p_h^n - p_h^{n-1})/\triangle t$. It follows from (3.9) and the definition of R_h that

$$\begin{aligned} A_1(\eta_1^n, v_h) &= A_1(p_h^n - R_h p(t_n), v_h) \\ &= -(\overline{\partial}_t p_h^n, v_h) + (s_1^n, v_h) - A_1(p(t_n), v_h) \\ &= (p_t(t_n) - \overline{\partial}_t p_h^n, v_h) \end{aligned}$$

Hence

$$(\overline{\partial}_t \eta_1^n, \eta_1^n) + A_1(\eta_1^n, \eta_1^n) = (p_t(t_n) - R_h\overline{\partial}_t p(t_n), \eta_1^n) = (w_1^n + w_2^n, \eta_1^n), \tag{4.7}$$

Where

$$w_1^n = p_t(t_n) - \overline{\partial}_t p(t_n), w_2^n = \overline{\partial}_t p(t_n) - R_h\overline{\partial}_t p(t_n).$$

Notice

$$(\eta_1^n)^+|_{(\partial I)_-} = 0, A_1(\eta_1^n, \eta_1^n) \ge 0.$$

So by (4.7) we have

$$\|\eta_1^n\|_0 \le \|\eta_1^{n-1}\|_0 + \triangle t\|w_1^n + w_2^n\|_0 \le \|\eta_1^0\|_0 + \triangle t\sum_{j=1}^n \|w_1^j + w_2^j\|_0. \tag{4.8}$$

It is obvious that

$$\begin{aligned} \|\eta_1^0\|_0 = \|p_h^0 - R_h p(x,0)\|_0 &\le \|p_h^0 - p(x,0)\|_0 + \|p(x,0) - R_h p(x,0)\|_0 \\ &\le \|p_h^0 - p(x,0)\|_0 + Ch^{\frac{3}{2}}\|p(x,0)\|_2. \end{aligned} \tag{4.9}$$

Also note

$$w_1^j = p_t(t_j) - \triangle t^{-1}(p(t_j) - p(t_{j-1})) = \triangle t^{-1}\int_{t_{j-1}}^{t_j}(t - t_j)p_{tt}(t)dt,$$

$$w_2^j = (I - R_h)\overline{\partial}_t p(t_j) = \triangle t^{-1} \int_{t_{j-1}}^{t_j} (I - R_h) p_t(t) dt.$$

So we have:

$$\triangle t \sum_{j=1}^{n} \|w_1^j + w_2^j\|_0 \leq \triangle t \int_0^{t_n} \|p_{tt}(t)\|_0 dt + Ch^{\frac{3}{2}} \int_0^{t_n} \|p_t(t)\|^2 dt. \tag{4.10}$$

Inserting (4.9) and (4.10) into (4.8) yields

$$\|\eta_1^n\|_0 \leq \|p^0 - p_h^0\|_0 + Ch^{\frac{3}{2}} \|p^0\|_2 + \triangle t \int_0^{t_n} \|p_{tt}(t)\|_0 dt + Ch^{\frac{3}{2}} \int_0^{t_n} \|p_t(t)\|_2 dt. \tag{4.11}$$

Finally, a combination of (4.6) and (4.11) leads to the desired estimate

$$\begin{aligned} \|p_h^n - p(t_n)\|_0 &\leq \|\eta_1^n\|_0 + \|\xi_1^n\|_0 \\ &\leq \|p^0 - p_h^0\|_0 + Ch^{\frac{3}{2}} \|p^0\|_2 + \triangle t \int_0^{t_n} \|p_{tt}(t)\|_0 dt \\ &\quad + Ch^{\frac{3}{2}} \int_0^{t_n} \|p_t(t)\|_2 dt. \end{aligned} \tag{4.12}$$

This completes (4.4). The same to (4.5).

5. NUMERICAL EXAMPLES

In this section, we try to illustrate the effectiveness and the stability of the upwind finite volume scheme by numerical experiments, comparing the upwind finite volume scheme with the simple center scheme and giving some figures to illustrate the merits of the upwind finite volume method.

We give simple center scheme:

$$p_{h,j}^n = p_{h,j}^{n-1} - \frac{\sigma_1}{2}(p_{h,j+1}^{n-1} - p_{h,j-1}^{n-1}) + \frac{\sigma_1^2}{2}(p_{h,j+1}^{n-1} - 2p_{h,j}^{n-1} + p_{h,j-1}^{n-1}),$$

$$q_{h,j}^n = q_{h,j}^{n-1} - \frac{\sigma_2}{2}(q_{h,j+1}^{n-1} - q_{h,j-1}^{n-1}) + \frac{\sigma_2^2}{2}(q_{h,j+1}^{n-1} - 2q_{h,j}^{n-1} + q_{h,j-1}^{n-1}).$$

where $\sigma_i = a_i \Delta t / \Delta x, i = 1, 2$. $\Delta t = t_{i+1} - t_i$, is time step, $\Delta x = x_{i+1} - x_i$ is space step.

Example 1: Static water. The initial value is zero, and the initial depth of water is $1m$. See figure 1. The depth of water keeps immovable.

Example 2: This example is the classical break dam problem. Let $s = 0$, the initial value is:

$$H(x, 0) = \begin{cases} 1.0, & x \leq 0, \\ 0.5, & x > 0. \end{cases}$$

$$u(x, 0) = 0.$$

In these examples, we set $I = [-100, 100]m$ as the space area. The space step is $2m$. The time step is $0.2s$. In figure 2, 3 and 4, we compare the values of H that are

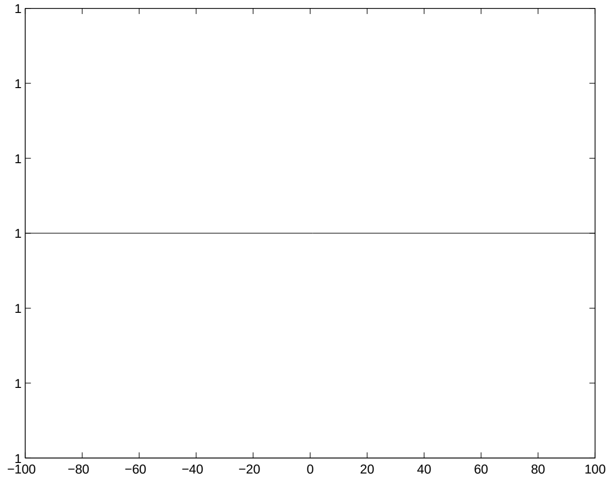

Figure 1. static water

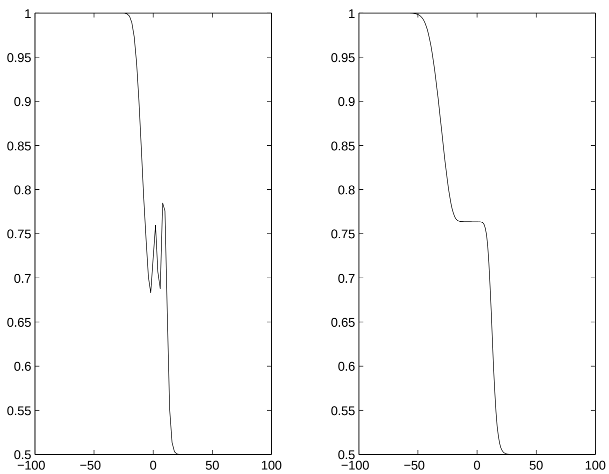

Figure 2. (l)simple center method,(r)GUDM,t=5s

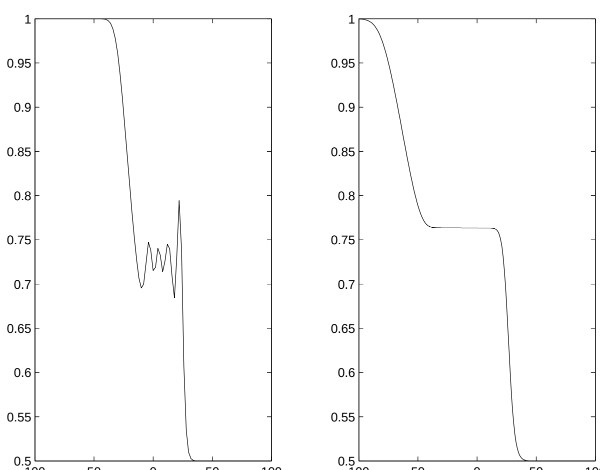

Figure 3.(l)simple center method,(r)GUDM,t=10s

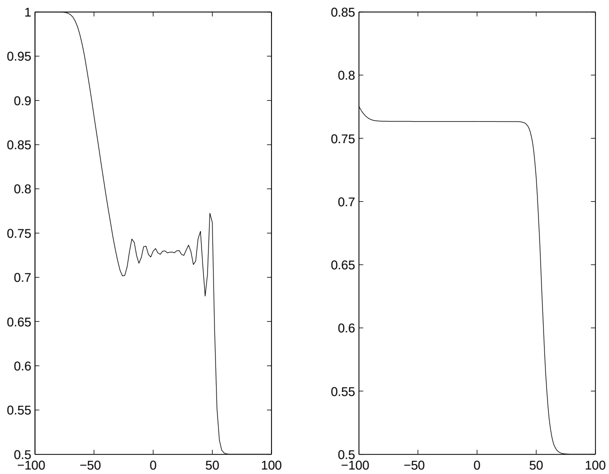

Figure 4. (l)simple center method,(r)GUDM,t=20s

obtained by applying different schemes at three time level. We have also given some tables to illustrate the efficiency of our method.

Table 1.Compare the simple center solution and UFVM solution of H at $T = 5s$

notes	$x = -8$	$x = -4$	$x = 0$	$x = 4$	$x = 8$
simple center solu.	0.84664	0.74436	0.68315	0.75943	0.6879
UFVM solu.	0.76354	0.76353	0.7635	0.76339	0.76216

Table 2.Compare the simple center solution and UFVM solution of H at $T = 10s$

notes	$x = -8$	$x = -4$	$x = 0$	$x = 4$	$x = 8$
simple center solu.	0.69552	0.72352	0.73847	0.71905	0.7328
UFVM solu.	0.7634	0.7634	0.76339	0.76338	0.76338

Table 3.Compare the simple center solution and UFVM solution of H at $T = 20s$

notes	$x = -8$	$x = -4$	$x = 0$	$x = 4$	$x = 8$
simple center solu.	0.72254	0.73524	0.72288	0.73249	0.72598
UFVM solu.	0.76334	0.76334	0.76333	0.76333	0.76333

In these figures and tables, we can see that, at the same time, the simple center scheme has nonphysical oscillations at jump point, but the method of this paper eliminates nonphysical oscillations. From figures 2(r) and 3(r) we can see the depth of water is fall.

REFERENCES

[1] M. Heniche, Y. Secretan, P. Boudreau, M. Leclerc (2000). A two-dimensional finite element drying-wetting shallow water model for rivers and estuaries, Adv. Water Resour. 23, 359–372.

[2] M. González, M. Seaïd (2005). Finite element modified method of characteristics for shallow water flows: application to the Strait of Gibraltar, Prog. Indust. Math. 8, 518–522.

[3] A. A. Akanbi, N. D. Katopodes (1988). Models for flood propagation on initially dry land, J. Hydraul. Eng. (ASCE). 114, 686–695.

[4] M. S. Horrit (2000). Development of Physically Based Neshes for Two-Dimensional Models of Meandering Channel flow, Int. J. Numer. Methods Eng. 47, 2019–2037.

[5] E. F. Toro (2001). Shock-Capturing Methods for Free-Surface Shallow Waters, Wiley, Chichester.

[6] F. Alcrudo, P. Garcia-Navarro (1993). A high resolution Godunov-type scheme in finite volumes for the 2D shallow water equation, Int. J. Numer. Meth. Fluids 16, 489–505.

[7] F. Alcrudo, F. Benkhaldoun (2001). Exact solutions to the Riemann problem of the shallow water equations with a bottom step, Comput. Fluids. 30, 643–671.

[8] R. J. Leveque (1998). Balancing source terms and flux gradients in high-resolution Godunov methods: the quasi-steady wave-propagation algorithm, J. Comput. Phys. 146, 346–365.

[9] J. G. Zhou, D. M. Causon, C. G. Mingham, D. M. Ingram (2001). The surface gradient method for the treatment of source terms in the shallow water equations, J. Comp. Phys. 20, 1–25.

[10] T. Ming-Hen (2003). The improved surface gradient method for flows simulation in variable bed topography channel using TVD MacCormack scheme, Int. J. Numer. Methods Fluids 43, 71–91.

[11] E. Audusse, F. Bouchut, M. O. Bristeau, R. Klein, B. Perthame (2004). A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows, SIAM J. Sci. Comp. 25, 2050–2065.

[12] Y. Xing, C. Shu (2006). High order well-balanced finite volume WENO schemes and discontinuous Galerkin methods for a class of hyperbolic systems with source terms, J. Comp. Phys. 214, 567–598.

[13] A. Kurganov, D. Levy (2002). Central-upwind schemes for the Saint-Venant system, Math. Model. Numer. Anal. 36, 397–425.

[14] S. Vukovic, L. Sopta (2002) ENO and WENO schemes with the exact conservation property for one-dimensional shallow-water equations, J. Comp. Phys. 179, 593–621.

[15] B. Perthame, C. Simeoni (2001). A kinetic scheme for the Saint-Venant system with a source term, CALCOLO 38, 201–231.

[16] A. Birman, J. Falcovitz b (2007). Application of the GRP scheme to open channel flow equations, J. Comp. Phys. 222, 131–154.

[17] M. Seaïd (2004). Non-oscillatory relaxation methods for the shallow water equations in one and two space dimensions, Int. J. Numer. Methods Fluids 46, 457–484.

[18] R. H. Li, Z. Y. Chen and W. Wu (1999). Generalized difference methods for differential equations:Numerical analysis of finite volume methods, NewYork: Marcel Dekker.