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COUETTE FLOWS OF AN OLDROYD 8-CONSTANT FLUID WITH MAGNETIC FIELD

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ABSTRACT. We develop a generalized approximation method (GAM) to obtain a solution of steady unidirectional flows of an Oldroyd 8-constant magneto-hydrodynamic (MHD) fluid in bounded domain. The fluid is electrically conducting in the presence of a uniform magnetic field. The nonlinear equation which describe the velocity field is solved via GAM. The GAM generates a bounded monotone sequence of solutions of linear problems. The sequence of solutions of linear problems converges monotonically and rapidly to a solution of the original nonlinear problem. We study the effect of fluid parameters on the velocity field and present some numerical simulation to illustrate and confirm our results.

Key Words: Oldroyd fluid; upper and lower solutions; approximation method.

Subject, Classification 44.05.+e; 47.50-d.

1. INTRODUCTION

An important class of fluids commonly used in industries is Non-Newtonian fluids. Due to their importance in various industrial applications, flows of non-Newtonian fluids have been the subject of numerous theoretical and experimental studies. For example, plastics and polymers are handled extensively by chemical industries. Recently, Baris [1] studied series method to discuss an Oldroyd 8-constant fluid in a convergent channel. Hayat, et al [6] studied series solution by using homotopy analysis method [19, 20] to discuss an Oldroyd 8-constant magneto-hydrodynamic (MHD) fluid in bounded domain. The homotopy analysis method is used to a variety of problems, see for example, [7, 8, 17, 18, 22] and the references therein.

Motivated by the work in [1, 6, 7], in this paper, we study the flow of an electrically conducting fluid in the presence of a magnetic field $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, where $\mathbf{b}$ is the induced magnetic field which is assumed to be negligibly small compared to $\mathbf{B}_0$. Assume that the electric conductivity, the magnetic permeability are constant and $\mathbf{B}$ is perpendicular to the velocity field. Assume the fluid is an Oldroyd 8-constant fluid between two parallel plates and the motion of the fluid is induced due to applied

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constant pressure and motion of the top plate. The bottom plate is assumed to be at rest. Assume that the flow is steady state and uniform, the two plates are at positions $y = 0$, $y = d$ and the top plate is moving with constant velocity U. The governing equation that describe the velocity field is given by the following nonlinear boundary value problem

$$\begin{aligned}\frac{d^2u}{dy^2} + \Big[(3\alpha_1 - \alpha_2) + \alpha_1\alpha_2(\frac{du}{dy})^2\Big](\frac{du}{dy})^2\frac{d^2u}{dy^2} - \frac{\sigma B_0^2}{\mu}u\Big[1 + \alpha_2(\frac{du}{dy})^2\Big]^2 = 0,\\ u = 0 \text{ at } y = 0,\ u = U \text{ at } y = d,\end{aligned} \tag{1.1}$$

see [6], where ρ is the density, μ is the dynamic viscosity, α_1, α_2 are material moduli of Oldroyd 8-constant fluid defined by

$$\alpha_1 = \lambda_1(\lambda_4 + \lambda_7) - (\lambda_3 + \lambda_5)(\lambda_4 + \lambda_7 - \lambda_2) - \frac{\lambda_5\lambda_7}{2}$$
$$\alpha_2 = \lambda_1(\lambda_3 + \lambda_6) - (\lambda_3 + \lambda_5)(\lambda_3 + \lambda_6 - \lambda_1) - \frac{\lambda_5\lambda_6}{2},$$

$\lambda_1, \ldots, \lambda_7$ are material constants.

If $\alpha_1 = \alpha_2$, the BVP (1.1) is linear whose exact solution is given by

$$u(y) = U\frac{\sinh\sqrt{\frac{\sigma}{\mu}}(B_0 dy)}{\sinh\sqrt{\frac{\sigma}{\mu}}(B_0 d)},\ y \in [0, d].$$

However, if $\alpha_1 \neq \alpha_2$, boundary value problem (1.1) is nonlinear and exact analytic solutions of such nonlinear problems are not available in general.

Introducing the dimensionless quantities [6],

$$y^* = \frac{y}{d},\ x^* = \frac{x}{d},\ u^* = \frac{u}{U},\ \alpha_1^* = \frac{\alpha_1}{(\frac{d}{U})^2},\ \alpha_2^* = \frac{\alpha_2}{(\frac{d}{U})^2} \text{ and } m^* = \frac{\sigma B_0^2 d^2}{\mu}$$

and dropping $'*'$, the problem (1.1) takes the following form

$$\begin{aligned}\frac{d^2u}{dy^2}\Big[1 + (3\alpha_1 - \alpha_2)(\frac{du}{dy})^2 + \alpha_1\alpha_2(\frac{du}{dy})^4\Big] = mu\Big[1 + \alpha_2(\frac{du}{dx})^2\Big]^2,\ y \in [0, 1]\\ u(0) = 0,\ u(1) = 1.\end{aligned} \tag{1.2}$$

For simplicity, we write the boundary value problem (1.2) as follows

$$\begin{aligned}\frac{d^2u}{dy^2} = f(u, u'),\ y \in [0, 1] = I,\\ u(0) = 0,\ u(1) = 1,\end{aligned} \tag{1.3}$$

where $f(u, u') = \frac{mu[1+\alpha_2(\frac{du}{dy})^2]^2}{1+(3\alpha_1-\alpha_2)(\frac{du}{dy})^2+\alpha_1\alpha_2(\frac{du}{dy})^4}$.

In this paper, we provide estimates for the actual solution of the problem. These estimates determine the region of existence of solution of the problem. Based on these estimates, we apply a new analytical technique, the generalized approximation method (GAM), [9, 10, 11, 12], a kind of quasilinearization method [3, 4, 5, 13, 14, 15, 16, 21] which uses linear iterations to deal with nonlinear problems, to approximate solution of the problem. We shall show that only few iterations lead to an accurate solution of the problem. GAM generates a bounded monotone sequence of solutions of linear problems that converges uniformly and rapidly to a solution of the original problem. Moreover, the solution is bracketed between the iterates and a fixed upper solution. We shall show that our results are consistent and accurately represents the actual solution of the problem for any value of the parameters. For the numerical simulations, we use the computer programme, Mathematica.

2. UPPER AND LOWER SOLUTIONS

Recall the concept of lower and upper solutions corresponding to the BVP (1.3).

Definition 2.1. A function $\alpha \in C^1(I)$ is called a lower solution of the BVP (1.3), if it satisfies the following inequalities,

$$\begin{aligned}&\alpha''(y) \geq f(\alpha(y), \alpha'(y)), \quad y \in (0,1)\\ &\alpha(0) \leq 0,\ \alpha(1) \leq 1.\end{aligned}$$

An upper solution $\beta \in C^1(I)$ of the BVP (1.3) is defined similarly by reversing the inequalities.

For example, $\alpha = 0$ and $\beta = y$ are lower and upper solutions of the BVP (1.3) respectively and these functions provide estimates for the exact solution of the problem. To give an estimate for the derivative u' of the possible solution, we recall the concept of Nagumo function.

Definition 2.2. A continuous function $\omega : (0,\infty) \to (0,\infty)$ is called a Nagumo function if

$$\int_{\lambda}^{\infty} \frac{s ds}{\omega(s)} = \infty,$$

where $\lambda = \max\{|\alpha(0) - \beta(1)|, |\alpha(1) - \beta(0)|\} = 1$. We say that $f \in C[\mathbb{R} \times \mathbb{R}]$ satisfies a Nagumo condition relative to α, β if for $x \in [\min \alpha, \max \beta]$, there exists a Nagumo function ω such that $|f(u, u')| \leq \omega(|u'|)$.

For the present problem, since

$$f(u,u') = \frac{4mu[1+\alpha_2(\frac{du}{dy})^2]^2}{(1+\alpha_2(\frac{du}{dy})^2)^2 + 3 + 6(2\alpha_1 - \alpha_2)(\frac{du}{dy})^2 + \alpha_2(4\alpha_1 - \alpha_2)(\frac{du}{dy})^4} \leq 4m,$$

for $u \in [0,1]$, where α_1, α_2 are selected such that $3 + 6(2\alpha_1 - \alpha_2)(\frac{du}{dx})^2 + \alpha_2(4\alpha_1 - \alpha_2)(\frac{du}{dx})^4 \geq 0$. Hence, f is continuous and bounded and therefore satisfies a Nagomo condition with $\omega(s) = 4m$ as a Nagumo function. Hence by Theorem 1.4.1 of [2] (page 14) , there exists a constant $C > \lambda$ such that any solution u of the the BVP (1.3) which satisfy $\alpha \le u \le \beta$ must satisfy $|u'| \le C$. The constant C that provide estimate for the derivative of a solution can explicitly be computed using the following relation

$$\int_1^C \frac{sds}{\omega(s)} \ge \max\beta - \min\alpha = 1,$$

which implies that $C \ge \sqrt{1+8m}$. In particular, we choose $C = \sqrt{1+8m}$. The following result is known [2] (Theorem 1.5.1, Page 31).

Theorem 2.3. *Assume that $\alpha, \beta \in C^1(I)$ are lower and upper solutions of the BVP (1.3) such that $\alpha \le \beta$ on I. Assume that $f : \mathbb{R} \times \mathbb{R} \to (0, \infty)$ is continuous and satisfies a Nagumo's condition on I relative to α, β. Then the BVP (1.3) has a solution $u \in C^1(I)$ such that $\alpha \le u \le \beta$ and $|u'| \le C$ on I, where C depends only on α, β and ω.*

We note that the BVP (1.3) satisfies the conditions of Theorem 2.3 with $\alpha = 0$ and $\beta = y$ as lower and upper solutions. We write the boundary value problem (1.3) as an equivalent integral equation

$$u(y) = y + \int_0^1 G(y,s) f(u(s), u'(s)) ds,$$

where

$$G(y,s) = \begin{cases} y(s-1), & 0 \le y < s \le 1 \\ s(y-1), & 0 \le s < y \le 1, \end{cases}$$

is the Green's function. Clearly, $G(y,s) < 0$ on $(0,1) \times (0,1)$. Define $T : C^1(I) \to C^1(I)$ by

$$Tu(y) = y + \int_0^1 G(y,s) f(u(s), u'(s)) ds.$$

By a solution of the BVP (1.3), we mean a fixed point of T. Since, f is continuous and bounded, by Schauder's fixed point theorem, T has a fixed point.

3. GENERALIZED APPROXIMATION METHOD (GAM)

Differentiating f with respect to u, u', we obtain

$$f_{uu} = 0, f_{uu'} = \frac{2mu'(\alpha_2 - \alpha_1)(1 + \alpha_2 u'^2)(3 - \alpha_2 u'^2)}{(1 + (3\alpha_1 - \alpha_2)u'^2 + \alpha_1\alpha_2 u'^4)^2}$$

and

$$f_{u'u'} = \frac{2mu(\alpha_2 - \alpha_1)}{(1 + (3\alpha_1 - \alpha_2)u'^2 + \alpha_1\alpha_2 u'^4)^3} \Big\{ 3 - 3(9\alpha_1 - 5\alpha_2)u'^2 - 3\alpha_1(\alpha_1 + 9\alpha_2 u'^4) - \alpha_2^2(13\alpha_1 - \alpha_2)u'^6 + 3\alpha_1\alpha_2^3 u'^8 \Big\}.$$

Let

$$\rho_1 = \max\{f_{uu'}(u,u') : u \in [0,1],\ u' \in [-C, C]\},$$
$$\rho_2 = \max\{f_{u'u'}(u,u') : u \in [0,1],\ u' \in [-C, C]\}.$$

Define

$$m_1 = \begin{cases} 0, & \text{if } \rho_1 \le 0, \\ \rho_1, & \text{if } \rho_1 \ge 0 \end{cases}, \; m_2 = \begin{cases} 0, & \text{if } \rho_2 \le 0, \\ \rho_2, & \text{if } \rho_2 \ge 0, \end{cases}$$

and $\phi(u,u') = -\frac{1}{2}(2m_1uu' + m_2u'^2)$. Then,

$$\phi_{uu}(u,u') = 0,\; \phi_{uu'}(u,u') = -m_1,\; \phi_{u'u'}(u,u') = -m_2.$$

Define $F(u,u') = f(u,u') + \phi(u,u')$. Using the definitions of m_1 and m_2, we have

$$F_{uu}(u,u') = 0,\; F_{uu'}(u,u') = f_{uu'}(u,u') - m_1 \le 0,\; F_{u'u'}(u,u') = f_{u'u'}(u,u') - m_2 \le 0.$$

Hence, the quadratic form

(3.1)
$$v^T H(F) v = (u-z)^2 F_{uu}(z,z') + 2(u-z)(u'-z')F_{uu'}(z,z') + (u'-z')^2 F_{u'u'}(z,z') \le 0,$$

where $H(F) = \begin{pmatrix} F_{uu} & F_{uu'} \\ F_{uu'} & F_{u'u'} \end{pmatrix}$ is the Hessian matrix and $v = \begin{pmatrix} u-z \\ u'-z' \end{pmatrix}$. Using (3.1), we obtain

$$F(u,u') \le F(z,z') + F_u(z,z')(u-z) + F_{u'}(z,z')(u'-z'),\; z,\; z' \in \mathbb{R},$$

which implies that

(3.2)
$$f(u,u') \le f(z,z') + F_u(z,z')(u-z) + F_{u'}(z,z')(u'-z') - (\phi(u,u') - \phi(z,z')),\; z,\; z' \in \mathbb{R}.$$

Using the mean value theorem, we have

(3.3)
$$\begin{aligned} &\phi(u,u') - \phi(z,z') = \phi_u(z,z')(u-z) + \phi_{u'}(z,z')(u'-z') + v^T H(\phi) v \\ &= \phi_u(z,z')(u-z) + \phi_{u'}(z,z')(u'-z') - \frac{1}{2}\big(m_1(u-z)(u'-z') + m_2(u'-z')^2\big) \\ &\ge \phi_u(z,z')(u-z) + \phi_{u'}(z,z')(u'-z') - \frac{1}{2}\big(m_1(u-z)(C-z') + m_2(u'-z')^2\big), \end{aligned}$$

for $u \ge z$.

For computational purpose, we use the following approximation in (3.3),

$$(u'-z')^2 \approx (u'-z')(z'-w') \text{ for some } w' \in [-C, C].$$

Hence,

$$\begin{aligned} \phi(u,u') - \phi(z,z') &\ge \phi_u(z,z')(u-z) + \phi_{u'}(z,z')(u'-z') \\ &\quad - \frac{1}{2}\big(m_1(u-z)(C-z') + m_2(u'-z')(z'-w')\big). \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.2), we obtain

$$f(u,u') \le f(z,z') + A(z,z')(u-z) + B(z,z',w')(u'-z'),\ \text{for } u \ge z, \tag{3.5}$$

where $A(z,z') = f_u(z,z') + m_1(C-z')$ and $B(z,z',w') = f_{u'}(z,z') + m_2(z'-w')$.

Define $g : \mathbb{R}^5 \to \mathbb{R}$ by

$$g(u, u'; z, z'; w') = f(z, z') + A(z, z')(u - z) + B(z, z', w')(u' - z').$$

Clearly, g is continuous and satisfies the following relations

$$\begin{cases} f(u, u') \leq g(u, u'; z, z'; w'), \ u \geq z \\ f(u, u') = g(u, u'; u, u'; w'). \end{cases} \tag{3.6}$$

Now, consider the following linear BVP

$$\begin{aligned} u''(y) &= g(u(y), u'(y); z(y), z'(y); w'(y)) \\ &= p(z(y), z'(y)) + A(z(y), z'(y))u(y) + B(z(y), z'(y), w'(y))u'(y), \\ u(0) &= 0, \quad u(1) = 1, \end{aligned} \tag{3.7}$$

where $p(z, z') = f(z, z') + A(z, z')z + B(z, z', w')z'$.

To develop the iterative scheme, we choose $w_0 = \alpha = 0$ as an initial approximation to the exact solution and consider the following linear BVP

$$\begin{aligned} u''(y) = g(u(y), u'(y); w_0'(y), w_0'(y); w_0'(y)), \\ u(0) = 0, \quad u(1) = 1. \end{aligned} \tag{3.8}$$

Using (3.6) and the definition of lower and upper solutions, we obtain

$$g(w_0(y), w_0'(y);\ w_0(y), w_0'(y); w_0'(y)) = f(w_0(y), w_0'(y)) \leq w_0''(y),\ y \in I$$
$$g(\beta(y), \beta'(y);\ w_0(y), w_0'(y); w_0'(y)) \geq f(\beta(y), \beta'(y)) \geq \beta''(y),\ y \in I,$$

which imply that w_0 and β are lower and upper solutions of (3.8). Hence, by Theorem 2.3, the solution w_1 of (3.8) satisfies $w_0 \leq w_1 \leq \beta$ on I. Moreover, in view of (3.6) and the fact that w_1 is a solution of (3.8), we obtain

$$w_1''(y) = g(w_1(y), w_1'(y);\ w_0(y); w_0'(y), w_0'(y)) \geq f(w_1(y), w_1'(y)),\ y \in I \tag{3.9}$$

which implies that w_1 is a lower solution of (1.3).

Now, we consider the following linear BVP

$$\begin{aligned} u''(y) = g(u(y), u'(y); w_1'(y), w_1'(y); w_0'(y)), \\ u(0) = 0, \quad u(1) = 1. \end{aligned} \tag{3.10}$$

By the same process as above and in view of (3.9), we can show that w_1 and β are lower and upper solutions of (3.10). Hence, there exists a solution w_2 of (3.10) such that $w_1 \leq w_2 \leq \beta$ on I.

Continuing this process we obtain a monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$\alpha = w_0 \leq w_1 \leq w_2 \leq w_3 \leq \cdots \leq w_{n-1} \leq w_n \leq \beta \text{ on } I,$$

where w_n is a solution of the following linear problem

$$u''(y) = g(u(y), u'(y); w_{n-1}(y), w'_{n-1}(y); w'_{n-2}(y)),\ y \in I \text{ and } n \geq 2,$$
$$u(0) = 0, \quad u(1) = 1,$$

and is given by

$$w_n(y) = y + \int_0^1 G(y,s) g(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s); w'_{n-2}(y)) ds,\ y \in I. \tag{3.11}$$

The sequence of functions w_n is uniformly bounded and equicontinuous. The monotonicity and uniform boundedness of the sequence $\{w_n\}$ implies the existence of a pointwise limit ω on I. From the boundary conditions, we have

$$0 = w_n(0) \to \omega(0) \text{ and } 1 = w_n(1) \to \omega(1).$$

Hence ω satisfy the boundary conditions. Moreover, by the dominated convergence theorem, for any $y \in I$, we have

$$\int_0^1 G(y,s) g(w_n(s), w'_n(s); w_{n-1}(s), w'_{n-1}(s); w'_{n-2}(y)) ds \to \int_0^1 G(y,s) f(\omega(s), \omega'(s)) ds.$$

Passing to the limit as $n \to \infty$, (3.11) yields

$$\omega(y) = y + \int_0^1 G(y,s) f(\omega(s), \omega'(s)) ds,\ y \in I,$$

which implies that ω is a solution of (1.3). Hence, the sequence of approximants $\{w_n\}$ converges to a solution of the nonlinear BVP (1.3).

4. NUMERICAL RESULTS AND DISCUSSION

Starting with the initial approximation $w_0 = 0$, results via GAM for different values of the parameters α_1, α_2 and m are obtained. Numerical simulation shows that only few iterations generated by the GAM lead to the exact solution of the problem independent of the choices of the parameters involved and the convergence is very fast. For example, see Tables $1, 2, 3$ and Figures $1, 2, 3$. For fixed $m = 0.5$ and $\alpha_2 = 0.3$, results via GAM for $\alpha_1 = 0.2$ and $\alpha_1 = 0.4$ are shown in the left and right of Fig. 1 and for $\alpha_1 = 0.8$ and $\alpha_1 = 1$, are shown in the left and right of Fig. 2. The GAM produces excellent results even for larger values of the parameters. For example, results via GAM for $m = 1.5$, $\alpha_1 = 0.4$, $\alpha_2 = 0.3$ and $m = 1.5$, $\alpha_1 = 1.5$, $\alpha_2 = 0.8$ are shown in the left and right of Fig. 3.

Finally, we study the effect of fluid parameters of the velocity field u. We observer that for fixed m and α_2, the velocity of the fluid increases as the value of the fluid parameter α_1 increases. For example, see Table 4 and Fig. 4, for the results via GAM for $m = 0.5$, $\alpha_2 = 0.3$ and $\alpha_1 = 0.2$, 0.4, 0.6, 0.8, 1. For fixed m and α_1, the velocity of the fluid decreases as the value of the fluid parameter α_2 increases. This fact is shown in Table 5 and in Fig. 5, for $m = 0.5$, $\alpha_1 = 0.4$ and $\alpha_2 = 0.3$, 0.5, 0.7, 0.9, 1.2.

Further, for fixed α_1, α_2, we observe that the velocity of the fluid decreases as the value of k increases, for example, see Table 6 and Fig. 6 for $\alpha_1 = 0.4$, $\alpha_2 = 0.3$ and $m = 0.5$, 1, 1.5, 2.2.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.0862405	0.173279	0.26192	0.352984	0.447313	0.54578	0.649296	0.758819	0.875361
w_2	0.090153	0.180961	0.272977	0.366765	0.462894	0.561953	0.664547	0.771308	0.882894
w_3	0.090464	0.181476	0.273588	0.367366	0.463395	0.562283	0.664674	0.771255	0.882764
w_4	0.0904687	0.181484	0.273599	0.367378	0.463406	0.562292	0.664682	0.77126	0.882766
w_5	0.0904688	0.181484	0.273599	0.367378	0.463406	0.562293	0.664682	0.771261	0.882766

Table 1; GAM for $m = 0.5$, $\alpha_1 = 0.2$ and $\alpha_2 = 0.3$.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.0887656	0.178188	0.26893	0.361663	0.457073	0.555867	0.658777	0.766563	0.880024
w_2	0.0932642	0.186967	0.281509	0.377287	0.4747	0.574142	0.676005	0.780679	0.88855
w_3	0.0934084	0.187213	0.281811	0.377601	0.474982	0.574358	0.676135	0.780723	0.888538
w_4	0.0934097	0.187216	0.281815	0.377605	0.474986	0.574362	0.676138	0.780725	0.888538
w_5	0.0934098	0.187216	0.281815	0.377605	0.474986	0.574362	0.676138	0.780725	0.888538

Table 2; GAM for $m = 0.5$, $\alpha_1 = 0.4$ and $\alpha_2 = 0.3$.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
w_1	0.0789724	0.159131	0.241679	0.327857	0.418959	0.516354	0.621503	0.735987	0.861524
w_2	0.00744577	0.0302052	0.0693221	0.126391	0.203565	0.303545	0.429551	0.585209	0.774275
w_3	0.0875309	0.175813	0.265599	0.35764	0.452689	0.551505	0.654851	0.76351	0.878282
w_4	0.0875309	0.175813	0.265599	0.35764	0.45269	0.551505	0.654851	0.76351	0.878282
w_5	0.0875309	0.175813	0.265599	0.35764	0.45269	0.551505	0.654851	0.76351	0.878282

Table 3; GAM for $m = 1.5$, $\alpha_1 = 1.5$ and $\alpha_2 = 0.8$.

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
u	0.0904688	0.181484	0.273599	0.367378	0.463406	0.562293	0.664682	0.771261	0.882766
u	0.0934098	0.187216	0.281815	0.377605	0.474986	0.574362	0.676138	0.780725	0.888538
u	0.0949697	0.190246	0.286135	0.382942	0.480971	0.580526	0.681907	0.785415	0.891347
u	0.0959343	0.192118	0.288798	0.386223	0.484638	0.584287	0.685411	0.788247	0.893033
u	0.0965893	0.193388	0.290604	0.388445	0.487118	0.586826	0.68777	0.790151	0.894163

Table 4; Results for u via GAM with fixed $m = 0.5$, $\alpha_2 = 0.3$ and different values of α_1, that is, $\alpha_1 = 0.2$ (1st row), $\alpha_1 = 0.4$ (2nd row), $\alpha_1 = 0.6$ (3rd row), $\alpha_1 = 0.8$ (4th row) and $\alpha_1 = 1$ (5th row).

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
u	0.0934098	0.187216	0.281815	0.377605	0.474986	0.574362	0.676138	0.780725	0.888538
u	0.0908795	0.182289	0.274762	0.368843	0.465091	0.564082	0.666419	0.772733	0.883691
u	0.0878576	0.176388	0.266278	0.358237	0.453016	0.551417	0.654312	0.762654	0.877496
u	0.0843484	0.169514	0.256339	0.345719	0.438628	0.536156	0.639539	0.75019	0.869732
u	0.0782829	0.157575	0.238941	0.323563	0.412815	0.508348	0.612181	0.726773	0.854997

Table 5; Results for u via GAM with $m = 0.5$, $\alpha_1 = 0.4$ and different values of α_2, that is, $\alpha_2 = 0.3$ (1st row), $\alpha_2 = 0.5$ (2nd row), $\alpha_2 = 0.7$ (3rd row), $\alpha_2 = 0.9$ (4th row) and $\alpha_2 = 1.2$ (5th row).

y	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
u	0.0934098	0.187216	0.281815	0.377605	0.474986	0.574362	0.676138	0.780725	0.888538
u	0.0873915	0.175534	0.265181	0.357093	0.452035	0.550786	0.654138	0.7629	0.877902
$u)$	0.0818773	0.164823	0.249913	0.338233	0.430889	0.529009	0.633754	0.746325	0.86797
u	0.0768109	0.154974	0.235854	0.320835	0.411335	0.508813	0.614786	0.730838	0.858646

Table 6; Results for u via GAM with $\alpha_1 = 0.4$, $\alpha_2 = 0.3$ and different values of m, that is, $m = 0.5$ (1st row), $m = 1$ (2nd row), $m = 1.5$ (3rd row) and $m = 2$ (4th row).

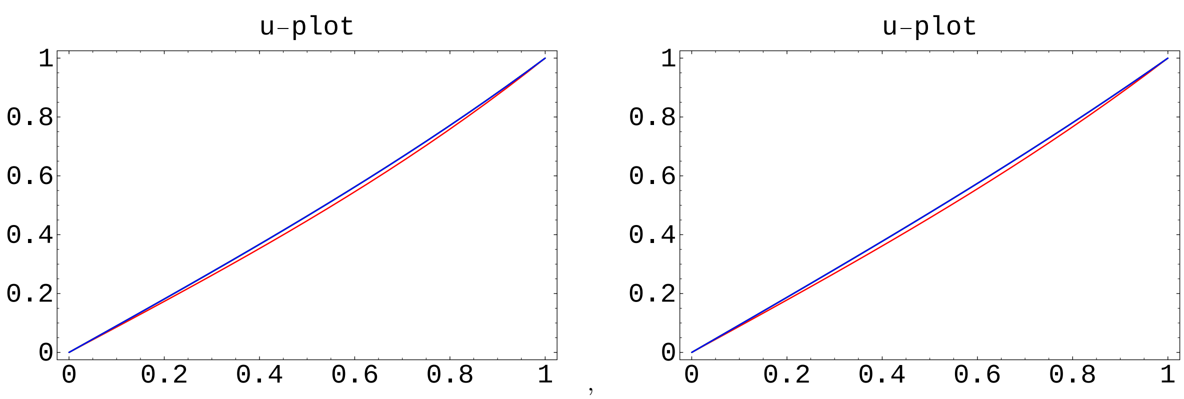

Fig. 1, GAM for $m = 0.5$, $\alpha_2 = 0.3$, $\alpha_1 = 0.2$ (left graph),$\alpha_1 = 0.4$ (right graph)

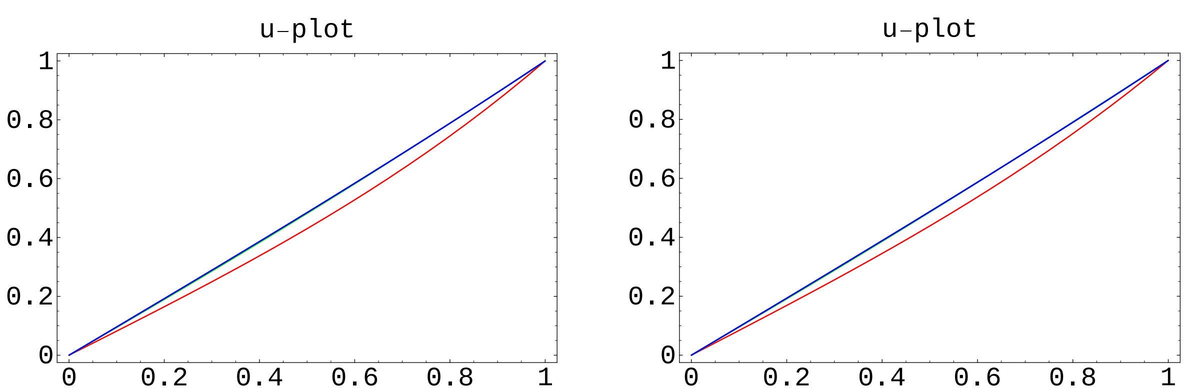

Fig. 2, GAM for $m = 0.5$, $\alpha_2 = 0.3$, $\alpha_1 = 0.8$ (left graph), $\alpha_1 = 1$ (right graph)

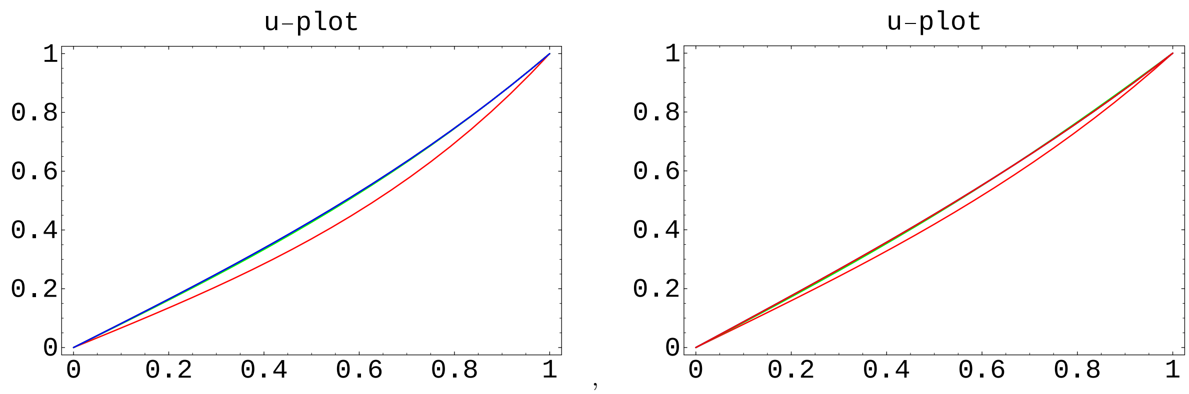

Fig. 3, GAM for $m = 1.5$ and $\alpha_1 = 0.4$, $\alpha_2 = 0.3$(left graph), $\alpha_1 = 1.5$, $\alpha_2 = 0.8$ (right graph)

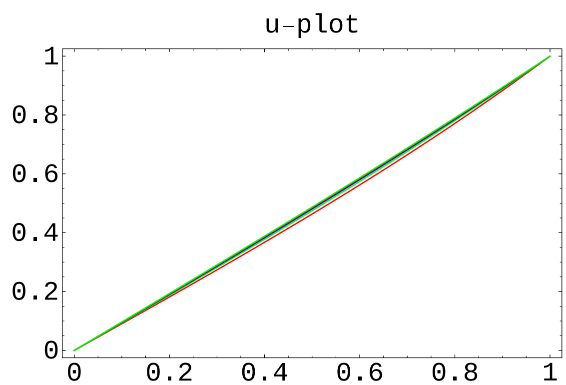

Fig. 4, GAM for $m = 0.5, \alpha_2 = 0.3$ and $\alpha_1 = 0.2$, 0.4, 0.6, 0.8, 1,

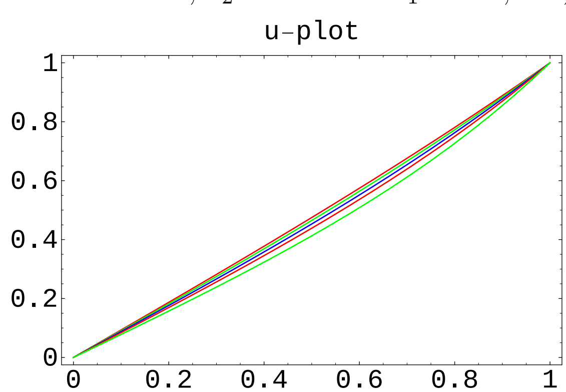

Fig. 5, GAM for $m = 0.5, \alpha_1 = 0.4$ and $\alpha_2 = 0.3$, 0.5, 0.7, 0.9, 1.2,

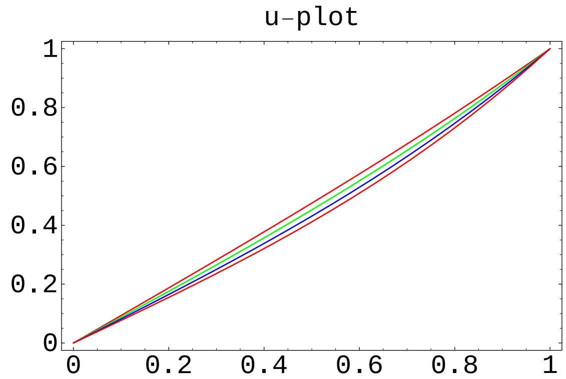

Fig. 6, GAM for $\alpha_1 = 0.4$, $\alpha_2 = 0.3$ and $m = 0.5$, 1, 1.5, 2

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