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Rank Properties of a Sequence of Block Bidiagonal Toeplitz Matrices

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Abstract. In the present paper, we proposed a new efficient rank updating methodology for evaluating the rank (or equivalently the nullity) of a sequence of block diagonal Toeplitz matrices. The results are applied to a variation of the partial realization problem. Characteristically, this sequence of block matrices is a basis for the computation of the Weierstrass canonical form of a matrix pencil that appeared in many practical numerical applications in control theory.

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1. Introduction-Preliminary Results

In several fields of applied mathematics, for instance in matrix pencil and control theory, in numerical linear algebra, even in Markov Chains, in financial and actuarial models (see, for instance the Bonus-Malus pricing policy model), a sequence of block of Toeplitz matrices are often appeared.

Consider the sequence of block Toeplitz matrices, see [1] (known in the literature also as "Matrices of the *Weyr* characteristics"), [9] and [2], as follows

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$$\mathrm{J}_0 \equiv A,\ \mathrm{J}_1 = \begin{bmatrix} A & \\ B & A \end{bmatrix},\ \mathrm{J}_2 = \begin{bmatrix} A & & \\ B & A & \\ & B & A \end{bmatrix},\ \ldots,\ \mathrm{J}_i = \begin{bmatrix} A & & & & & \\ B & A & & & & \\ & B & A & & & \\ & & \ddots & \ddots & & \\ & & & & B & A \end{bmatrix}$$

where $A,\ B \in \mathrm{F}^{m\times n}$ ($\mathrm{F}=\square\ \ or\ \square$) and $\mathrm{J}_i \in \mathrm{F}^{(i+1)m\times(i+1)n}$ for $i \in \square$. In this paper, for notion's simplicity, we denote J_i, for $i \in \square$, to be the (A,B)-sequence of matrices. In what it follows, we suppose that $m \geq n$.

It is not meaningless to say that the computation of rank of the (A,B) -sequence of matrices is a very important and a computationally challenging aspect, as well. However, it should be clarified that the efficient formulation of the algebraic to an equivalent numerical problem is more than essential. Consequently, in the present paper, we specify an appropriate numerical transformation of the theoretical notion of the rank (or equivalently of the nullity) of a matrix to an equivalent computational framework (see next section). This approach will form a basis of the proposed numerical procedure which computes the required rank (nullity) of a given sequence of block bidiagonal Toeplitz matrix. Due to the nature of this sequence, we try to exploit firstly its structure and afterwards to propose a *Rank Updating Technique* (RUT) that attains the required computation of rank achieving the lowest complexity compared with other well-established methods.

Finally, the new results may also be compared, see the results of [4], where a method of computing the ε-rank and ε-nullity of a matrix through the Singular Value Decomposition (**SVD**) method is proposed. This methodology is very standard, numerically stable and it is being adopted by many Computer packages, although it has a remarkable computational complexity. Precisely, for $A \in \mathrm{F}^{m\times n}$, $m \geq n$, it requires $2mn^2 - \frac{2}{3}n^3$ flops; see [3].

The following interesting results are appeared on [4] and [5].

Definition 1.1 The numerical ε-rank of a $m\times n$-matrix A is defined by

$$r_\varepsilon(A) = \min_B \left\{ rank(B):\ \|A-B\| \leq \varepsilon \right\} \tag{1.1}$$

and the numerical ε-nullity is defined by

$$n_\varepsilon(A) = \max\{nullity(B):\ \|A-B\| \le \varepsilon\}. \tag{1.2}$$

Considering to the authors' knowledge, the following theorem describes the simplest condition for the computational determination of the ε-rank ($r_\varepsilon(A)$) and the ε-nullity ($n_\varepsilon(A)$) of a given matrix $A \in \mathrm{F}^{m\times n}$.

Theorem 1.1 [5] For $A \in \mathrm{F}^{m\times n}$ and a given accuracy ε, it holds

a) $r_\varepsilon(A)$: Number of singular values of A that is greater to ε.

b) $n_\varepsilon(A)$: Number of singular values of A that is less or equal to ε.

c) $r_\varepsilon(A) = n - n_\varepsilon(A)$, when $m \ge n$.

2. The main results

In order to present the new algorithm, see session 3, the following theoretical results are required. For the (A,B)-sequence of matrices, we define the sequence of ranks, r_k for $k \in \square$

$$r_0 = rank(\mathrm{J}_0),$$

$$r_1 = rank(\mathrm{J}_1) - rank(\mathrm{J}_0),$$

$$\dots$$

$$r_k = rank(\mathrm{J}_k) - rank(\mathrm{J}_{k-1}).$$

In what it follows, the next two lemmas are important.

Lemma 2.1 $\{r_k\}_{k\in\square}$ is an increasing sequence.

Proof. Remind that $r_0 = rank(\mathrm{J}_0)$.

Now, the row space of $\mathrm{J}_0^t \in \square^{n\times m}$ (i.e. the transpose matrix of $\mathrm{J}_0 \equiv A$) is produced,

$$\mathrm{R}(\mathrm{J}_0^t) = span(\underline{\gamma}_1^0, \underline{\gamma}_2^0, \dots, \underline{\gamma}_{r_0}^0)$$

where $\underline{\gamma}_i^j$ are independent row vectors, for $i = 1, 2, \ldots, r_0$ and the $k \in \mathbb{N}$ step.

Due to the construction of (A, B)-sequence of matrices, the rows of J_1^t are linear independent and

$$\mathrm{R}\left(\mathrm{J}_1^t\right) = span\left(\underline{\gamma}_1^0, \underline{\gamma}_2^0, \ldots, \underline{\gamma}_{r_0}^0, \underline{\gamma}_1^1, \underline{\gamma}_2^1, \ldots, \underline{\gamma}_{r_o}^1, \ldots, \underline{\gamma}_{r_1}^1\right).$$

Thus, it is derived that $r_1 \geq r_0$. Analogously,

$$\mathrm{R}\left(\mathrm{J}_2^t\right) = span\left(\underline{\gamma}_1^0, \underline{\gamma}_2^0, \ldots, \underline{\gamma}_{r_0}^0, \underline{\gamma}_1^1, \underline{\gamma}_2^1, \ldots, \underline{\gamma}_{r_o}^1, \ldots, \underline{\gamma}_{r_1}^1, \ldots, \underline{\gamma}_{r_2}^2\right),$$

and $r_2 \geq r_1$. Working analogously, we finally obtain that $r_{k+1} \geq r_k$ for every step $k \in \mathbb{N}$. $\square$

In the next lemma, without lost of generality, we assume that the first k-row vectors are linear dependent.

Lemma 2.2 $A, B \in \mathrm{F}^{m \times n}$ and $\underline{v}_o, \ldots, \underline{v}_k \in \mathrm{F}^{1 \times n}$ are dependent and $\underline{v}_{k+1}, \ldots, \underline{v}_s \in \mathrm{F}^{1 \times n}$ are independent row-vectors, respectively, such as

$$\underline{v}_i A^t = \underline{v}_{i-1} B^t, \text{ for } i = 1, 2, \ldots, s. \tag{2.1}$$

Then, for an arbitrary $N > 0$, a sequence of vectors $\underline{u}_o, \underline{u}_1, \ldots, \underline{u}_{N-1}, \underline{u}_N \in \mathrm{F}^{1 \times n}$ with $\underline{u}_N \equiv \underline{\lambda}_s$ is always constructed satisfying the following recursive formula

$$\underline{u}_j A^t = \underline{u}_{j-1} B^t, \text{ for every } j = N, N-1, \ldots, 2, 1, \tag{2.2}$$

Proof. When $\underline{v}_s = \underline{0}^t$, we merely set $\underline{u}_k = \underline{0}^t$ for every $1 \leq k \leq s$. Profoundly, this case is not interesting. Consequently, we examine the case where $\underline{v}_s \neq \underline{0}^t$.

Since $\underline{v}_1, \ldots, \underline{v}_k \in \mathrm{F}^{1 \times n}$ are dependent and $\underline{v}_{k+1}, \ldots, \underline{v}_s \in \mathrm{F}^{1 \times n}$ are independent row - vectors, respectively, then

$$\underline{v}_k = \sum_{i=k+1}^{s} a_i \underline{v}_i \text{ for } k = 1, 2, \ldots, s \text{ and } a_i \in \mathrm{F}.$$

Afterwards, we denote V which is a sub-space of $\mathrm{F}^{1 \times n}$ spanned from the independent row vectors $\underline{v}_{k+1}$, $\underline{v}_{k+2}, \ldots, \underline{v}_s$. Note that $V \neq 0$, since $\underline{v}_s \neq \underline{0}^t$.

Then, we prove that for every $\underline{v} \in V$, there exists $\underline{u} \in V$, such as $\underline{v} A^t = \underline{u} B^t$.

Analytically, since $\underline{v} \in V$, then $\underline{v} = \sum_{i=k+1}^{s} a_i \underline{v}_i = a_{k+1}\underline{v}_{k+1} + a_{k+2}\underline{v}_{k+2} + \ldots + a_s \underline{v}_s$.

By considering also expressions (2.1), it is derived

$$\begin{aligned}\underline{v}A^t &= a_{k+1}\underline{v}_{k+1}A^t + a_{k+2}\underline{v}_{k+2}A^t + \ldots + a_s\underline{v}_s A^t = a_{k+1}\underline{v}_k B^t + a_{k+2}\underline{v}_{k+1}B^t + \ldots + a_s\underline{v}_{s-1}B^t \\ &= \left[a_{k+1}\left(\tilde{a}_{k+1}\underline{v}_{k+1} + \tilde{a}_{k+2}\underline{v}_{k+2} + \ldots + \tilde{a}_s\underline{v}_s\right) + a_{k+2}\underline{v}_{k+1} + \ldots + a_s\underline{v}_{s-1}\right]B^t\end{aligned}$$

$$\begin{aligned}&= \left[\left(a_{k+1}\tilde{a}_{k+1} + a_{k+2}\right)\underline{v}_{k+1} + \left(a_{k+1}\tilde{a}_{k+2} + a_{k+3}\right)\underline{v}_{k+2} + \ldots + \left(a_{k+1}\tilde{a}_{s-1} + a_s\right)\underline{v}_{s-1} + a_{k+1}\tilde{a}_s\underline{v}_s\right]B^t \\ &= \underline{u}B^t\end{aligned}$$

where $\underline{u} = \sum_{i=k+1}^{s} b_i \underline{v}_i$,

$$b_i = \left(a_{k+1}\tilde{a}_i + a_{i+1}\right), \text{ for } i = k+1, k+2, \ldots, s-1,$$

and $$b_i = a_{k+1}\tilde{a}_s, \text{ for } i = s.$$

Now, we are ready to construct the sequence of vectors $\underline{u}_j \in \mathrm{F}^{1\times n}$, as follows.

Firstly, we denote

$$\underline{u}_N \triangleq \underline{v}_s \in V,$$

then

$$\underline{u}_{N-1} \in V, \text{ such as } \underline{u}_N A^t = \underline{u}_{N-1} B^t.$$

Successively, since $\underline{u}_{N-1} \in V$, we can find all the other vectors satisfying (2.2).□

Now, we have all the necessary supplementary tools to introduce the follo-wing theorem. It can be proven that whenever the rank of the (A, B)-sequence of matrices at $k+1$ step is equal to the k-th step, i.e. $r_{\mathbf{k}+1} = \mathbf{k}$, afterwards the sequence of ranks stays constant and it is equal to k.

Theorem 2.1 If $r_{k+1} = k$, for $k \geq 0$ then $r_{k+j} = k$ for every $j \geq 2$.

Proof. Since $r_{k+1} = k$ for $k \geq 0$, then the rows

$$\left(\underline{\gamma}_1^0, \underline{\gamma}_2^0, \ldots, \underline{\gamma}_k^0, \underline{\gamma}_1^1, \underline{\gamma}_2^1, \ldots, \underline{\gamma}_k^1, \ldots, \underline{\gamma}_1^{k+1}, \underline{\gamma}_2^{k+1}, \ldots, \underline{\gamma}_k^{k+1}\right)$$

span the row space of J_{k+1}^{t}.

We will prove that for every $j \geq 2$, the

$$\left(\underline{\gamma}_1^0,\underline{\gamma}_2^0,\ldots,\underline{\gamma}_k^0,\underline{\gamma}_1^1,\underline{\gamma}_2^1\ldots,\underline{\gamma}_k^1,\ldots,\underline{\gamma}_1^{k+j},\underline{\gamma}_2^{k+j},\ldots,\underline{\gamma}_k^{k+j}\right)$$

span the row space of J_{k+j}^{t}, so $r_{k+j} \le r_{k+1}$. Consequently, considering Lemma 2.1, it is proven that $r_{k+j} = r_{k+1} = k$ for every $j \ge 2$ which concludes the proof.

Analytically, in order to go further it is sufficient to write the row vector $\underline{\gamma}_{k+1}^{k+1}$, as a linear combination of the independent rows $\underline{\gamma}_1^i,\underline{\gamma}_2^i,\ldots,\underline{\gamma}_k^i$, for every $0 \le i \le k+1$.

Moreover, using the coefficients and the results of Lemma 2.2, an expression for the row vector of J_{k+j}^{t}, i.e. $\underline{\gamma}_{k+1}^{k+j}$, for $j \ge 2$ is determined as a linear combination of the independent row vectors,

$$\left(\underline{\gamma}_1^0,\underline{\gamma}_2^0,\ldots,\underline{\gamma}_k^0,\underline{\gamma}_1^1,\underline{\gamma}_2^1,\ldots,\underline{\gamma}_k^1,\ldots,\underline{\gamma}_1^{k+j},\underline{\gamma}_2^{k+j},\ldots,\underline{\gamma}_k^{k+j}\right).$$

In order to simplify the presentation of the proof, we introduce the following notation.

We take $$A^t = \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_k \\ \vdots \\ \underline{a}_n \end{bmatrix} \in \square^{n\times m}, \text{ where } \underline{a}_i \in \square^{m},\ i=1,2,\ldots,n$$

and $$B^t = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_k \\ \vdots \\ \underline{b}_n \end{bmatrix} \in \square^{n\times m}, \text{ where } \underline{b}_i \in \square^{m},\ i=1,2,\ldots,n.$$

Moreover, we take

$\underline{\mathbf{a}} = \begin{bmatrix} \underline{a}_1 \\ \underline{a}_2 \\ \vdots \\ \underline{a}_k \end{bmatrix}$, i.e. the independent k-row vectors of matrix A^t, and $\underline{\mathbf{b}} = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_k \end{bmatrix}$, i.e. the independent k-row vectors of matrix B^t

(Without lost of generality, we may assume that the first k- rows vectors are independent).

It is also important to denote the sub-matrices of J_{i}^{t} as follows

$$[\underline{\mathbf{a}};\underline{\mathbf{b}}]_0 = \begin{bmatrix} \underline{\gamma}_1^0 \\ \vdots \\ \underline{\gamma}_k^0 \end{bmatrix} \in □^{\rho\times m},\ [\underline{\mathbf{a}};\underline{\mathbf{b}}]_1 = \begin{bmatrix} \underline{\gamma}_1^1 \\ \vdots \\ \underline{\gamma}_k^1 \end{bmatrix}, \ldots, [\underline{\mathbf{a}};\underline{\mathbf{b}}]_i = \begin{bmatrix} \underline{\gamma}_1^i \\ \vdots \\ \underline{\gamma}_k^i \end{bmatrix}, \text{ for } i \in □ .$$

Since the $r_{k+1} = k$, the row vector is written as

$$\underline{\gamma}_{k+1}^{k+1} = \underline{v}_0 [\underline{\mathbf{a}};\underline{\mathbf{b}}]_0 + \underline{v}_1 [\underline{\mathbf{a}};\underline{\mathbf{b}}]_1 + \ldots + \underline{v}_{k+1} [\underline{\mathbf{a}};\underline{\mathbf{b}}]_{k+1} . \tag{2.3}$$

Now, the following expressions are derived by considering (2.3) and making so-me algebraic calculations.

$$\underline{a}_{k+1} = \underline{v}_{k+1}\underline{\mathbf{a}} , \text{ and } \underline{b}_{k+1} - \underline{v}_{k+1}\underline{\mathbf{b}} = \underline{v}_k \underline{\mathbf{a}} . \tag{2.4}$$

Furthermore, the following sequence are also obtained,

$$\underline{v}_k \underline{\mathbf{b}} = \underline{v}_{k-1}\underline{\mathbf{a}} ,\ \underline{v}_{k-1}\underline{\mathbf{b}} = \underline{v}_{k-2}\underline{\mathbf{a}} , \ldots ,\ \underline{v}_1\underline{\mathbf{b}} = \underline{v}_0\underline{\mathbf{a}} . \tag{2.5}$$

Note that the $k+1$-row vectors, $\underline{v}_0, \underline{v}_1, \ldots, \underline{v}_k$ in $□^{1\times m}$, are linear dependent. Thus, if we apply the results of Lemma 2.2 by considering the sub-matrices $\underline{\mathbf{a}}$, $\underline{\mathbf{b}}$, we construct the sequence of row vectors $\underline{v}_k \equiv \underline{u}_{k+j-1}, \underline{u}_{k+j-2}, \underline{u}_{k+j-3}, \ldots, \underline{u}_1, \underline{u}_0$, such as $\underline{u}_i\underline{\mathbf{b}} = \underline{u}_{i-1}\underline{\mathbf{a}}$ for every $i = k+j-1, k+j-2, \ldots, 2, 1$.

Consequently, the expressions (2.4) also hold and it is obtained the following se-quence,

$$\underline{u}_{k+j-1}\underline{\mathbf{b}} = \underline{u}_{k+j-2}\underline{\mathbf{a}} ,\ \underline{u}_{k+j-2}\underline{\mathbf{b}} = \underline{u}_{k+j-3}\underline{\mathbf{a}} , \ldots ,\ \underline{u}_1\underline{\mathbf{b}} = \underline{u}_0\underline{\mathbf{a}} . \tag{2.6}$$

Due to the structure of matrix J_{k+j}^{t}, we observe that $\underline{u}_{k+j} = \underline{u}_{k+1}$ and $\underline{v}_{k+j} = \underline{v}_{k+1}$ for every $j \geq 2$. Moreover, combing the expression (2.3), (2.4) and (2.6), an analytical formula for the vectors $\underline{\gamma}_{k+1}^{k+j}$ is constructed

$$\underline{\gamma}_{k+1}^{k+j} = \underline{v}_0 [\underline{\mathbf{a}};\underline{\mathbf{b}}]_0 + \underline{v}_1 [\underline{\mathbf{a}};\underline{\mathbf{b}}]_1 + \ldots + \underline{v}_{k+j-2} [\underline{\mathbf{a}};\underline{\mathbf{b}}]_{k+j-2} + \underline{v}_{k+j-1} [\underline{\mathbf{a}};\underline{\mathbf{b}}]_{k+j-1} + \underline{v}_{k+1} [\underline{\mathbf{a}};\underline{\mathbf{b}}]_{k+j} . \tag{2.7}$$

Thus, it has been proved that $\underline{\gamma}_{k+1}^{k+j}$, for $j \geq 2$, is a linear combination of the rows

$$\left(\underline{\gamma}_1^0, \underline{\gamma}_2^0, \ldots, \underline{\gamma}_k^0, \underline{\gamma}_1^1, \underline{\gamma}_2^1, \ldots, \underline{\gamma}_k^1, \ldots, \underline{\gamma}_1^{k+j}, \underline{\gamma}_2^{k+j}, \ldots, \underline{\gamma}_k^{k+j}\right).$$

The theorem is fully completed. □

Remark 2.1 Since, in the next session, an algorithm is being constructed for matrices A, B, it should be noted that the results of Theorem 2.1 also holds for $m = n$. This is a straightforward result, given that $r_{n+1} \le n < n+1$.

The next examples clarify the importance of Theorem 2.1. Thus, in Example 2.1, it should be stressed out that even if two sequential ranks are equal, i.e. $r_0 = r_1 = 2$, not earlier than the 4th step, i.e. $\boldsymbol{r_{3+1}}$, we can assume that the final rank is being determined. In Example 2.2, a case where $r_2 \ne r_3 = 8$ is provided. However, we can not determine the rank of the sequence until the 9th step.

Example 2.1 Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

We construct the sequence of matrices as follows.

$\mathrm{J}_0 = A$ and $r_0 = rank(\mathrm{J}_0) = 2$.

$$\mathrm{J}_1 = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \hline 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \square \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ 0 & 0 & 0 & 0 & & & & \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

with $rank(\mathrm{J}_1) = 4$ and $r_1 = 2$.

$$
J_2 = \left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & & & & & & & & \\
0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
& & & & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
& & & & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
& & & & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & & & & & & & & \\
0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
0 & 0 & 0 & 0 & & & & & & & & \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & & & & \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
\hline
& & & & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
& & & & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
& & & & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right],
$$

with $rank(J_2) = 7$ and $r_2 = 3$.

Following the same procedure, we obtain

$$rank(J_3) = 10 \text{ and } r_3 = 3,$$

$$rank(J_4) = 13 \text{ and } r_{\mathbf{3+1}} = \mathbf{3},$$

$$rank(J_5) = 16 \text{ and } r_{3+2} = 3, \text{ etc.}$$

Example 2.2 Let

$$
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
2 & 1 & -2 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -3 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

We construct the sequence of matrices as follows.

$$r_0 = rank(\mathrm{J}_0) = 5;\ rank(\mathrm{J}_1) = 11 \text{ and } r_1 = 6;\ rank(\mathrm{J}_2) = 19 \text{ and } r_2 = 8,$$

$$rank(\mathrm{J}_3) = 27 \text{ and } r_3 = 8;\ rank(\mathrm{J}_4) = 35 \text{ and } r_4 = 8,$$

$$rank(\mathrm{J}_5) = 43 \text{ and } r_5 = 8;\ rank(\mathrm{J}_6) = 51 \text{ and } r_6 = 8,$$

$$rank(\mathrm{J}_7) = 59 \text{ and } r_7 = 8;\ rank(\mathrm{J}_8) = 67 \text{ and } r_8 = 8,$$

$$\text{and since } rank(\mathrm{J}_9) = 75 \text{ and } r_{\mathbf{8+1}} = \mathbf{8}.$$

Unfortunately, only after the 9th step, we are sure about the rank of the sequence, i.e. $r_9 = 8$.

Remark 2.2 (Tree Diagram) The whole procedure is presented densely through the following diagram.

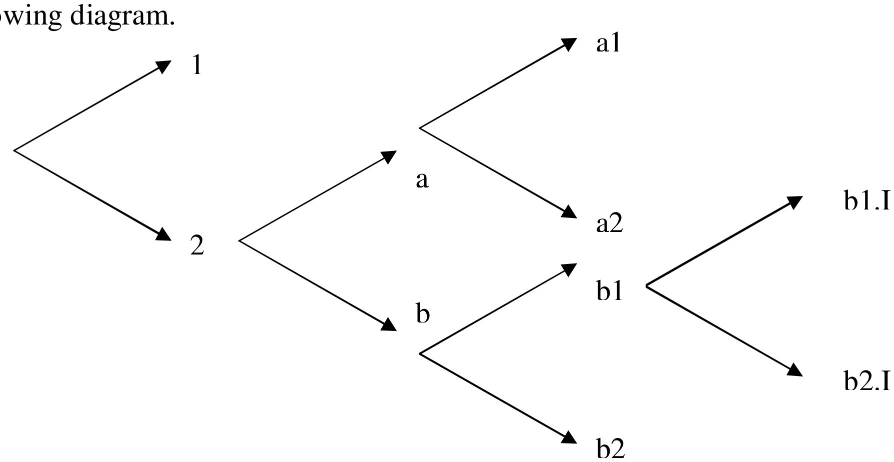

figure: Tree Diagram of the entire cases.

In more details, for the (A,B)-sequence of matrices, we suppose that $A,B \in \square^{n\times n}$ and the $rank(A) = \rho_1 < n$, $rank(B) = \rho_2 < n$, respectively.

Furthermore, the row vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{\rho_1}$ and $\underline{\eta}_1, \underline{\eta}_2, \ldots, \underline{\eta}_{\rho_2}$ span matrix A and B, respectively. Then, the following cases are only appeared:

1) **If** the row vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{\rho_1}$ are independent with $\underline{\eta}_1, \underline{\eta}_2, \ldots, \underline{\eta}_{\rho_2}$, then $r_1 = r_2 = r_3 = \ldots$ and $r_n = rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

2) **If** some of the row vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{\rho_1}$ are dependent with some of the row vectors of $\underline{\eta}_1, \underline{\eta}_2, \ldots, \underline{\eta}_{\rho_2}$, then the following two cases should be consider:

a) If the row vectors of $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{\rho_1}$ (span the A) are in different positions with the rows vectors of $\underline{\eta}_1, \underline{\eta}_2, \ldots, \underline{\eta}_{\rho_2}$ (span the B), then, after some algebraic transformations, we obtain the equivalent matrix

$$\begin{bmatrix} A & \\ B & A \end{bmatrix} \sim \begin{bmatrix} A & \\ \tilde{B} & \tilde{A} \end{bmatrix}.$$

Afterwards, two sub-cases should be considered. These sub-cases are related to matrix $\tilde{A}$.

a1) If the row vectors that span matrix $\tilde{A}$ are in different positions with the rows vectors that span matrix $\tilde{B}$, then $r_1 = r_2 = r_3 = \ldots$ and $r_n = rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

a2) Else, we obtain that $r_n \neq rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

b) Now, if some of the row vectors $\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_{\rho_1}$ (span the A) are in the same position with some of the row vectors $\underline{\eta}_1, \underline{\eta}_2, \ldots, \underline{\eta}_{\rho_2}$ (span the B), then two more sub-cases should be considered.

b1) If the row vectors that span matrix B are linear dependent with the rows vectors that span matrix A, then two sub-cases should be considered.

b1.I) If the row vectors that span matrix $\tilde{A}$ are in different position with the row vectors that span matrix $\tilde{B}$, then $r_n =\ rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

b1.II) Else, we obtain that $r_n \neq rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

b2) Finally, if the row vectors that span matrix B are linear dependent and are in different position, as well, with the row vectors that span matrix A, then $r_n \neq rank\left(\begin{bmatrix} B & A \end{bmatrix}\right)$ for every $n \geq 1$.

Consequently, in some cases due to Remark 2.2, the number of steps that needed according to equality, $r_{\mathbf{k}+1} = \mathbf{k}$, is already known from the very beginning. This remark can affect positively the speed of the algorithm.

3. Rank computation of the (A, B) sequence

In this section, the algorithm for computing the (A,B)-sequence of matrices is presented. However, to avoid transposing matrices A,B, it is supposed that $A,B \in \mathbb{R}^{n\times n}$ and $rank(A)=\rho_1<n$, $rank(B)=\rho_2<n$, respectively. Note that the results for rectangular matrices can easily derive. For the sequence above of matrices, two things are required to be computed:

a) the rank of matrices J_i, for $i \in \mathbb{N}$, and

b) the quantities $r_i = rank(\mathrm{J}_i) - rank(\mathrm{J}_{i-1})$.

In the vast literature of Numerical linear Algebra, the well known proposed methods, see [3], [4], and the references within, for the computations above are

- Gaussian Elimination Operations (LU or Gauss-Jordan)
- Rank Revealing QR
- Singular Value Decomposition (SVD).

Adopting the SVD approach for each matrix J_i and for a given accuracy ε, the ε-rank is computed by the relation (c); see Theorem 1.1. Although the special structure of (A,B)-sequence of matrices, i.e. J_i, can be exploited by SVD, the rank computation of each matrix J_i does not positively effected by the already known rank of matrix J_{i-1}. Thus, for each matrix J_i, its SVD is computed from the beginning and this approach requires much time and enough memory (cost) as the sequential steps are being increased.

More specifically, for the rank computation of a matrix $A \in \mathbb{R}^{n\times n}$ through SVD are required $\frac{4}{3}n^3$ flops, see [3]. Thus, for the rank calculation of each matrix J_i of order $in \times in$ are required $\frac{4}{3}(in)^3$ flops. Consequently, if we compute the ranks of the (A,B) sequence of matrices till the k-th step we obtain

$$\frac{4}{3}\left[n^3+2^3n^3+\ldots+2k^3n^3\right]=\frac{4}{3}n^3\left(1+2^3+\ldots+k^3\right) \text{ flops,}$$

thus k^4n^3 flops, totally.

An algorithm based on rank revealing QR decomposition (RRQR) has a better performance, since it exploits the structure of each matrix J_i, see [10]. The RRQR algorithm is applied each time to matrices having order size $O(n)$ and it needs n^3 flops for the calculation of rank for each J_i. Consequently, in the k-th step of the computation of rank for the (A,B)-sequence, kn^3 flops are required. Obviously, the RRQR approach is theoretically k^3 times faster than SVD.

In the next lines, a more efficient way for the rank computation based on the theoretical results of section 2 and the Gauss-Jordan transformation is presented. Before we go further, the next classical theorem is presented.

Theorem 3.1 (Gauss-Jordan reduction) [3] Let $A \in \mathbb{R}^{n\times n}$, by applying the sequence of Gauss-Jordan transformations G_i, A is transformed to the following diagonal matrix

$$G_A A \triangleq \underbrace{G_n G_{n-1} \cdots G_2 G_1}_{G_A} A \tag{3.1}$$

where
$$G_i = \begin{bmatrix} 1 & 0 & \cdots & -\dfrac{a_1}{a_i} & \cdots & 0 \\ 0 & 1 & \cdots & -\dfrac{a_2}{a_i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & -\dfrac{a_n}{a_i} & \cdots & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}.$$

Now, the algorithm is presented analytically, see also [8].

Rank Updating Technique (RUT)

Alg. Step 1: Transform A to diagonal form $G_A A$ using Gauss-Jordan transformations

$$A \xrightarrow[transformations]{Gauss-Jordan} G_A A,$$

and $\rho_1 = rank(A)$ (the none zero diagonal entries of $G_A A$).

Alg. Step 2: Transform also B to diagonal form,

$$B \xrightarrow[transformations]{Gauss-Jordan} G_B B,$$

and $\rho_2 = rank(B)$ (the none zero diagonal entries of $G_A B$).

***Alg. Step 3**: Apply the above transformation to matrix* $\mathrm{J}_1 = \begin{bmatrix} A & \mathrm{O} \\ B & A \end{bmatrix}$*, it derives*

$$\begin{bmatrix} G_A & \mathrm{O} \\ \mathrm{O} & G_B \end{bmatrix} \begin{bmatrix} A & \mathrm{O} \\ B & A \end{bmatrix} = \begin{bmatrix} G_A A & \mathrm{O} \\ G_B B & \tilde{A} \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$

and

$$\mathrm{J}_1^{\mathrm{M}} = \left[\begin{array}{cccc|ccc} * & & & & 0 & \cdots & 0 \\ & * & & & 0 & \cdots & 0 \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & * & 0 & \cdots & 0 \\ \hline * & & & & * & \cdots & * \\ & * & & & * & \cdots & * \\ & & \ddots & & \vdots & \ddots & \vdots \\ & & & * & * & \cdots & * \end{array}\right].$$

Making the Gauss-Jordan reduction, we can proceed by applying the next step.

Alg. Step 4:

If in matrix $G_B B$ exists rows proportional to the $G_A A$ rows, we can zero them. Then, in sub-matrix $\begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}$ the nonzero rows are gathered in the beginning. Thus,

$$M^{(1)} = \begin{bmatrix} G_B B^{(1)} & \tilde{A}^{(1)} \end{bmatrix} = \overset{M_{UL}^{(1)}}{\underset{M_{LR}^{(1)}}{\left[\begin{array}{ccc|ccc} * & & & * & \cdots & * \\ & \ddots & & \vdots & & \vdots \\ & & * & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right]}}$$

Compute the rank of the matrix J_1, using the relations

$$rank(\mathrm{J}_1) = rank(\mathrm{J}_1^{M}) = rank(G_A A) + rank(M^{(1)}),$$

Where

$$rank\left(M^{(1)}\right) = \underbrace{rank\left(M_{UL}^{(1)}\right)}_{\substack{number\ of\\ diagonal\ entries}} + rank\left(M_{LR}^{(1)}\right),$$

and $M_{LR}^{(1)} \in \mathbb{R}^{p_1 \times n}$, $p_1 \leq n$.

Moreover, since $r_0 = rank\left(\mathrm{J}_0\right) = rank\left(G_A A\right)$, then $r_1 = rank\left(M^{(1)}\right)$.

Else

$$M^{(1)} = \begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}.$$

Then, the $r_1 = rank\left(M^{(1)}\right) = rank\left(\begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}\right)$.

Remark 3.1 From **Alg. Step 4**, the rank computation of matrix J_1 requires the rank computation of a lower dimension matrix.

This remark will be further enriched in the discussion of the computational algorithm. Consequently, each time we will add a block of type $M^{(1)} = \begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}$ we will check its dependence with the nonzero upper part of the matrix. This idea leads to the following step.

Alg. Step 5: Set $k = 1$ (1st step)

While $r_k \neq k - 1$

If $G_B B$ has rows that are linear combinations of rows of $M_{LR}^{(1)} \in \mathbb{R}^{p_1 \times n}$ perform **Alg. Step. 6.**

Else perform **Alg. Step 7**.

end if.

Alg. Step 6: Zero the rows and reorder $\begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}$, such as the zero rows are appeared in the end. Construct the sub-matrix

$$M^{(k+1)} = \begin{bmatrix} G_B B^{(k)} & \tilde{A}^{(k)} & \\ & G_B B^{(k+1)} & \tilde{A}^{(k+1)} \end{bmatrix},$$

then

$$rank\left(\mathrm{J}_{k+1}\right) = rank\left(\mathrm{J}_k\right) + rank\left(\begin{bmatrix} G_B B^{(k+1)} & \tilde{A}^{(k+1)} \end{bmatrix}\right),$$

and

$$r_{k+1} = rank\left(M^{(k+1)}\right) = rank\left(\left[G_B B^{(k+1)} \quad \tilde{A}^{(k+1)}\right]\right),$$

where

$$M^{(k+1)} = \left[G_B B^{(k+1)} \quad \tilde{A}^{(k+1)}\right] = \begin{array}{c} M_{UL}^{(k+1)} \\ \left[\begin{array}{ccc|ccc} * & & & * & \cdots & * \\ & \ddots & & \vdots & & \vdots \\ & & * & * & \cdots & * \\ - & - & - & - & - & - \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right] \\ M_{LR}^{(k+1)} \end{array}.$$

Compute $rank\left(M_{UL}^{(k+1)}\right)$, $M_{LR}^{(k+1)} \in \mathbb{R}^{p_{k+1} \times n}$, $p_{k+1} \le n$.

$$r_{k+1} = rank\left(M^{(k+1)}\right) = rank\left(\left[G_B B^{(k+1)} \quad \tilde{A}^{(k+1)}\right]\right),$$

$$k := k+1.$$

Alg. Step 7: The part of matrix

$$M^{(k+1)} = \begin{bmatrix} G_B B^{(k)} & \tilde{A}^{(k)} & \\ & G_B B^{(k+1)} & \tilde{A}^{(k+1)} \end{bmatrix}$$

remains unchanged in the form

$$M^{(k+1)} = \begin{bmatrix} G_B B^{(k)} & \tilde{A}^{(k)} & \\ & G_B B & \tilde{A} \end{bmatrix},$$

then

$$rank\left(\mathrm{J}_{k+1}\right) = rank\left(\mathrm{J}_{k}\right) + rank\left(\left[G_B B \quad \tilde{A}\right]\right),$$

and

$$r_{k+1} = rank\left(\left[G_B B \quad \tilde{A}\right]\right).$$

end while.

end Algorithm.

However, in order to understand better that the algorithm is finally ended, the following lemma is required.

First of all, note that the $r_n = r_{k+1}$ for every $n \geq k+1$ and

$$rank\left(\mathrm{J}_{n}\right) = rank\left(\mathrm{J}_{k}\right) + (n-k)\, rank\left(\begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}\right),$$

or equivalently

$$rank\left(\mathrm{J}_{n}\right) = rank\left(\mathrm{J}_{k}\right) + (n-k)\, r_{k+1}.$$

Then, Lemma 2.3 is derived.

Lemma 2.3 It holds that

$$r_n = rank\left(\begin{bmatrix} G_B B & \tilde{A} \end{bmatrix}\right), \text{ for } n \geq k+1.$$

Proof. Consider that

$$\mathrm{J}_{k+1} = \begin{bmatrix} G_A A & & & & & \\ G_B B^{(1)} & \tilde{A}^{(1)} & & & & \\ & G_B B^{(2)} & \tilde{A}^{(2)} & & & \\ & & \ddots & \ddots & & \\ & & & G_B B^{(k)} & \tilde{A}^{(k)} & \\ & & & & G_B B & \tilde{A} \end{bmatrix}$$

and that the rows of $G_B B$ can not be written as a linear combination of rows of $M_{LR}^{(k+1)} \in \square^{p_{k+1} \times n}$. Thus, its nonzero rows are those that gave the nonzero rows of $G_B B^{(k)}$ and so on. Let us examine the matrix

$$\mathrm{J}_{k+2} = \begin{bmatrix} G_A A & & & & & & \\ G_B B^{(1)} & \tilde{A}^{(1)} & & & & & \\ & G_B B^{(2)} & \tilde{A}^{(2)} & & & & \\ & & \ddots & \ddots & & & \\ & & & G_B B^{(k)} & \tilde{A}^{(k)} & & \\ & & & & G_B B & \tilde{A} & \\ & & & & & G_B B & \tilde{A} \end{bmatrix},$$

and particularly, the lower 2×2- block sub-matrix

$$\begin{bmatrix} G_B B & \tilde{A} & \\ & G_B B & \tilde{A} \end{bmatrix}.$$

We can not also find rows of matrix $G_B B$ that can be expressed as linear combinations of those of $\tilde{A}$ which correspond to zero rows of $G_B B$. If such rows exist, then same things should appear to the 2×2 - block sub-matrix

$$\begin{bmatrix} G_B B^{(k)} & \tilde{A}^{(k)} & \\ & G_B B & \tilde{A} \end{bmatrix}$$

of matrix J_{k+1}. However, the non zero rows of $G_B B^{(k)}$ are fewer than those of $G_B B$, since the matrix $\tilde{A}^{(k)}$ has been produced by $\tilde{A}$, see for more details **Alg. Step 7**.

Remark 2.2 The following relation does also hold

$$rank\left(\begin{bmatrix} G_B B^{(k)} & \tilde{A}^{(k)} \end{bmatrix}\right) = rank\left(M_{UL}^{(k)}\right) + rank\left(M_{LR}^{(k)}\right).$$

Remark 2.3 (Computational Complexity) The transformation of A and B to $G_A A$ and $G_B B$ require $\frac{n^3}{2}$ flops each, respectively. Each computation of $rank\left(M_{LR}^{(i)}\right)$, for matrix $M_{LR}^{(i)} \in \mathbb{R}^{p_i \times n}$ requires $\frac{p_i^3}{3}$, for $p_i \leq n$, $i = 1, 2, \ldots.$ Consequently, the total flops are $n^3 + k\frac{p^3}{3}$, where $p = \max\{p_i\} \leq n$.

In the end of this section, the following table collects the number of flops required from the above methods.

TABLE

Matrix	**Rank Updating Technique (RUT)**	**SVD**	**RRQR**
J_k	$n^3 + k\frac{p^3}{3}$, $p << n$	$\frac{4}{3}\left(k^4 n^3\right)$	kn^3
J_n	$n^3 + n\frac{p^3}{3}$, $p << n$ (efficient)	$\frac{4}{3}\left(n^4 n^3\right)$ (NOT efficient)	n^4 (NOT efficient)

4. Conclusions

We have presented an updating algorithm for the computation of rank for a sequence of block bidiagonal Toeplitz matrices, J_i for $i = 1, 2, \ldots$. It should be stressed that the computational complexity of the proposed algorithm has indicated that performs better when it is compared with the SVD and RRQR techniques. Moreover, due to the nature of the method, RUT can also be applied symbolically.

Finally, the main direction of our future work is the embedding of the pro-posed algorithm to other related problems that are also under consideration. For instance, in Control theory, the computation of the Weierstrass and Kronecker canonical form for regular and singular matrices, respectively; see [7-9], is more than important.

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