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A UNIFORMLY CONVERGENT EXPONENTIAL SPLINE DIFFERENCE SCHEME FOR SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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ABSTRACT. We consider a Dirichlet boundary value problem for singularly perturbed reaction-diffusion equation. The problem is discretized using an exponential spline difference scheme derived on the basis of splines in tension. The fitted mesh technique is employed to generate piecewise-uniform Shishkin type mesh, condensed in the neighborhood of the boundary layers. The convergence analysis is given and the method is shown to have second order ε-uniform convergence on piecewise-uniform Shishkin type mesh. Numerical experiments are conducted to demonstrate the theoretical results.

Key Words. Singular perturbation problems, Boundary value problems, Piecewise-uniform Shishkin mesh, ε-Uniform convergence, Exponential splines, Splines in tension.

AMS Subject Classification. 65L10, 65L12

1. INTRODUCTION

Consider the singularly perturbed reaction-diffusion problem

$$L_\varepsilon u \equiv -\varepsilon u''(x) + b(x)u(x) = f(x), \quad \text{for } x \in (0,1), \tag{1.1}$$

with boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 0, \tag{1.2}$$

where ε is a small parameter and $f(x), b(x)$ are smooth functions and satisfy $b(x) \geqslant \beta^2 > 0 > 0$, $\forall x \in [0,1]$ for some positive constant β. The boundary value problem (1.1)–(1.2) under these assumptions possesses unique solution $u(x)$. In general, as ε tends to zero, the solution $u(x)$ may exhibit exponential boundary layers of width $O(\sqrt{\varepsilon}\ln(1/\sqrt{\varepsilon}))$ at both ends of the interval $[0,1]$. We consider the boundary value problem (1.1)–(1.2) with homogenous boundary conditions. As it is well known, by a simple substitution the boundary value problem with non-homogenous boundary conditions can be reduced to a boundary value problem with homogenous boundary conditions.

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The classical numerical methods on uniform mesh for solving singularly perturbed problems may give rise to difficulties when the singular perturbation parameter ε is sufficiently small. This leads to the development of the numerical methods that are uniformly convergent with respect to the perturbation parameter ε, that is, numerical methods for which there exists an N_0, independent of ε, such that for all $N \geq N_0$, where N is the number of mesh elements, the error constant and the rate of convergence in maximum norm are independent of ε. Thus a numerical method is said to be ε-uniform convergence of order k on the mesh $X_N = \{x_i, i = 0, 1, \ldots, N\}$ if there exists an N_0 independent of ε such that for all $N \geq N_0$

$$\sup_{0<\varepsilon\leq 1} \max_{X_N} |u - U_N| \leq CN^{-k},$$

where u is the exact solution, U_N is the numerical approximate to u, C and $k > 0$ are independent of ε and N.

In general, there are two types of ε-uniform numerical methods: fitted operator methods and fitted mesh methods. For the numerical solution of singularly perturbed boundary value problem (1.1)–(1.2), ε-uniform numerical methods consisting of fitted operators on uniform meshes have been constructed and analyzed in [2, 4–6, 12, 15]; and several types of special fitted mesh methods have been introduced and analyzed in [2, 6–7, 9, 13]. The application of exponential splines for the numerical solution of singularly perturbed boundary value problem (1.1)–(1.2) has been described [4–5, 10–12, 15]. In general exponentially fitted scheme is used to achieve the uniform convergence on uniform meshes. But some exponentially fitted schemes on non-uniform meshes were also derived in [3, 14, 16–17]. In these schemes the problem is how to change the mesh points when ε changes was not considered. This was addressed first in [1] for the classical difference schemes and this idea of Bakhvalov was further used in [17]. The determination of fitting factor for non-uniform meshes is more complicated. The fitting factor is determined in such a way that the truncation error of the difference scheme for the boundary layer functions should be zero (by assuming $b(x)$ is constant); and a special condition on the meshes is determined to prove the required accuracy of the difference scheme.

In the present paper, for the solution of singulary perturbed reaction-diffusion problem (1.1)–(1.2) we construct an exponential spline difference scheme based on spline in tension on piecewise-uniform Shishkin mesh. Since the spline difference scheme has the same order of accuracy and the similar matrix structure on both uniform and non-uniform meshes, we use an exponential spline difference scheme on piecewise-uniform Shishkin mesh to have more nodal points within the layers region. The present scheme is derived by combining the exponential spline identity relation with the exponential spline approximation of (1.1)–(1.2) at nodal points $x_i's$ of the mesh. Let $T(x) \in C^2[0,1]$ be the exponential spline and let $T(x)|_{[x_{i-1},x_i]} \in$

span$\{1, x, \exp(p_i x), \exp(-p_i x)\}$. When $b(x)$ is constant, the exponential spline collocation method, in which $T(x)$ collocate (1.1)–(1.2) at the nodal points $x_i' s$, can be interpreted as the present exponential spline difference scheme (see [10, 12]).

This paper is arranged as follows. The exponential spline difference scheme on piecewise-uniform Shishkin mesh for singularly perturbed boundary value problem (1.1)–(1.2) is described in section 2. The discrete comparison principle and uniform stability result of the present exponential spline difference scheme are also presented. In section 3, ε-uniform convergence result of the exponential spline difference scheme on piecewise uniform Shishkin mesh is given. Numerical experiments are conducted to demonstrate the efficiency of the proposed method in section 4. Results of experiments are discussed also discussed. Finally, the conclusions are included in section 6.

Let $0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1$, be the partition of $[0,1]$ and $h_i = x_{i+1} - x_i$, for $i = 0, 1, \ldots, N-2, N-1$, be the mesh spacing. We use the following notations in the remaining parts of the paper. $p = \min_{\forall i} p_i$, $h_{\max} = \max_{\forall i} h_i > 0$, $s_i = \sinh(p_i h_i)$, $c_i = \cosh(p_i h_i)$ and C is a generic positive constant independent of h_i and ε. $\|\cdot\|$ is used for the maximum norm, that is, $\|u\| = \max_{\forall i} |u_i|$.

2. DISCRETIZATION

In this section, we introduce an exponential spline difference scheme to discretize the singulary perturbed reaction-diffusion problem (1.1)–(1.2). Let Π be the partition of the given domain $[0,1]$, defined by

$$\Pi : 0 = x_0 < x_1 < x_2 < \cdots < x_{N-2} < x_{N-1} < x_N = 1,$$

with mesh width $h_i = x_{i+1} - x_i$.

Suppose p_is are non-negative tension parameters defined on each subinterval $[x_{i-1}, x_i]$, $i = 1(1)N$. Define the exponential spline $T(x) \in \text{span}\{1, x, \exp(p_i x), \exp(-p_i x)\}$, as the solution to the boundary value problems on each subinterval $[x_{i-1}, x_i]$, $i = 1(1)N$.

$$(D^4 - p_i^2 D^2)T(x) = 0, \quad \text{for } x \in (x_{i-1}, x_i),$$
$$T(x_{i-1}) = U_{i-1}, \quad T(x_i) = U_i, \quad T''(x_{i-1}) = T''_{i-1}, \quad T''(x_i) = T''_i,$$

with T''_i $(i = 0, \ldots, N)$ yet to be determined. Note that $p_i \to 0$ implies that $D^4 T(x) \to 0$, $x \in (x_{i-1}, x_i)$, which gives the cubic spline $S(x)$.

Using $T(x)$ from the above boundary value problem we obtain the following exponential spline identity relation

$$e_{i-1} T''_{i-1} + (d_{i-1} + d_i) T''_i + e_i T''_{i+1} = \frac{U_{i+1} - U_i}{h_i} - \frac{U_i - U_{i-1}}{h_{i-1}}, \quad i = 1(1)N - 1, \tag{2.1}$$

where

$$e_i = \frac{s_i - p_i h_i}{p_i^2 s_i h_i}, \quad d_i = \frac{p_i h_i c_i - s_i}{p_i^2 s_i h_i},$$

$$d_{i-1} = \frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{p_{i-1}^2 s_{i-1}h_{i-1}}, \quad e_{i-1} = \frac{s_{i-1} - p_{i-1}h_{i-1}}{p_{i-1}^2 s_{i-1}h_{i-1}},$$

which ensures the continuity of $T'(x)$ at the interior nodes.

Approximate the solution $u(x)$ of (1.1)–(1.2) with the exponential spline $T(x)$ and use $T_j'' = \frac{1}{\varepsilon}(b(x_j)U_j - f(x_j))$, $j = i, i \pm 1$, in (2.1) to obtain the following exponential spline difference scheme

$$[L_\varepsilon U]_i = -\varepsilon[\widetilde{D}U]_i + [PbU]_i = [Pf]_i, \quad \text{for } i = 1(1)N-1, \qquad U_0 = 0, \quad U_N = 0. \tag{2.2}$$

where

$$[\widetilde{D}v]_i = \frac{1}{\hbar_i}\Big(\frac{v_{i+1} - v_i}{h_i} - \frac{v_i - v_{i-1}}{h_{i-1}}\Big), \quad [Pv]_i = P_i^- v_{i-1} + P_i^c v_i + P_i^+ v_{i+1}$$

where

$$P_i^- = \frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1}h_{i-1}}, \quad P_i^+ = \frac{s_i - p_i h_i}{\hbar_i p_i^2 (s_i h_i},$$

$$P_i^c = \frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1}h_{i-1}} + \frac{p_i h_i c_i - s_i}{\hbar_i p_i^2 s_i h_i},$$

with

$$h_i = x_{i+1} - x_i \quad \text{and} \quad \hbar_i = \frac{h_{i-1} + h_i}{2}.$$

Note that, if the tension parameters p_is become zero then the present exponential spline difference scheme (2.2) reduces to the following cubic spline difference scheme

$$[L_\varepsilon U]_i = -\varepsilon[\widetilde{D}U]_i + [PbU]_i = [Pf]_i, \quad \text{for } i = 1(1)N-1, \qquad U_0 = 0, \quad U_N = 0, \tag{2.3}$$

where

$$[\widetilde{D}v]_i = \frac{1}{\hbar_i}\Big(\frac{v_{i+1} - v_i}{h_i} - \frac{v_i - v_{i-1}}{h_{i-1}}\Big), \quad [Pv]_i = P_i^- v_{i-1} + P_i^c v_i + P_i^+ v_{i+1},$$

where

$$P_i^- = \frac{h_{i-1}}{6\hbar_i}, \quad P_i^c = \frac{2}{3}, \quad P_i^+ = \frac{h_i}{6\hbar_i},$$

with

$$h_i = x_{i+1} - x_i \quad \text{and} \quad \hbar_i = \frac{h_{i-1} + h_i}{2}.$$

We apply the exponential spline difference scheme (2.2) on a piecewise uniform Shishkin mesh X_s^N. For this, the mesh X_s^N is constructed as follows.

Define the transition parameter

$$\tau = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon}}{\beta}\ln N\right\}. \tag{2.4}$$

Assuming that N is an integer and N is divisible by 4, we divide each of the two intervals $[0, \tau]$ and $[1 - \tau, 1]$ uniformly into $N/4$ subintervals and $[\tau, 1 - \tau]$ into $N/2$ subintervals of equal length.

The resulting piecewise-uniform Shishkin mesh may be represented as

$$(2.5)\quad h_i = \begin{cases} H_1 = \frac{4\tau}{N}, & x_i = x_{i-1} + H_1,\ \text{for } i = 1(1)\frac{N}{4} \text{ and } i = \frac{3N}{4} + 1(1)N; \\ H_2 = \frac{2(1-2\tau)}{N}, & x_i = x_{i-1} + H_2,\ \text{for } i = \frac{N}{4} + 1(1)\frac{3N}{4}. \end{cases}$$

If $\tau = \frac{1}{4}$, that is, $\frac{1}{4} \le \frac{2\sqrt{\varepsilon}}{\beta}\ln N$, then N^{-1} is very small relative to ε. This is unlikely in practice and in this case the method can be analyzed using the classical technique. We therefore assume $\tau = \frac{2\sqrt{\varepsilon}}{\beta}\ln N$. From (2.5), these mesh widths H_1 and H_2 satisfy

$$H_1 = \frac{8}{\beta}\sqrt{\varepsilon}N^{-1}\ln N, \quad N^{-1} \leqslant H_2 \leqslant 2N^{-1}.$$

The coefficient matrix of (2.2) is of size $(N+1)\times(N+1)$ and the unknowns are $U_0, U_1, \ldots, U_{N-1}, U_N$. On eliminating the U_0 and U_N using the boundary conditions, the coefficient matrix of (5) reduces to the size $(N-1)\times(N-1)$ with the unknowns $U_1, \ldots, U_{N-1}$. This system may be represented as

$$(2.6)\qquad A\alpha = d,$$

where A is an $(N-1)\times(N-1)$ tridiagonal matrix, $\alpha = [U_1, \ldots, U_{N-1}]^T$ and $d = [d_1, \ldots, d_{N-1}]^T$.

The elements of tridiagonal matrix $A = [a_{ij}]$ are

$$(2.7)\quad \begin{cases} a_{1,1} = \frac{2\varepsilon}{h_0 h_1} + (\frac{p_0 h_0 c_0 - s_0}{\hbar_1 p_0^2 s_0 h_0} + \frac{p_1 h_1 c_1 - s_1}{\hbar_1 p_1^2 s_1 h_1}) b_1; \\ a_{1,2} = -\frac{\varepsilon}{h_1 \hbar_1} + (\frac{s_1 - p_1 h_1}{\hbar_1 p_1^2 s_1 h_1}) b_2; \\ a_{i,i-1} = -\frac{\varepsilon}{h_{i-1}\hbar_i} + (\frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}}) b_{i-1},\ i = 2, \ldots, N-2; \\ a_{i,i} = \frac{2\varepsilon}{h_{i-1}h_i} + (\frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} + \frac{p_i h_i c_i - s_i}{\hbar_i p_i^2 s_i h_i}) b_i,\ i = 2, \ldots, N-2; \\ a_{i,i+1} = -\frac{\varepsilon}{h_i \hbar_i} + (\frac{s_i - p_i h_i}{\hbar_i p_i^2 s_i h_i}) b_{i+1},\ i = 2, \ldots, N-2; \\ a_{N-1,N-2} = -\frac{\varepsilon}{h_{N-2}\hbar_{N-1}} + (\frac{s_{N-2} - p_{N-2}h_{N-2}}{\hbar_{N-1} p_{N-2}^2 s_{N-2} h_{N-2}}) b_{N-2} \\ a_{N-1,N-1} = \frac{2\varepsilon}{h_{N-2}h_{N-1}} + (\frac{p_{N-2}h_{N-2}c_{N-2} - s_{N-2}}{\hbar_{N-1} p_{N-2}^2 s_{N-2} h_{N-2}} + \frac{p_{N-1}h_{N-1}c_{N-1} - s_{N-1}}{\hbar_{N-1} p_{N-1}^2 s_{N-1} h_{N-1}}) b_{N-1} \\ a_{i,j} = 0,\ \forall |i-j| > 1. \end{cases}$$

The elements d_i of column vector d are

$$d_i = \left(\frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}}\right) f_{i-1} + \left(\frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{\hbar_i p_{i-1}^2 (s_{i-1}h_{i-1})} + \frac{p_i h_i c_i - s_i}{\hbar_i p_i^2 s_i h_i}\right) f_i + \left(\frac{s_i - p_i h_i}{\hbar_i p_i^2 s_i h_i}\right) f_{i+1}, \quad i = 1, \ldots, N-1.$$

As

$$|a_{i,i}| - (|a_{i,i-1}| + |a_{i,i+1}|) > 0,$$

where $b(x_i) \geqslant \beta^2 > 0$, $p_i > 0$ and $h_i > 0$, the matrix A is strictly diagonally dominant. So the linear systems $A\alpha = d$ can be solved uniquely for the unknowns $U_1, \ldots, U_{N-1}$.

Define $p = \min_{\forall i} p_i$ is a uniform tension parameter for our present scheme (4). Now we use this uniform tension parameter p on each $[x_{i-1}, x_i]$. In general the matrix A is not an M-matrix. However now we choose a uniform tension parameters p as given in the following lemma that ensures the coefficient matrix A in (2.6) is an M-matrix.

Lemma 2.1. *If the tension parameter* $p \geq \max_{\forall i} \sqrt{b_i/\varepsilon}$, $i = 1, \ldots, N-1$, *then*

$$a_{i,i-1} < 0, \quad a_{i,i+1} < 0, \quad a_{i,i} > 0,$$

and

$$a_{i,i} + a_{i,i-1} + a_{i,i+1} \geq K > 0,$$

where K is a positive constant independent on h_i and ε.

Proof. From (2.7), We can write $a_{i,i-1}$, $a_{i,i}$ and $a_{i,i+1}$ as

$$a_{i,i-1} = -\frac{\varepsilon}{h_{i-1}\hbar_i} + \left(\frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}}\right) b_{i-1} = \frac{1}{h_{i-1}\hbar_i}\left(-\varepsilon + \frac{b_{i-1}}{p_{i-1}^2}\right) - \frac{b_{i-1}}{p_{i-1}s_{i-1}\hbar_i},$$

$$a_{i,i+1} = -\frac{\varepsilon}{h_i\hbar_i} + \left(\frac{s_i - p_i h_i}{\hbar_i p_i^2 s_i h_i}\right) b_{i+1} = \frac{1}{h_i\hbar_i}\left(-\varepsilon + \frac{b_{i+1}}{p_i^2}\right) - \frac{b_{i+1}}{p_i s_i \hbar_i},$$

$$\begin{aligned}
a_{i,i} &= \frac{2\varepsilon}{h_{i-1}h_i} + \left(\frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} + \frac{p_i h_i c_i - s_i}{\hbar_i p_i^2 s_i h_i}\right) b_i, \\
&= \frac{2\varepsilon}{h_{i-1}h_i} + \left(\frac{p_{i-1}h_{i-1}(c_{i-1}-1) + (p_{i-1}h_{i-1} - s_{i-1})}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} + \frac{p_i h_i (c_i - 1) + (p_i h_i - s_i)}{\hbar_i p_i^2 s_i h_i}\right) b_i \\
&= \frac{1}{h_{i-1}\hbar_i}\left(\varepsilon - \frac{b_i}{p_{i-1}^2}\right) + \Big(\frac{p_{i-1}h_{i-1}(c_{i-1}-1) + (p_{i-1}h_{i-1})}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} \\
&\quad + \frac{p_i h_i (c_i - 1) + (p_i h_i)}{\hbar_i p_i^2 s_i h_i}\Big) b_i + \frac{1}{h_i\hbar_i}\left(\varepsilon - \frac{b_i}{p_i^2}\right).
\end{aligned}$$

Choose the uniform tension parameter $p = \min_{\forall i} p_i \geq \max_{\forall i} \sqrt{b(x_i)/\varepsilon}$, this implies that $a_{i,i-1} < 0$, $a_{i,i-1} < 0$ and $a_{i,i} > 0$. While

$$\begin{aligned}
a_{i,i} + a_{i,i-1} + a_{i,i+1} &= \left(\frac{p_{i-1}h_{i-1}c_{i-1} - s_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} + \frac{p_i h_i c_i - s_i}{\hbar_i p_i^2 s_i h_i}\right) b_i \\
&\quad + \left(\frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}}\right) b_{i-1} + \left(\frac{s_i - p_i h_i}{\hbar_i p_i^2 s_i h_i}\right) b_{i+1}, \\
&\geq \left(\frac{c_{i-1} - 1}{\hbar_i p_{i-1} s_{i-1}} + \frac{c_i - 1}{\hbar_i p_i s_i}\right) \beta^2 > 0, \quad \text{as } b_i = b(x_i) \geq \beta^2 > 0.
\end{aligned}$$

Also,

$$c_i - 1 = \cosh(p_i h_i) - 1 = \frac{(p_i h_i)^2}{2} + \frac{(p_i h_i)^4}{24} + \cdots$$

$$p_i h_i s_i = p_i h_i \sinh(p_i h_i) = (p_i h_i)^2 + \frac{(p_i h_i)^4}{6} + \cdots$$

This implies that

$$a_{i,i} + a_{i,i-1} + a_{i,i+1} \geq K(say) > 0,$$

where K is a constant independent of h_i and ε. ◻

With the suitable choice of uniform tension parameter p as mentioned in the above lemma, the coefficient matrix A is an M-matrix, and hence, $[L_\varepsilon U]_i$ satisfies the following discrete comparison principle.

Lemma 2.2 (Discrete Comparison Principle)**.** *Let V and W be two mesh functions and satisfy $[L_\varepsilon V]_i \geq [L_\varepsilon W]_i$, $i = 1, 2, \ldots, N-1$, $V_0 \geq W_0$ and $V_N \geq W_N$, then $V_i \geq W_i$ $i = 0, 1, \ldots, N$.*

Using the above discrete comparison principle we obtain the following discrete stability estimate.

Lemma 2.3 (Stability Estimate)**.** *Let V be the mesh function with $V_0 = V_N = 0$. Then there exist a positive constant $C = 1/\min\{K, 1\}$ such that*

$$||V|| \leq C||L_\varepsilon V||,$$

where C is independent of h_i and ε.

We use the uniform tension parameter p on each subinterval $[x_{i-1}, x_i]\forall i = 1(1)N$, which is given by $p = \sqrt{\beta^*/\varepsilon}$, where $\beta^* \geq b(x)$ for $x \in [0, 1]$. This criterion of choosing the uniform tension parameter p leads to the ε-uniform stability for the exponential spline difference scheme.

Remarks

(i) In the present scheme $P_i^- \geq 0$, $P_i^+ \geq 0$ and $P_i^c \geq 0$.

$$P_i^- = \frac{s_{i-1} - p_{i-1}h_{i-1}}{\hbar_i p_{i-1}^2 s_{i-1} h_{i-1}} = \frac{h_{i-1}}{3(h_{i-1}+h_i)} \left\{ \frac{1 + \frac{(p_{i-1}h_{i-1})^2}{20} + \cdots}{1 + \frac{(p_{i-1}h_{i-1})^2}{6} + \cdots} \right\}$$
$$< \frac{h_{i-1}}{3(h_{i-1}+h_i)}, \quad \forall p_{i-1}h_{i-1} > 0,$$
$$P_i^+ = \frac{s_i - p_i h_i}{\hbar_i p_i^2 s_i h_i} = \frac{h_i}{3(h_{i-1}+h_i)} \left\{ \frac{1 + \frac{(p_i h_i)^2}{20} + \cdots}{1 + \frac{(p_i h_i)^2}{6} + \cdots} \right\} < \frac{h_i}{3(h_{i-1}+h_i)}, \quad \forall p_i h_i > 0,$$

similarly

$$P_i^c < \frac{2}{3}, \quad \forall p_{i-1}h_{i-1} > 0, \quad p_i h_i > 0.$$

These bounds on P_i^-, P_i^+ and P_i^c are useful in establishing the truncation error estimate (in the next section) at the transition points τ and $1-\tau$ of piecewise uniform Shishkin mesh X_N^S.

(ii) In the present scheme, as the tension parameters $p_i' s \to 0$, the exponential spline $T(x)$ reduces to the cubic spline $S(x)$. Consequently, the present exponential spline difference scheme (2.2) reduces to a non-monotone cubic spline difference scheme (2.3) which can also be derived via FEM discretization (see [7]).

(iii) The present exponential spline difference scheme (2.2), on uniform mesh $h_i = h_{i-1} = h = 1/N$, with the tension parameters $p_i = \sqrt{(\frac{b_i}{\varepsilon})}$, $\forall\ i$, reduces to the similar difference scheme derived via spline in tension on uniform mesh in [18]; in which the error of the form $O(h\min(h, \sqrt{\varepsilon}))$ is proved and the second order global uniform convergence result is established when $b(x)$ is constant. The numerical results in section 4 show that the above theoretical results are also true for the present scheme (4) on uniform mesh.

3. UNIFORM CONVERGENCE

We investigate the truncation error estimate of the present scheme (2.2) when applied to the problem (1.1)–(1.2). For the analysis of the truncation error we need sharp error bounds on the exact solution $u(x)$ of (1.1)–(1.2) and its derivatives. Vulanović [17] proved the following lemma for a priori estimates of the solution of the problem (1.1)–(1.2).

Lemma 3.1. *Let the exact solution $u(x) \in C^3[0,1]$ and it can be represented as*

$$u(x) = v(x) + w(x),$$

where, for $k = 0, 1, 2, 3 \quad \forall x \in [0,1]$

$$|v^{(k)}(x)| \leqslant C,$$

$$|w^{(k)}(x)| \leqslant C\varepsilon^{-k/2}(\exp(-\beta x/\varepsilon + \exp(-\beta(1-x)/\varepsilon))\}.$$

Proof. See Vulanović [17]. □

Now we discuss the convergence of the exponential spline difference scheme (2.2) on a piecewise-uniform Shishkin mesh X_s^N. For the purpose, we have $h_{max} \leq 2N^{-1}$. Let $\eta = u - U$ denote the error of the scheme. We consider the two distinct cases : $\tau = \frac{1}{4}$ and $\tau < \frac{1}{4}$.

In the first case, when $\tau = \frac{1}{4}$ the mesh is uniform with uniform spacing $h_i = \frac{1}{N}$ for $i = 1(1)N$. Moreover $\varepsilon^{-1} \leq (\frac{8}{\beta}\ln N)^2$.

If $u(x) \in C^3[0,1]$, then by a Taylor's expansion, using Lemma 3.1 with the value of $p = \sqrt{\beta^*/\varepsilon}$ (as discussed in the last section), we obtain

(3.1)
$$|[L_\varepsilon \eta]_i| = \varepsilon|[Pu'' - \widetilde{D}u]_i| \leq C\varepsilon h_i^2 p^2 \|u''\|_{[x_{i-1},x_{i+1}]} \leq CN^{-2}\ln^2 N, \quad i = 1(1)N-1.$$

While in the second case, when $\tau < \frac{1}{4}$, the mesh is piecewise-uniform and the mesh spacing h_i's is given by equation (2.5). The error analysis on piecewise-uniform Shishkin mesh X_s^N is discuss in the following cases.

(i) For $x_i \in (0,\tau)\cup(1-\tau,1)$, we have $h_i = h_{i-1} = \frac{8\sqrt{\varepsilon}}{\beta N}\ln N$. If $u(x)\in C^3[0,1]$, then by Taylor's expansion, using the value of $p=\sqrt{\beta^*/\varepsilon}$ with $p^2\|u''\|_{[x_{i-1},x_{i+1}]} \le C\varepsilon^{-2}$, we obtain

$$\varepsilon|[Pu''-\widetilde{D}u]_i| \le C\varepsilon h_i^2 p^2\|u''\|_{[x_{i-1},x_{i+1}]} \le CN^{-2}\ln^2 N, \quad \text{for } x_i \in (0,\tau)\cup(1-\tau,1).$$

(ii) For $x_i \in (\tau, 1-\tau)$, according to the decomposition of $u = v+w$, split the truncation error into two parts as

$$\varepsilon|[Pu''-\widetilde{D}u]_i| \le \varepsilon|[Pv''-\widetilde{D}v]_i| + \varepsilon|[Pw''-\widetilde{D}w]_i|. \tag{3.2}$$

If $u(x)\in C^3[0,1]$, then by Taylor's expansion, $|[L_\varepsilon\eta]_i| \le C\varepsilon h_i^2p^2\|u''\|_{[x_{i-1},x_{i+1}]}$, using the value of $p=\sqrt{\beta^*/\varepsilon}$ with $\varepsilon p^2\|v''\|\le C$ and $h_i = h_{i-1} \le 2N^{-1}$, the bound on the first term of the right hand side of (3.2) is given by

$$\varepsilon|[Pv''-\widetilde{D}v]_i| \le C\varepsilon h_i^2p^2\|v''\|_{[x_{i-1},x_{i+1}]} \le CN^{-2}, \quad \text{for } x_i\in(\tau,1-\tau). \tag{3.3}$$

It can be observed that the estimate $|[L_\varepsilon\eta]_i| \le C\varepsilon h_i^2p^2\|u''\|_{[x_{i-1},x_{i+1}]}$, together with Lemma 3.1 do not yield a bound on $\varepsilon|[Pw''-\widetilde{D}w]_i|$, that is uniform in ε. Therefore we use a Taylor's expansion with integral remainder to control $w(x)$. Using the estimate $|[L_\varepsilon\eta]_i| \le C\varepsilon\|u''\|_{[x_{i-1},x_{i+1}]}$, together with Lemma 3.1, the bound on the second term on the right hand side of (3.2) is given by

$$\varepsilon|[Pw''-\widetilde{D}w]_i| \le \varepsilon\|w''\|_{[x_{i-1},x_{i+1}]} \le CN^{-2}. \tag{3.4}$$

Using (3.3)–(3.4) in (3.2), we obtain

$$|[L_\varepsilon\eta]_i| = \varepsilon|[Pu''-\widetilde{D}u]_i| \le CN^{-2}, \quad \text{for } x_i\in(\tau,1-\tau).$$

(iii) For $x_i\in\{\tau,1-\tau\}$, according to the decomposition of $u(x)$, split the truncation error into two parts as

$$\varepsilon|[Pu''-\widetilde{D}u]_i| \le \varepsilon|[Pv''-\widetilde{D}v]_i| + \varepsilon|[Pw''-\widetilde{D}w]_i|. \tag{3.5}$$

Using the estimate $|[L_\varepsilon\eta]_i| \le C\varepsilon p^2(h_{i-1}+h_i)^2||u''||_{[x_{i-1},x_{i+1}]}$, for $h_{i-1}\ne h_i$, with the value of $p=\sqrt{\beta^*/\varepsilon}$ and Lemma 3.1, the bound on the first term of the right hand side of (3.5) is given by

$$\varepsilon|[Pv''-\widetilde{D}v]_i| \le CN^{-2}, \quad \text{for } i=\tau,1-\tau.$$

It can be observed that the estimate $|[L_\varepsilon\eta]_i| \le C\varepsilon p^2(h_{i-1}+h_i)^2\|u''\|_{[x_{i-1},x_{i+1}]}$ together with Lemma 3.1 do not yield a bound on $\varepsilon|[Pw''-\widetilde{D}w]_i|$, that is uniform in ε. Therefore we use Taylor's expansion with integral remainder to control $w(x)$. Using the estimate $|[L_\varepsilon\eta]_i| \le C\varepsilon\|u''\|_{[x_i,x_{i+1}]} + C\varepsilon h_{i-1}\|u^{(3)}\|_{[x_{i-1},x_i]}$, for $h_{i-1} \neq h_i$, together with Lemma 3.1, the bound on the second term of the right hand side of (3.5) is given by

$$\begin{aligned}\varepsilon|[Pw'' - \widetilde{D}w]_i| &\leq C\{\exp(-\beta x_{N/4}/\sqrt{\varepsilon}) + \exp(-\beta(1 - x_{3N/4})/\sqrt{\varepsilon})\} \\ &\quad + CN^{-1}\ln N\{\exp(-\beta x_{N/4-1}/\sqrt{\varepsilon}) + \exp(-\beta(1 - x_{3N/4+1})/\sqrt{\varepsilon})\} \\ &\leq C\{\exp(-\beta x_{N/4}/\sqrt{\varepsilon})\} + CN^{-1}\ln N\{\exp(-\beta x_{N/4-1}/\sqrt{\varepsilon})\}, \\ &\leq C\{\exp(-\beta x_{N/4}/\sqrt{\varepsilon})\} \\ &\quad + CN^{-1}\ln N\{\exp(-\beta x_{N/4}/\sqrt{\varepsilon})\}\{\exp(\beta H_1/\sqrt{\varepsilon})\}, \\ &\leq CN^{-2} + CN^{8/N}N^{-3}\ln N,\end{aligned}$$

since $x_{N/4} = \frac{2}{\beta}\sqrt{\varepsilon}\ln N$, $H_1 = \frac{4x_{N/4}}{N}$. As $N^{1/N} < \infty$ for $N \geq 1$, we obtain

$$\varepsilon|[Pw'' - \widetilde{D}w]_i| \leq CN^{-2} + CN^{-3}\ln N.$$

With the analogous estimate for $i = 3N/4$, we have

$$\varepsilon|[Pw'' - \widetilde{D}w]_i| \leq CN^{-2} + CN^{-3}\ln N.$$

On combining the various bounds for the truncation error, we get (for $\tau < \frac{1}{4}$)

$$\varepsilon|[Pu'' - \widetilde{D}u]_i| \leq \varepsilon|[Pv'' - \widetilde{D}v]_i| + \varepsilon|[Pw'' - \widetilde{D}w]_i| \leq CN^{-2}\ln^2 N. \tag{3.6}$$

From the above three cases (i), (ii) and (iii); we obtain

$$\varepsilon|[Pu'' - \widetilde{D}u]_i| \leq CN^{-2}\ln^2 N, \quad \text{for } \tau < \frac{1}{4}. \tag{3.7}$$

Thus combining (3.1) with (3.7), we obtain the truncation error estimate for the exponential spline difference scheme (4) on piecewise-uniform Shishkin mesh X_N^S is given by

$$\|L_\varepsilon \eta\| \leq CN^{-2}\ln^2 N. \tag{3.8}$$

On combining the above truncation error estimate (3.8) with the uniform stability estimate (Lemma 2.3), we conclude the section with following main theorem.

Theorem 3.2. *Let $u(x)$ be the exact solution of the singularly perturbed reaction-diffusion problem* (1.1)–(1.2) *and it satisfies Lemma* 3.1*. Let U be the approximate solution obtained by the exponential spline difference scheme* (2.2) *with a uniform tension parameter $p = \sqrt{\beta^*/\varepsilon}$, where $\beta^* \geq b(x) > 0$ for $x \in [0, 1]$, on piecewise uniform Shishkin mesh X_N^S. Then*

$$\|u - U\| \leq CN^{-2}\ln^2 N,$$

where C is independent of N and ε.

4. NUMERICAL EXPERIMENTS

The proposed exponential spline difference scheme (2.2) on a piecewise-uniform Shishkin mesh X_N^S is implemented on two test examples. For the present method, the maximum error, $e_{\varepsilon,i}^N = \max_{0\leqslant i\leqslant N} |u_i - U_i|$, at the nodal points is calculated and the numerical order of convergence is computed for different values of ε and N.

Example 4.1 Consider the following singularly perturbed boundary value problem

$$-\varepsilon u'' + u(x) = -\cos^2(\pi x) - 2\varepsilon\pi^2\cos(2\pi x),$$

$$u(0) = 0, \qquad u(1) = 0,$$

whose exact solution is

$$u(x) = (\exp(-(1-x)/\sqrt{\varepsilon}) + \exp(-x/\sqrt{\varepsilon}))/(1+\exp(-1/\sqrt{\varepsilon})) - \cos^2(\pi x)$$

and having the boundary layer of width $\delta = O(\sqrt{\varepsilon})$ at both ends of the given interval $[0,1]$.

Example 4.2 Consider the following singularly perturbed boundary value problem

$$-\varepsilon u'' + b(x)u(x) = f(x),$$

$$u(0) = 0, \qquad u(1) = 0,$$

where $b(x) = 1 + x(1-x)$ and $f(x) = 1 + x(1-x) + [2\sqrt{\varepsilon} - x(1-x)^2]\exp(-x/\sqrt{\varepsilon}) + [2\sqrt{\varepsilon} - x^2(1-x)]\exp(-(1-x)/\sqrt{\varepsilon})$, whose exact solution is

$$u(x) = 1 + (x-1)\exp(-x/\sqrt{\varepsilon}) - x\exp(-(1-x)/\sqrt{\varepsilon})$$

and having the boundary layer of width $\delta = O(\sqrt{\varepsilon})$ at both ends of the given interval $[0,1]$.

The numerical rate of convergence R is calculated using the formula

$$R = \frac{\log(e_\varepsilon^N/e_\varepsilon^{2N})}{\log 2},$$

for different value of ε and N, where $e_{\varepsilon,i}^N = \max_{0\leqslant i\leqslant N} |u_i - U_i|$ denotes the maximum error. Table 4.1 and Table 4.4 show the maximum error $e_{\varepsilon,i}^N$ and the numerical rate of convergence R of the present method on piecewise-uniform Shishkin mesh for the Example 4.1 and for the Example 4.2 respectively. It can be observed from these tables that the present method is uniformly convergent with respect to singular perturbation parameter ε and the computed order of convergence is almost close to theoretical order of convergence that is proved in section 3.

Table 4.3 and Table 4.6 show the numerical rate of convergence R of the present method on uniform mesh for the Example 4.1 and for the Example 4.2 respectively. For the smaller values of ε, it can be seen from the Table 4.3 and the Table 4.6 that the exponential spline difference scheme on uniform mesh have second order convergent

Table 4.1: Maximum error of the present method for the Example 4.1 on piecewise-uniform Shishkin mesh.

$\varepsilon = 10^{-k}$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
10^0	$3.04E-03$ 2.00	$7.61E-04$ 2.00	$1.90E-04$ 2.00	$4.76E-05$ 2.00	$1.19E-05$ 2.00	$2.98E-06$
10^{-1}	$2.28E-03$ 1.99	$5.73E-04$ 2.00	$1.43E-04$ 2.00	$3.58E-05$ 2.00	$8.96E-06$ 2.00	$2.24E-06$
10^{-2}	$1.84E-03$ 2.00	$4.60E-04$ 2.00	$1.15E-04$ 2.00	$2.88E-05$ 2.0	$7.21E-06$ 2.00	$1.80E-06$
10^{-3}	$3.65E-03$ 1.59	$1.21E-04$ 1.98	$3.06E-04$ 2.00	$7.66E-05$ 2.00	$1.91E-05$ 2.00	$4.79E-06$
10^{-4}	$3.93E-03$ 1.69	$1.22E-03$ 1.79	$3.53E-04$ 1.61	$1.16E-05$ 1.66	$3.66E-05$ 1.70	$1.13E-06$
10^{-5}	$2.82E-03$ 1.48	$1.01E-04$ 1.56	$3.43E-04$ 1.61	$1.12E-05$ 1.66	$3.55E-05$ 1.70	$1.09E-06$
10^{-6}	$2.81E-03$ 1.48	$1.01E-04$ 1.56	$3.42E-04$ 1.61	$1.12E-05$ 1.66	$3.54E-05$ 1.70	$1.09E-06$

Table 4.2: Comparison of maximum error of the present scheme (4) and non-monotone scheme (5) for $N = 32$ on piecewise-uniform Shishkin mesh for the Example 4.1.

$\varepsilon = 10^{-k}$	Non-monotone scheme (5)	Present scheme (4)
10^0	$2.96E-03$	$3.04E-03$
10^{-1}	$2.01E-03$	$2.28E-03$
10^{-2}	$1.31E-03$	$1.85E-03$
10^{-3}	$1.22E-02$	$3.65E-03$
10^{-4}	$1.23E-02$	$3.93E-03$
10^{-5}	$1.23E-02$	$2.82E-03$
10^{-6}	$1.23E-02$	$2.81E-03$

Table 4.3: Numerical rate of convergence R of the present scheme for the Example 4.1 on uniform mesh.

$\varepsilon = 10^{-k}$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
10^0	2.00	2.00	2.00	2.00	2.00
10^{-1}	1.99	2.00	2.00	2.00	2.00
10^{-2}	2.00	2.00	2.00	2.0	2.00
10^{-3}	1.97	1.98	2.00	2.00	2.00
10^{-4}	1.84	1.99	1.99	1.99	2.00
10^{-5}	1.38	1.70	1.90	1.96	2.10
10^{-6}	1.08	1.22	1.49	1.78	1.94

rate only when $h \leq \sqrt{\varepsilon}$. This supports the error estimate stated in Remark (iii) of section 2. In general in fitted scheme the fitting factor is determined in such a way that the truncation error of the difference scheme (under the assumption that

Table 4.4: Maximum error of the present method for the Example 4.2 on piecewise-uniform Shishkin mesh.

$\varepsilon = 10^{-k}$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
10^{0}	$1.58E-05$ 2.04	$3.85E-06$ 2.01	$9.56E-07$ 2.01	$2.38E-07$ 1.99	$6.00E-07$ 2.00	$1.50E-07$
10^{-1}	$7.67E-05$ 2.11	$1.78E-05$ 2.02	$4.38E-06$ 2.01	$1.09E-06$ 2.00	$2.72E-07$ 2.00	$6.80E-08$
10^{-2}	$5.11E-04$ 3.77	$3.74E-05$ 2.46	$6.80E-06$ 2.54	$1.70E-06$ 2.00	$4.25E-07$ 2.00	$1.06E-07$
10^{-3}	$2.42E-03$ 1.75	$7.18E-04$ 2.01	$1.78E-04$ 1.99	$4.47E-05$ 2.00	$1.12E-05$ 2.00	$2.79E-06$
10^{-4}	$2.67E-03$ 1.48	$9.60E-04$ 1.56	$3.25E-04$ 1.62	$1.06E-04$ 1.66	$3.36E-05$ 1.71	$1.03E-05$
10^{-5}	$2.81E-03$ 1.48	$1.01E-03$ 1.56	$3.42E-04$ 1.61	$1.12E-04$ 1.66	$3.55E-05$ 1.69	$1.10E-05$
10^{-6}	$2.86E-03$ 1.46	$1.03E-03$ 1.57	$3.48E-04$ 1.61	$1.14E-04$ 1.66	$3.61E-05$ 1.70	$1.11E-05$

Table 4.5: Comparison of maximum error of the present scheme (4) and non-monotone scheme (5) for $N = 32$ on piecewise-uniform Shishkin mesh for the Example 4.2.

$\varepsilon = 10^{-k}$	Non-monotone scheme (5)	Present scheme (4)
10^{0}	$5.10E-05$	$1.58E-05$
10^{-1}	$4.14E-04$	$7.67E-05$
10^{-2}	$2.18E-03$	$5.11E-04$
10^{-3}	$1.82E-02$	$2.42E-03$
10^{-4}	$1.42E-02$	$2.67E-03$
10^{-5}	$1.28E-02$	$2.81E-03$
10^{-6}	$1.24E-02$	$2.86E-03$

Table 4.6: Numerical rate of convergence R of the present scheme for the Example 4.2 on uniform mesh.

$\varepsilon = 10^{-k}$	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
10^{0}	2.04	2.01	2.01	1.99	2.00
10^{-1}	2.11	2.02	2.01	2.00	2.00
10^{-2}	3.77	2.46	2.54	2.00	2.00
10^{-3}	2.35	2.01	1.99	2.00	2.00
10^{-4}	1.45	1.92	1.98	1.98	1.99
10^{-5}	1.00	1.12	1.53	1.66	1.69
10^{-6}	1.00	0.98	1.01	1.03	1.21

$b(x)$ is constant) for the boundary layer functions should be zero. In the case of variable coefficient problems (1.1)–(1.2) the same fitting factor corresponding to the constant coefficient b is used after replacing b with $b_i = b(x_i)$. The determination of the fitting factor for exponentially fitted scheme on nonuniform meshes is more complicated. Also the error analysis of the fitted scheme on non-uniform meshes is complicated. The present scheme belongs to the class of fitted schemes and the fitting factor appears in an implicit form which can be obtained after rearranging the term of the scheme, under the assumption that b is constant. The error analysis of the present scheme on piecewise-uniform Shishkin is given and the scheme is shown to have almost second order uniform convergence in section 3. This theoretical rate of convergence can be verified from the results shown in Table 4.1 and Table 4.4 for the Example 4.1 and Example 4.2 respectively.

As the uniform tension parameter $p \to 0$, the tension spline $T(x)$ reduces to the cubic spline $S(x)$. Consequently the present exponential spline difference scheme (4) reduces to a non-monotone cubic spline difference scheme (5). In general the non-monotone cubic spline difference scheme (5) does not satisfy the discrete maximum principle. Recently Linss [7] derived the same non-monotone difference scheme (5) using FEM discretization and gives the stability estimate with the condition that $h_{\max}$ is smaller than some threshold value which is independent of ε. The second order ε-uniform convergence is proved via. Green's function technique. However, in the present exponential spline difference scheme based on spline in tension on piecewise-uniform Shishkin mesh, the uniform tension parameter p is so chosen that the coefficient matrix corresponding to the present scheme results in an M-matrix. The second order ε-uniform convergence in maximum norm on piecewise-uniform Shishkin mesh is proved using the truncation error and the barrier function technique. Table 4.2 and Table 4.5 show better approximation results of the present method in comparison to the non-monotone cubic spline difference scheme (5) on piecewise-uniform Shishkin mesh for small value of ε, for the Example 4.1 and for the Example 4.2 respectively.

5. **CONCLUSIONS**

We presented an exponential spline difference scheme based on spline in tension on a piecewise uniform Shishkin mesh for singularly perturbed Dirichlet boundary value problem (1.1)–(1.2). The essential idea in this method is to use exponential spline identity relation based on second derivative formulation to approximate the solution of given problems via. spline in tension that results in a tridiagonal system which can be solved using standard algorithm. The convergence analysis in maximum norm of the present exponential spline difference scheme is given and method is shown to have second order ε-uniform convergence on piecewise-uniform Shishkin mesh. The numerical experiments are presented to verify the uniform convergence of the exponential spline difference scheme with respect to singular perturbation parameter ε. Also this method produces a exponential spline function which is useful to obtain the solution at any point of the interval whereas the finite difference method gives the solution only at the selected nodal points.

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