Neural, Parallel, and Scientific Computations 18 (2010) 207-220

VARIABLE MESH SPLINE APPROXIMATION METHOD FOR SOLVING SINGULARLY PERTURBED TURNING POINT PROBLEMS HAVING INTERIOR LAYER

MOHAN K. KADALBAJOO AND KAILASH C. PATIDAR

Department of Mathematics, Indian Institute of Technology
Kanpur-208016, India
Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, Bellville 7535, South Africa

ABSTRACT. We present a second order numerical method based on cubic spline on a non-uniform mesh for the singularly perturbed two-point boundary value problems having interior layer in the turning point region. As opposed to our previous work dealing with the turning point problems having boundary layers [Kadalbajoo, M. K. & Patidar, K. C., (2001). Variable mesh spline approximation method for solving singularly perturbed turning point problems having boundary layer(s), Comp. Math. Appl., v. 42(10-11), 1439–1453], the distinctive feature of the problem considered in this paper lies in the fact that the layer appears in the interior of the region around the turning point. Rather than using a piecewise uniform mesh of Shishkin type [Miller, J. J. H., O'Riordan, E. & Shishkin, G. I., (1996). Fitted numerical methods for singular perturbation problems, Singapore: Word Scientific] which can not resolve the interior layer problems efficiently, we design a fully nonuniform mesh in the interior layer region. We then extend the classical approach of Berger et al. [Berger, A. E., Solomon, J. M. & Ciment, M., (1981). An analysis of a uniformly accurate difference method for a singular perturbation problem, Math. Comp., v. 37, 79–94] and Kellogg and Tsan [Kellogg, R. B. & Tsan, A., (1978). Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comp., v. 32, 1025–1039] to analyze our numerical method. Some numerical results confirming the theoretical estimates are also provided.

Key Words Boundary value problems; ordinary differential equations; singular perturbations; cubic splines; turning point problems; interior layers

AMS Subject Classification (2000): 34E20, 41A15, 65D07, 65L10

1. INTRODUCTION

We consider the following class of turning point problems (TPPs) for a singularly perturbed two-point boundary value problem:

$$(1.1)\qquad \left.\begin{array}{c} Ly \equiv \varepsilon y'' + a(x)y' - b(x)y = f(x) \quad \text{on } [p_1, p_2] \\ y(p_1) = \eta_1, \quad y(p_2) = \eta_2 \end{array}\right\}$$

Corresponding author: Kailash C. Patidar, E-mail: kpatidar@uwc.ac.za, Fax.: +27-21-9591241.
Received March 15, 2010

where $a(x)$ is assumed to be in $C^2[p_1, p_2]$; $b(x)$ and $f(x)$ are required to be in $C^1[p_1, p_2]$; $\eta_1,\ \eta_2$ are given constants; $p_1 \le 0$, $p_2 > 0$ (usually $p_1 = -1$ and $p_2 = 1$) and $0 < \varepsilon \ll 1$. Moreover

$$a(0) = 0, \quad a'(0) > 0. \tag{1.2}$$

For the solution of (1.1) to satisfy a maximum principle, we require that

$$b(x) \ge 0, \quad b(0) > 0. \tag{1.3}$$

Furthermore, in order to avoid resonance [1], the coefficient function $b(x)$ is required to be bounded below by some positive constant b, i. e.,

$$b(x) \ge b > 0. \tag{1.4}$$

We also impose the following condition which ensures that there are no other turning points in the interval $[p_1, p_2]$

$$|a'(x)| \ge \left|\frac{a'(0)}{2}\right|, \quad x \in [p_1, p_2]. \tag{1.5}$$

Under conditions (1.2)–(1.5), the turning point problem (1.1) has a unique solution having interior layer (a region of rapid variation in y or its derivatives) in the turning point region (c.f. [36]).

The turning point is simple if $a(x)$ vanishes at $x = 0$ and is called a multiple turning point if not only $a(x)$ but also its first derivative vanishes at $x = 0$. Simple turning point problems have attracted the most attention of all turning point problems, both analytically and numerically. The present work deals with the simple turning point problems. However, for multiple turning point problems, the readers are referred to [6, 42] and the references therein.

To the best of our knowledge, there are three principle approaches to solve such problems numerically, namely, the Finite-Difference Methods, the Finite-Element Methods and the Spline Approximation Methods. Probably Berger et al. ([6]) were the first to analyze TPPs using comparison function approach for finite difference techniques. Kellogg [28] solved these type of problems using Allen-Southwell difference schemes. Farrell [10] derived sufficient conditions for uniform convergence for some difference schemes for TPPs. Sun and Stynes [39] used Galerkin finite element methods for such problems, whereas Surla and Uzelac [41] solved them by taking a linear combination of the two spline difference schemes: El Mistikawy and Werle (EMW) scheme of [6] and Improved El Mistikawy and Werle (IEMW) scheme of [40]. More results based on numerical methods for turning point problems have been obtained in [31, 33, 37].

In this paper we have used the third approach, namely, the Spline Approximation Method which involves the use of a cubic spline on a non-uniform mesh in the interior layer region and uniform outside the region. Such spline techniques are used in the past to solve two-point boundary value problems for singularly perturbed non-turning point problems, see, e.g., [3, 12, 20, 21, 22, 23, 24, 25, 26] and the references therein (for older works) and those by the authors [14, 15, 16, 17, 18, 19]; or singularly perturbed turning point problems whose solutions possess boundary layers (see, e.g., [13]). However, to the best of our knowledge, this is the first work using spline techniques to solve interior layer problems. As far as non-spline techniques are concerned, some researchers did attempt to solve these interior layer problems (see, e.g., [11, 30, 32, 34] but the success (in terms of efficiency and overall order of convergence) was very limited.

The rest of the paper is organized as follows. In Section 2 we derive the difference scheme. Meshes are chosen according to the mesh selection strategy described in Section 3. The proposed numerical method has been analyzed for convergence in Section 4. Some comparative numerical results are presented in Section 5. We summarize the main outcomes of this paper in Section 6 where we also indicate the scope for future research.

2. **DERIVATION OF THE SCHEME**

The approximate solution of the problem (1.1) is sought in the form of the cubic spline function, which on each interval $[x_{j-1}, x_j]$, denoted by $S_j(x)$ and will be defined as follows. Let

$$x_0 = p_1, \quad x_j = p_1 + \sum_{m=1}^{j} h_m, \quad j = 1(1)n, \quad h_m = x_m - x_{m-1}, \quad x_n = p_2.$$

For the values $y(x_0), y(x_1), \ldots, y(x_n)$, there exists an interpolating cubic spline with the following properties

(i) $S_j(x)$ coincides with a polynomial of degree 3 on each interval $[x_{j-1}, x_j]$, $j = 1(1)n$,
(ii) $S_j(x) \in C^2[0,1]$,
(iii) $S_j(x_j) = y(x_j)$, $j = 0(1)n$.

Hence as in [2], the cubic spline can be given by

$$\begin{aligned} S_j(x) = {} & \frac{(x_j - x)^3}{6h_j} U_{j-1} + \frac{(x - x_{j-1})^3}{6h_j} U_j + \left(y_{j-1} - \frac{h_j^2 U_{j-1}}{6} \right) \left(\frac{x_j - x}{h_j} \right) \\ & + \left(y_j - \frac{h_j^2 U_j}{6} \right) \left(\frac{x - x_{j-1}}{h_j} \right), \end{aligned} \tag{2.1}$$

where

$$x \in [x_{j-1}, x_j], \quad h_j = x_j - x_{j-1}, \quad j = 1(1)n$$

and

$$U_j = S_j''(x_j), \quad j = 0(1)n.$$

Using this spline function we will derive the difference scheme in Section 3, which will give us the approximate solution of $y(x)$.

Differentiating (2.1) and denoting the nodal interpolants of $y(x)$ by u_j's, we get

$$S_j'(x) = -\frac{(x_j - x)^2}{2h_j}U_{j-1} + \frac{(x - x_{j-1})^2}{2h_j}U_j + \left(\frac{u_j - u_{j-1}}{h_j}\right) - \left(\frac{U_j - U_{j-1}}{6}\right) h_j. \tag{2.2}$$

Since $S_j(x) \in C^2[0, 1]$, therefore we have

$$S_j'(x_j) = S_{j+1}'(x_j). \tag{2.3}$$

This implies

$$\frac{h_j}{6}U_{j-1} + \frac{h_j + h_{j+1}}{6}U_j + \frac{h_{j+1}}{6}U_{j+1} = \frac{u_{j+1} - u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j}, \tag{2.4}$$

where

$$U_j = \frac{1}{\varepsilon}\left(f_j - a_j u_j' + b_j u_j\right). \tag{2.5}$$

Taking the Taylor series expansion for u around x_j and neglecting the terms containing third and higher order terms, we get the following approximations for u_{j+1} and u_{j-1}:

$$u_{j+1} \approx u_j + h_{j+1}u_j' + \frac{h_{j+1}^2}{2}u_j'' \tag{2.6}$$

$$u_{j-1} \approx u_j - h_j u_j' + \frac{h_j^2}{2}u_j'' \tag{2.7}$$

Simplifying (2.6) and (2.7), we obtain

$$u_j' \approx \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}\left[h_j^2 u_{j+1} - \left(h_j^2 - h_{j+1}^2\right) u_j - h_{j+1}^2 u_{j-1}\right] \tag{2.8}$$

and

$$u_j'' \approx \frac{2}{h_j h_{j+1}(h_j + h_{j+1})}\left[h_j u_{j+1} - (h_j + h_{j+1}) u_j - h_{j+1} u_{j-1}\right]. \tag{2.9}$$

On the other hand, we have

$$u_{j+1}' \approx u_j' + h_{j+1}u_j'' \tag{2.10}$$

and

$$u_{j-1}' \approx u_j' - h_j u_j''. \tag{2.11}$$

From equations (2.8), (2.9) and (2.10), we get

$$u_{j+1}' \approx \frac{1}{h_j h_{j+1}(h_j + h_{j+1})}\left[h_{j+1}^2 u_{j-1} - (h_j + h_{j+1})^2 u_j + \left(h_j^2 + 2h_j h_{j+1}\right) u_{j+1}\right] \tag{2.12}$$

and from equations (2.8), (2.9) and (2.11), we get

(2.13)
$$u'_{j-1} \approx \frac{1}{h_j h_{j+1}\left(h_j + h_{j+1}\right)}\left[-\left(h_{j+1}^2 + 2h_j h_{j+1}\right)u_{j-1} - \left(h_j + h_{j+1}\right)^2 u_j - h_j^2 u_{j+1}\right].$$

Therefore, using (2.4), (2.5), (2.8), (2.12) and (2.13), we obtain the difference scheme

$$Ru_j = Qf_j, \quad j = 1(1)n-1, \tag{2.14}$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1},$$
$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1},$$
$$u_0 = \alpha_0, \quad u_n = \alpha_1,$$
$$r_j^- = \frac{2h_j + h_{j+1}}{6(h_j + h_{j+1})}\, a_{j-1} + \frac{h_{j+1}}{3h_j}\, a_j - \frac{h_{j+1}^2}{6h_j(h_j + h_{j+1})}\, a_{j+1} + \frac{h_j}{6}\, b_{j-1} - \frac{\varepsilon}{h_j},$$
$$r_j^+ = \frac{h_j^2}{6h_{j+1}(h_j + h_{j+1})}\, a_{j-1} - \frac{h_j}{3h_{j+1}}\, a_j - \frac{2h_{j+1} + h_j}{6(h_j + h_{j+1})}\, a_{j+1} + \frac{h_{j+1}}{6}\, b_{j+1} - \frac{\varepsilon}{h_{j+1}},$$
$$r_j^c = -\frac{h_j + h_{j+1}}{6h_{j+1}}\, a_{j-1} - \frac{h_{j+1}^2 - h_j^2}{3h_j h_{j+1}}\, a_j + \frac{h_j + h_{j+1}}{6h_j}\, a_{j+1} + \frac{h_j + h_{j+1}}{3}\, b_j + \frac{\varepsilon}{h_j} + \frac{\varepsilon}{h_{j+1}},$$
$$q_j^- = -\frac{h_j}{6}, \quad q_j^+ = -\frac{h_{j+1}}{6}, \quad q_j^c = -\frac{h_j + h_{j+1}}{3}. \tag{2.15}$$

3. MESH SELECTION STRATEGY

There are a number of different mesh selection strategies found in the literature for non-turning point problems but almost all of them can not be adopted for the turning point problems whose solutions possess interior layers. Some attempts were made using piecewise uniform meshes of Shishkin type but the results were not very promising. This is largely due to the fact that determining the location of the interior layer is not as straightforward as in the case of boundary layers. To this end, we design in this section a totally non-uniform mesh. We form it in such a way that more points are generated in the interior layer region than outside this.

Let the concerned interval on which the problem is to be solved be $[p_1, p_2]$, where $p_1 < 0$ and $p_2 > 0$. Let its midpoint, i.e., $(p_1 + p_2)/2$, be denoted by c_p. By means of singular perturbation techniques (see, e.g., [29, 35, 36], etc, for details) we know that the interior layer is located near to this point c_p for the problems of the type (1.1) with conditions (1.2)–(1.5). Let δ denotes the width of this interior layer. Thus we have four subintervals

$$\left[p_1, c_p - \frac{\delta}{2}\right], \quad \left[c_p - \frac{\delta}{2}, c_p\right], \quad \left[c_p, c_p + \frac{\delta}{2}\right] \quad \text{and} \quad \left[c_p + \frac{\delta}{2}, p_2\right].$$

Let n_1, n_2, n_3 and n_4 be the number of points in these subintervals, respectively, such that, $n_1 + n_2 + n_3 + n_4 = n$, $n_1 = n_4$ and $n_2 = n_3$. Further let the positive constants $\widetilde{h}_{n_1+n_2+1}$ and K are known. Then we generate the mesh as follows.

On the interval $\left[p_1, c_p - \frac{\delta}{2}\right]$, the mesh is uniform and is defined as $h_j = \frac{\left(c_p - \frac{\delta}{2}\right) - p_1}{n_1}$, $j = 1(1)n_1$. On the interval $\left[c_p, c_p + \frac{\delta}{2}\right]$, the mesh is non-uniform and is defined as

$$\widetilde{h}_{n_1+n_2+j} = \widetilde{h}_{n_1+n_2+j-1} + K\left[\frac{\widetilde{h}_{n_1+n_2+j-1}}{\varepsilon}\right]\min\left(\widetilde{h}^2_{n_1+n_2+j-1}, \varepsilon\right), \quad j = 2(1)n_3.$$

Now, let

$$\widetilde{q} = \sum_{j=1}^{n_3} \widetilde{h}_{n_1+n_2+j}, \quad q = \frac{\delta/2}{\widetilde{q}}, \; h_{n_1+n_2+j} = q\widetilde{h}_{n_1+n_2+j}, \; j = 1(1)n_3.$$

On the interval $\left[c_p - \frac{\delta}{2}, c_p\right]$ the mesh is the mirror image of the mesh on $\left[c_p, c_p + \frac{\delta}{2}\right]$, i.e., $h_{n_1+n_2-j+1} = h_{n_1+n_2+j}$,$j = 1(1)n_3$. Finally, on the interval $\left[c_p + \frac{\delta}{2}, p_2\right]$, the mesh is uniform and is defined as $h_j = \frac{p_2 - \left(c_p - \frac{\delta}{2}\right)}{n_4}$, $j = n_1 + n_2 + n_3 + 1, n$. and then we have $x_0 = p_1$, $x_j = x_{j-1} + h_j$, $j = 1(1)n$.

4. ANALYSIS OF THE NUMERICAL METHOD

For the sake of simplicity, we consider $p_1 = -1$ and $p_2 = 1$. Also throughout the paper M will be used to denote positive constants which may take different values in different equations (inequalities) but that are always independent of h and ε.

Before we proceed, it is to be acknowledged that for the error analysis, we have used the comparison functions method developed by Kellogg and Tsan [27] and Berger et al. [5]. These functions are used together with the maximum principle to convert the bounds on the truncation error to the bounds on the discretization error. This method uses the following two lemmas [5]:

Lemma 4.1 (Discrete maximum principle)**.** *Let $\{u_j\}$ be a set of values at the mesh points x_j, satisfying $u_0 \le 0$, $u_n \le 0$; $(u_0 = u(x = -1))$ and $Ru_j \ge 0$, $j = 1(1)n - 1$, then $u_j \le 0$, $j = 0(1)n$.*

This discrete maximum principle permits the use of the comparison function approach for an error analysis of the scheme ([4, 27]).

Lemma 4.2. *If $K_1(h, \varepsilon) \ge 0$ and $K_2(h, \varepsilon) \ge 0$ are such that*

$$R(K_1(h, \varepsilon)\phi_j + K_2(h, \varepsilon)\psi_j) \ge R(\pm e_j) = \pm\tau_j(y),$$

for each $j = 1, 2, \ldots, n - 1$, then the discrete maximum principle implies that

$$|e_j| \le K_1(h, \varepsilon)|\phi_j| + K_2(h, \varepsilon)|\psi_j|,$$

where $|e_j| = |y(x_j) - u_j|$, for each j and ϕ and ψ are two comparison functions.

We use the following lemma (which is analogous to the Lemma 2.4 in [27]), for the properties of the exact solution of (1.1).

Lemma 4.3. *If $y(x)$ satisfies (1.1), then it can be decomposed as*

$$y(x) = g_1(x) + v(x) + g_2(x),$$

where

$$v(x) = \left(-\frac{\varepsilon y'(\theta)}{a(\theta)}\right) \exp\left[-\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right)x\right],$$

$$\left|g_1^{(k)}(x)\right| \le M\left[1+\varepsilon^{-k+1}\exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x\right\}\right],$$

$$\left|g_2^{(k)}(x)\right| \le M\left[1+\varepsilon^{-k+1}\exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x\right\}\right],$$

$\theta = c_p - \frac{\delta}{2}$, $k = 0(1)4$, *c is some constant, M is a positive constant independent of h and ε and δ denotes the width of the interior layer.*

We use the two comparison functions:

$$\phi = C_1 \exp\left[-2\eta C_3 \frac{\left(1+\frac{\delta}{2}\right)x}{\varepsilon}\right] \quad \text{and} \quad \psi = C_2 \exp\left[-2\eta C_3 \frac{\left(1-\frac{\delta}{2}\right)x}{\varepsilon}\right],$$

where C_1, C_2, C_3 and η are constants independent of h and ε.

Remark 4.4. The following inequalities hold

$$R\phi_j \ge M, \quad R\psi_j \ge M \quad \text{when } Ch_c^2 \le \varepsilon$$

and

$$R\phi_j \ge Mh_c, R\psi_j \ge Mh_c \quad \text{when } Ch_c^2 \ge \varepsilon,$$

where $h_c = \max_j h_j$ ($=$ a constant) and C is a positive constant independent of h and ε.

Now we estimate the truncation error of the scheme (2.14) using (2.15).

First consider the case in which $Ch_c^2 \le \varepsilon$. We have

$$\tau_j(y) = T_0 y_j + T_1 y_j' + T_2 y_j'' + T_3 y'''(\xi),$$

where $\xi \in (x_{j-1}, x_{j+1})$ and

$$\begin{aligned}
T_0 &= \left(r_j^- + r_j^c + r_j^+\right) + \left(q_j^- b_{j-1} + q_j^c b_j + q_j^+ b_{j+1}\right), \\
T_1 &= \left(h_{j+1} r_j^+ - h_j r_j^-\right) - \left\{q_j^-(a_{j-1} + h_j b_{j-1}) + q_j^c a_j + q_j^+(a_{j+1} - h_{j+1} b_{j+1})\right\}, \\
T_2 &= \left(\frac{h_j^2}{2} r_j^- + \frac{h_{j+1}^2}{2} r_j^+\right) - \varepsilon\left(q_j^- + q_j^c + q_j^+\right) + \left\{q_j^-\left(h_j a_{j-1} + \frac{h_j^2}{2} b_{j-1}\right)\right. \\
&\quad \left. + q_j^+\left(-h_{j+1} a_{j+1} + \frac{h_{j+1}^2}{2} b_{j+1}\right)\right\}
\end{aligned}$$

and

$$T_3 = \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right) - q_j^-\left\{\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right\}$$
$$- q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right\}.$$

Using (2.15) we see that $T_0 = 0$, $T_1 = 0$, $T_2 = 0$ and $|T_3| \le Mh_c^3$. Now from Lemma 4.3, we have

$$v_j''' = \left(-\frac{a(\theta)}{\varepsilon}\right)^2\left(1+\frac{\delta}{2}\right)^3 y'(\theta)\exp\left\{-\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right)x_j\right\}.$$

Therefore

$$|\tau_j(v)| \le \frac{Mh_c^3}{\varepsilon^2}\exp\left\{-\frac{a(\theta)}{\varepsilon}\left(1+\frac{\delta}{2}\right)x_j\right\}, \quad Ch_c^2 \le \varepsilon.$$

Also

$$\left|g_1^{(3)}(x_j)\right| \le M\left[1+\varepsilon^{-2}\exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}\right], \quad Ch_c^2 \le \varepsilon$$

and

$$\left|g_2^{(3)}(x_j)\right| \le M\left[1+\varepsilon^{-2}\exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}\right], \quad Ch_c^2 \le \varepsilon$$

$$\Rightarrow |\tau_j(g_1)| \le Mh_c^3\left[1+\frac{1}{\varepsilon^2}\exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}\right], \quad Ch_c^2 \le \varepsilon$$

and

$$|\tau_j(g_2)| \le Mh_c^3\left[1+\frac{1}{\varepsilon^2}\exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}\right], \quad Ch_c^2 \le \varepsilon.$$

Since

$$\tau_j(y) = \tau_j(g_1) + \tau_j(v) + \tau_j(g_2),$$

We have

$$|\tau_j(y)| \le M\frac{h_c^3}{\varepsilon^2}\left[1+\exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}+\exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}\right], \quad Ch_c^2 \le \varepsilon.$$

In the opposite case, i.e., when $Ch_c^2 \ge \varepsilon$, we use the following expression for truncation error

$$\tau_j(y) = \left(\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right)y'''(\xi_1) + \varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)y'''(\xi_2)$$
$$- q_j^-\left\{\frac{h_j^2}{2}a_{j-1} + \frac{h_j^3}{6}b_{j-1}\right\}y'''(\xi_3) - q_j^+\left\{\frac{h_{j+1}^2}{2}a_{j+1} - \frac{h_{j+1}^3}{6}b_{j+1}\right\}y'''(\xi_4),$$
$$x_{j-1} < \xi_i < x_{j+1}, \quad i = 1(1)4.$$

After some algebraic manipulations, we find that

$$\left|\frac{h_{j+1}^3}{6}r_j^+ - \frac{h_j^3}{6}r_j^-\right| \le Mh_c^3,$$
$$\left|\varepsilon\left(q_j^- h_j - q_j^+ h_{j+1}\right)\right| \le Mh_c^3,$$

$$\left|q_j^-\left(\frac{h_j^2}{2}a_{j-1}+\frac{h_j^3}{6}b_{j-1}\right)\right|\le Mh_c^3,$$

and

$$\left|q_j^+\left(\frac{h_{j+1}^2}{2}a_{j+1}-\frac{h_{j+1}^3}{6}b_{j+1}\right)\right|\le Mh_c^3.$$

Using these estimates and the above expression for $\tau_j(y)$, we obtain the same estimates for $\tau_j(g_1)$, $\tau_j(v)$ and $\tau_j(g_2)$ as were in the case of $Ch_c^2\le\varepsilon$. Finally, choosing

$$K_1=h_c^2\exp\left\{-\frac{c\left(1+\frac{\delta}{2}\right)}{\varepsilon}x_j\right\}$$

and

$$K_2=h_c^2\exp\left\{-\frac{c\left(1-\frac{\delta}{2}\right)}{\varepsilon}x_j\right\},$$

we see that Lemma 4.2 is satisfied (in both the cases $Ch_c^2\le\varepsilon$ and $Ch_c^2\ge\varepsilon$) and thus we have proved the following main result.

Theorem 4.5. *Let $\{u_j\}$, $j=0(1)n$, be a set of values of the approximate solution to $y(x)$ of (1.1), obtained by using (2.14) and (2.15). Then there are positive constants $\widetilde{C}_1\left(=\frac{c(2+\delta)}{2\varepsilon}(1+2\eta C_3)\right)$, $\widetilde{C}_2\left(=\frac{c(2-\delta)}{2\varepsilon}(1+2\eta C_3)\right)$ and M (independent of h and ε) such that the following estimate holds*

$$\max_j|y_j-u_j|\le Mh_c^2\left[\exp\left\{-\frac{\widetilde{C}_1x_j}{\varepsilon}\right\}+\exp\left\{-\frac{\widetilde{C}_2x_j}{\varepsilon}\right\}\right],$$

where $h_c=\max_j h_j=$ a constant.

The above estimate shows that for a fixed ε, our method is second order accurate. We will demonstrate this through some numerical experiments in the next section.

5. TEST EXAMPLES AND NUMERICAL RESULTS

Example 5.1 ([8])**.** Consider

$$\varepsilon y''+xy'-0.5y=0;\quad y(-1)=1,\quad y(1)=2.$$

Exact solution is not available.

Characteristics: The equation has a turning point at $x=0$ and the solution has an interior layer of width $\mathcal{O}\left(\sqrt{\varepsilon}\right)$ in the turning point region ([8]).

Example 5.2 ([37])**.** Consider

$$\varepsilon y''+xy'=0;\quad y(-1)=0,\quad y(1)=2,$$

whose exact solution is given by

$$y(x)=1+\frac{erf\left(x/\sqrt{2\varepsilon}\right)}{erf\left(1/\sqrt{2\varepsilon}\right)}.$$

Characteristics: The equation has a turning point at $x = 0$ and the solution has an interior layer of width $\mathcal{O}(\sqrt{\varepsilon})$ in the turning point region ([37]).

Tables 1 and 2 contain maximum errors based on the double mesh principle (Dollan et al. [7]) (as for Example 5.1, the exact solution is not available):

$$\max_{0\le j\le n}\left|u_j^n - u_{2j}^{2n}\right|.$$

Tables 4 and 5 contain the maximum errors $\max_j |y_j - u_j|$, at all the mesh points for different n and ε, where u_j is the approximate solution of the problem considered in Example 5.2.

Tables 3 and 6 contain the numerical rate of uniform convergence which is determined as in [7]:

$$r_k \equiv r_{k,\varepsilon} := \log_2\left(z_{k,\varepsilon}/z_{k+1,\varepsilon}\right), \quad k = 0, 1, 2, \dots$$

where

$$z_{k,\varepsilon} = \max_j |u_j^{h_j/2^k} - u_{2j}^{h_j/2^{k+1}}|, \quad k = 0, 1, 2, \dots$$

denotes the maximum error for $n = n_k$ and $u_j^{h_j/2^k}$ denotes the value of u_j for the mesh length $h_j/2^k$.

6. SUMMARY AND SCOPE FOR FUTURE RESEARCH

We have described a second order numerical method using cubic spline on a non-uniform mesh to solve a class of singularly perturbed turning point problems whose solutions possess interior layer in the turning point region. The method is analyzed for convergence and we found that it is second order accurate which is duly verified by solving two numerical examples. Performance of the proposed numerical method on our non-uniform mesh can be seen further from the Tables 1, 2, 4 and 5 in which we have shown that the results obtained by using the non-uniform mesh are better than those obtained using the uniform mesh.

It should be noted that in our mesh selection procedure, we have chosen $\delta = O(\sqrt{\varepsilon})$ (see [8, 37]), $\widetilde{h}_{n_1+n_2+1} = 0.00001$ and $K = 1$. However the increase in the value of K will lead to more concentration of points in the interior layer region. Moreover, for a fixed K, the increase in the value of $\widetilde{h}_{n_1+n_2+1}$ leads to the same conclusion.

While a lot of work can be found in the literature using adaptive mesh methods for problems without interior layers, not much has been done towards interior layer problems. In fact it is seen how difficult it is to construct such methods when the solution has an interior layer (see , e.g., [9, 38]). To this end, the present work is not only a reasonable achievement but also the first attempt towards achieving higher

TABLE 1. Numerical Results for Example 5.1: Max. Error using uniform mesh

ε	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
2^{-10}	0.13E-01	0.29E-02	0.66E-03	0.17E-03	0.42E-04	0.10E-04
2^{-11}	0.21E-01	0.53E-02	0.12E-02	0.27E-03	0.71E-04	0.18E-04
2^{-12}	0.31E-01	0.93E-02	0.21E-02	0.47E-03	0.12E-03	0.30E-04

TABLE 2. Numerical Results for Example 5.1: Max. Error using about 25% mesh points in the interior layer region

ε	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
2^{-10}	0.34E-02	0.46E-03	0.10E-03	0.27E-04	0.72E-05	0.19E-05
2^{-11}	0.72E-02	0.12E-02	0.16E-03	0.42E-04	0.12E-04	0.31E-05
2^{-12}	0.13E-01	0.26E-02	0.37E-03	0.67E-04	0.18E-04	0.50E-05

TABLE 3. Numerical Results for Example 5.1: Rates of convergence using about 25% mesh points in the interior layer region $n_k = 32, 64, 128, 256, 512$

ε	r_1	r_2	r_3	r_4	r_5
2^{-8}	0.20E+01	0.19E+01	0.20E+01	0.20E+01	0.20E+01
2^{-9}	0.25E+01	0.20E+01	0.19E+01	0.20E+01	0.20E+01
2^{-10}	0.29E+01	0.22E+01	0.19E+01	0.19E+01	0.19E+01
2^{-11}	0.26E+01	0.29E+01	0.19E+01	0.19E+01	0.19E+01

TABLE 4. Numerical Results for Example 5.2: Max. Error using uniform mesh

ε	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
2^{-5}	0.29E-02	0.72E-03	0.18E-03	0.45E-04	0.11E-04	0.28E-05
2^{-6}	0.52E-02	0.14E-02	0.36E-03	0.90E-04	0.22E-04	0.56E-05
2^{-7}	0.12E-01	0.29E-02	0.72E-03	0.18E-03	0.45E-04	0.11E-04

TABLE 5. Numerical Results for Example 5.2: Max. Error using about 25% mesh points in the interior layer region

ε	$n=32$	$n=64$	$n=128$	$n=256$	$n=512$	$n=1024$
2^{-5}	0.18E-02	0.45E-03	0.11E-03	0.28E-04	0.69E-05	0.17E-05
2^{-6}	0.30E-02	0.67E-03	0.16E-03	0.41E-04	0.10E-04	0.25E-05
2^{-7}	0.65E-02	0.14E-02	0.33E-03	0.83E-04	0.21E-04	0.52E-05

TABLE 6. Numerical Results for Example 5.2: Rates of convergence using about 25% mesh points in the interior layer region $n_k = 32, 64, 128, 256, 512$

ε	r_1	r_2	r_3	r_4	r_5
2^{-4}	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
2^{-5}	0.20E+01	0.20E+01	0.20E+01	0.20E+01	0.20E+01
2^{-6}	0.22E+01	0.21E+01	0.20E+01	0.20E+01	0.20E+01
2^{-7}	0.23E+01	0.21E+01	0.20E+01	0.20E+01	0.20E+01

accuracy. However, there are still some concerns about parameter uniform results. Authors are currently investigating these issues.

REFERENCES

[1] Abrahamsson, L.R. (1977). A priori estimates for solutions of singular perturbations with a turning point, Stud. Appl. Math., v. 56, 51–69.

[2] Ahlberg, J.H., Nilson, E.N. & Walsh, J.L. (1967). The theory of splines and their applications, New York: Academic Press.

[3] Bawa, R.K. & Natesan, S. (2005). A computational method for self-adjoint singular perturbation problems using quintic spline, Comp. Math. Appl., v. 50 (8-9), 1371–1382.

[4] Berger, A.E., Solomon, J.M., Ciment, M., Leventhal, S.H. & Weinberg, B.C., (1980). Generalized operator compact implicit schemes for boundary layer problems, Math. Comp., v. 35, 695–731.

[5] Berger, A.E., Solomon, J.M. & Ciment, M., (1981). An analysis of a uniformly accurate difference method for a singular perturbation problem, Math. Comp., v. 37, 79–94.

[6] Berger, A.E. , Han, H. & Kellogg, R.B., (1984). A priori estimates and analysis of a numerical method for a turning point problem, Math. Comp., v. 42, 465–492.

[7] Doolan, E.P., Miller, J.J.H. & Schilders, W.H.A., (1980). Uniform numerical methods for problems with initial and boundary layers, Dublin: Boole Press.

[8] Hemker, P.W. (1977). A numerical study of stiff two-point boundary problems, Mathematisch Centrum, Amsterdam.

[9] Carey, G.F., Anderson, M. , Carnes, B. & Kirk, B., (2004). Some aspects of adaptive grid technology related to boundary and interior layers, J. Comput. Appl. Math., v. 166(1), 55–86.

[10] Farrell, P.A. (1988). Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem, SIAM J. Numer. Anal., v. 25(3), 618–643.

[11] Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E. & Shishkin, G.I., (2004). Singularly perturbed convectiondiffusion problems with boundary and weak interior layers, J. Comp. Appl. Math., v. 166(1), 133–151.

[12] Kadalbajoo, M.K. & Bawa, R.K., (1993). Third-order variable-mesh cubic spline methods for singularly-perturbed boundary-value problems, Appl. Math. Comput., v. 59(2-3), 117–129.

[13] Kadalbajoo, M.K. & Patidar, K.C., (2001). Variable mesh spline approximation method for solving singularly perturbed turning point problems having boundary layer(s), Comp. Math. Appl., v. 42(10-11), 1439–1453.

[14] Kadalbajoo, M.K. & Patidar, K.C., (2002). Numerical solution of singularly perturbed two-point boundary value problems by spline in tension, Appl. Math. Comput., v. 131(2-3), 299–320.

[15] Kadalbajoo, M.K. & Patidar, K.C., (2002). A survey of numerical techniques for solving singularly perturbed ordinary differential equations, Appl. Math. Comput., v. 130(2-3), 457–510.

[16] Kadalbajoo, M.K. & Patidar, K.C., (2002). Tension spline for the solution of self-adjoint singular perturbation problems, Int. J. Comput. Math., v. 79(7), 849–865.

[17] Kadalbajoo, M.K. & Patidar, K.C., (2002). Spline techniques for the numerical solution of singular perturbation problems, J. Optim. Theory Appl., v. 112(3), 575–594.

[18] Kadalbajoo, M.K. & Patidar, K.C., (2003). Exponentially fitted spline approximation method for solving selfadjoint singular perturbation problems, Int. J. Math. Math. Sci., v. 61, 3873–3891.

[19] Kadalbajoo, M.K. & Patidar, K.C., (2003). Spline approximation method for solving self-adjoint singular perturbation problems on non-uniform grids, J. Comput. Anal. Appl., v. 5(4), 425–451.

[20] Kadalbajoo, M.K. & Aggarwal, V.K., (2005). Fitted mesh B-spline collocation method for solving self-adjoint singularly perturbed boundary value problems, Appl. Math. Comput., v. 161(3), 973–987.

[21] Kadalbajoo, M.K. & Aggarwal, V.K., (2006). Fitted mesh B-spline collocation method for solving singularly perturbed reaction-diffusion problems, J. Concr. Appl. Math., v. 4(3), 349–365.

[22] Kadalbajoo, M.K. & Kumar, D., (2008). Parameter-uniform fitted operator B-spline collocation method for self-adjoint singularly perturbed two-point boundary value problems, Electron. Trans. Numer. Anal., v. 30, 346–358.

[23] M.K. Kadalbajoo, Yadaw, A.S. & Kumar, D., (2008). Comparative study of singularly perturbed two-point BVPs via: Fitted-mesh finite difference method, B-spline collocation method and finite element method, Appl. Math. Comput., v. 204(2), 713–725.

[24] Kadalbajoo, M.K. & Arora, P., (2009). B-spline collocation method for a singularly perturbed reaction-diffusion problem using artificial viscosity, Int. J. Comput. Methods, v. 6(1), 23–41.

[25] Kadalbajoo, M.K. & Arora, P., (2009). B-spline collocation method for the singular-perturbation problem using artificial viscosity, Comp. Math. Appl., v. 57(4), 650–663.

[26] Kadalbajoo, M.K. & Gupta, V., (2009). Numerical solution of singularly perturbed convection-diffusion problem using parameter uniform B-spline collocation method, J. Math. Anal. Appl., v. 355(1), 439–452.

[27] Kellogg, R.B. & Tsan, A., (1978). Analysis of some difference approximations for a singular perturbation problem without turning points, Math. Comp., v. 32, 1025–1039.

[28] Kellogg, R.B. (1981). Difference approximation for a singular perturbation problem with turning point, Analytical and numerical approaches to asymptotic problems in analysis, eds., Axelsson, S., Frank, L.S. & Van der Sluis, A., North-Holland Publishing Company, pp. 133–139.

[29] Kevorkian, J. & Cole, J.D., (1981). Perturbation methods in applied mathematics, New York: Springer-Verlag.

[30] Kopteva, N. (2004). How accurate is the streamline-diffusion FEM inside characteristic (boundary and interior) layers?, Comput. Meth. Appl. Mech. Engrg., v. 193(45-47), 4875–4889.

[31] Kreiss, B. & Kreiss, H.O., (1981). Numerical methods for singular perturbation problems, SIAM J. Numer. Anal., v. 18, 262–276.

[32] Miller, J.J.H., O'Riordan, E., Shishkin, G.I. & Wang, S., (2000). A parameter-uniform Schwarz method for a singularly perturbed reaction-diffusion problem with an interior layer, Appl. Numer. Math., v. 35(4), 323–337.

[33] Miranker, W.L. & Morreeuw, J.P., (1974). Semi-analytic numerical studies of turning point problems arising in stiff boundary value problems, Math. Comp., v. 28, 1017–1034.

[34] Natesan, S., Vigo-Aguiar, J. & Ramanujam, N., (2003). A numerical algorithm for singular perturbation problems exhibiting weak boundary layers, Comp. Math. Appl., v. 45(1-3), 469–479.

[35] Nayfeh, A.H. (1981). Introduction to perturbation techniques, New York: John Wiley & Sons.

[36] O'Malley, R.E. (1974). Introductions to singular perturbations, New York: Academic Press.

[37] Pearson, C.E. (1968). On a differential equation of boundary layer type, J. Math. Phy., v. 49, 134–154.

[38] Shishkin, G.I. (2004). Limitations of adaptive mesh refinement techniques for singularly perturbed problems with a moving interior layer, J. Comput. Appl. Math., v. 166(1), 267–280.

[39] Sun, G.F. & Stynes, M., (1994). Finite element methods on piecewise equidistant meshes for interior turning point problems, Numer. Algo., v. 8(1), 111–129.

[40] Surla, K. & Uzelac, Z., (1993). An analysis and improvement of El Mistikawy and Werle scheme, Publ. Math. Inst., v. 54(69), 144–155.

[41] Surla, K. & Uzelac, Z., (1995). A difference scheme for boundary value problems with turning point, Univ. u Novom Sadu Zb. Rad. Prirod. -Mat. Fak. Ser. Mat., v. 25(1), 67–73.

[42] Vulanović, R. & Farrell, P.A., (1993). Continuous and numerical analysis of a multiple boundary turning point problem, SIAM J. Numer. Anal., v. 30(5), 1400–1418.