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A STOCHASTIC APPROACH TO THE BONUS-MALUS SYSTEM

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ABSTRACT. This study is designed to provide an extension of well-known models of tarification in auto mobile insurance known as Bonus-Malus system. Bonus-Malus system, under certain conditions, can be interpreted as a Markov chain with constant transition matrix. The goal of this study is to find and investigate the stationary distribution of three Bonus-Malus systems used worldwide.
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1. INTRODUCTION

In automobile insurance the use of many *a priori* variables such as age, sex, occupation, vehicle type, location is a common practice in most developed countries. These variables are called a *priori* rating since these values are known during the inception of the policy. The variables are used to subdivide policyholders into homogeneous classes. Despite the use of many *a priori* variables, very heterogeneous driving behaviors are still observed in each tariff cell [1]. In the mid-1950s the idea of *posteriori* was introduced. These practices are known as experience rating, merit rating or Bonus-Malus (Latin for Good-Bad) systems (BMS). In BMS policyholders are assigned to merit classes, with potential movement up or down to other assigned classes, depending on the number of accident claims filed by the policyholder over the previous period. In BMS drivers are penalized for one or more accidents by an additional premium or *malus*, and claim-free drivers are rewarded a discount or *bonus*. In other words, a Bonus-Malus system is characterized among others by the fact that only the number of claims made by a policyholder does change the premium.The first class in a BMS is sometimes called super malus class and the last is called super bonus class. In automobile insurance, in particular in Europe, the BMS is widely used. Compared to a flat rate system, a BMS is better able to reflect risk levels in insurance premiums [3]. Mathematical definition of a Bonus-Malus system requires the system to be Markovian, or "memory-less" which states that the knowledge of the present class and the number of accidents during the policy year are sufficient to determine the class for the next year. Under certain conditions the BMS forms a Markov chain process. The goal of this study is to compute and compare the stationary distributions of several BMS used worldwide. The method of calculation used in this study is straightforward.

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2. DEFINITIONS AND NOTATION

As Loimaranta [4] and Lemaire [1] defined an insurance company uses a BMS if the following assumptions are valid:

1. All policies of a given risk can be partitioned into a finite number of classes, denoted C_i or simply $i(i = 1, ..., s)$, so that the premium of a policy for a given period depends solely on the class. Here s is the number of classes.
2. The actual class for a given period is uniquely defined by the class for the previous period and the number of claims occurred during the period.

There are two elements which determine the BMS:

1. Transition rules, i.e. the rules that determine the transfer from one class to another when the number of claims is known. These rules can be given in the form of transformations T_k, such that $T_k(i) = j$, if the policy is moved from class C_i to class C_j when k claims have been reported.
2. Bonus scale, which means the premium b_i for all bonus classes i. It has been assumed that these premiums are given as a vector B with components b_i.

The transformation T_k can also be written as a matrix

$$T_k = (t_{ij}^{(k)})$$

where

$$t_{ij}^{(k)} = \begin{cases} 1 & \text{if } T_k(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

Assume the claim frequency, i.e., the expected number of claims per period for a given policy, is λ and that the probability distribution of the number of claims during one period, $p_k(\lambda)$ is uniquely defined by the parameter λ. Moreover, it has been assumed that the value of λ is independent of time. The probability $p_{ij}(\lambda)$ of a policy moving from C_i into C_j in one period is given by

$$P_{ij}(\lambda) = \sum_{k=0}^{\infty} p_k(\lambda) t_{ij}^{(k)}$$

where $p_k(\lambda)$ is the probability that a driver with claim frequency λ has k claims in a year. Obviously $p_{ij}(\lambda) \geq 0$ and $\sum_{j=1}^{s} p_{ij}(\lambda) = 1$ since $p_{ij}(\lambda)$ is a probability. The matrix

$$\begin{aligned} M(\lambda) &= p_{ij}(\lambda) \\ &= \sum_{k=0}^{\infty} p_k(\lambda) T_k \end{aligned}$$

is then the transition matrix of this Markov chain. It has been assumed that the claim frequency is stationary in time and thus the chain is homogeneous. Thus, the BMS can be considered as homogeneous Markov chains with a finite state space E of bonus malus classes. A first order Markov chain is a stochastic process in which the future development depends only on the present state but not on the history of the process or the manner in which the present state was reached. It is a process without memory, such that the states of the chain are the different BMS classes. The knowledge of the present class and the number of claims for the year suffice to determine next year's class.

3. EIGENVECTOR OF THE STOCHASTIC MATRIX

A stochastic matrix or a transition matrix is used to describe the transitions of a Markov chain. There are two different types of stochastic matrices: namely, left stochastic matrix and right stochastic matrix. A right stochastic matrix is a square matrix whose rows consists of non-negative real numbers, with each row summing to 1. The stochastic matrices obtained from any Bonus-Malus system are the examples of right stochastic matrices. In order to execute the calculations it has been assumed that the bonus-malus has existed for a long period of time and that it has reached its stationary distribution. If a Bonus-Malus system is irreducible and aperiodic then there exist a unique limit distribution, and this distribution is stationary.In other words, the linear systems of equations

$$\mu = \mu M(\lambda)$$

has a unique, strictly non-zero solution μ, where $\mu = (\mu_j, j \in E)$. This solution μ is called the stationary distribution of the BMS, since

$$\mu_j = \lim_{t\to\infty} p_{ij}^{(t)}$$

and

$$\begin{aligned}\lim_{t\to\infty} p^{(0)} M^t &= p^{(0)} M^{\infty} \\ &= \mu\end{aligned}$$

for any starting distribution. μ is called the stationary distribution because $p^{(0)} = \mu$ implies that $Mp^{(0)} = \mu$ (i.e. the state probabilities are time constant). In a BMS, the stationary distribution represents the percentage of drivers in the different classes after the BMS has been run for a long time.

4. EXAMPLES

Toughness of Bonus-Malus systems towards consumers is an important topic and it is measured by the coefficient of variation of the insureds premium, when the Bonus-Malus system has reached its stationary state. Lemaire and Zi [2] developed a technique to rank all 30 studied Bonus-Malus systems in "an index of toughness". A strong geographical patterns emerge from the ranking of BMS. It has been observed that countries from northern and central Europe use the toughest BMS whereas Asian countries for the most part adapt fairly mild BMS. Three systems have been chosen from that index and the systems are the Swiss BMS (the new version) which has been ranked 1, the Hong-Kong BMS which has been ranked 14 and the Brazilian BMS which has been ranked 29 in the index of toughness. Even though this index of toughness ranking does not necessarily imply about the quality of the systems there is a renewed interest among researchers to investigate the stationary distributions of those countries who ranks in the top most or bottom most or somewhere in the middle of the index of toughness. In this section stationary distributions of these three Bonus-Malus systems will be computed using the mathematical software Maple for different values of λ. A special attention has been given to those values of λ obtained by [5] by using the maximum likelihood estimation. Poisson distribution is the very frequently used distribution to model the transition probabilities with a BMS. In this study the Poisson distribution with intensity λ has been used to describe the number of claims for an individual.

4.1. **SWISS BMS.** Swiss insurers modified their BMS in the beginning of 1990. The new Swiss BMS consists of a scale with 22 steps. All users enter the system in class 9. In other words, under this system, the starting premium level is 100. Each claim is penalized by four classes. This made the Swiss system the toughest system in the world [2].

The transition rules are the following:

1. each year a one-class bonus is given.
2. each claim is penalized by four classes.
3. the maximal malus class is 21. This is the worst class in this system. The drivers in this class pay the maximum premium.
4. the maximal bonus class is 0. That means the best class in the Swiss BMS is 0.These drivers pay the least.

Let $p(x)$ be the probability that a driver with average claims frequency λ causes x claims during a given year. The transition probability matrix $M_1(\lambda)$ of this driver within the Swiss BMS is a stochastic matrix where every row stands for a probability distribution over states to be entered.

Table I: Swiss BMS transition probability matrix with class 21 at the top

$$M_1(\lambda) :=$$

$$\begin{pmatrix}
1-p_0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-p_0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-p_0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-p_0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-p_0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 \\
1-\sum p_i & p_4 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 \\
1-\sum p_i & 0 & p_4 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 \\
1-\sum p_i & 0 & 0 & p_4 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 & 0 \\
1-\sum p_i & 0 & 0 & 0 & p_4 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & 0 & p_0 \\
1-\sum p_i & p_5 & 0 & 0 & 0 & p_4 & 0 & 0 & 0 & p_3 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & p_0
\end{pmatrix}$$

Obviously, all elements of the above transition matrix are non-negative and all row sums $\sum_i p_i$ are equal to 1. Since the stationary distribution is a left eigenvector of the transition matrix with eigenvalue 1 the stationary distribution can be determined by finding a non-trivial solution for the linear system of equation and then by dividing it by the sum of its component to make the stationary distribution a probability distribution. The left eigenvector $(a_{21}, a_{20}, ..., a_0)$ of the Swiss BMS satisfies the following equations:

$$
\begin{aligned}
a_{21} &= a_{21}(1-p_0) + a_{20}(1-p_0) + a_{19}(1-p_0) + a_{18}(1-p_0) + a_{17}(1-p_0) \\
&\quad + a_{16}(1-p_0-p_1) + a_{15}(1-p_0-p_1) \\
&\quad + a_{14}(1-p_0-p_1) + a_{13}(1-p_0-p_1) \\
&\quad + a_{12}(1-p_0-p_1-p_2) + a_{11}(1-p_0-p_1-p_2) + a_{10}(1-p_0-p_1-p_2) \\
&\quad + a_9(1-p_0-p_1-p_2) + a_8(1-p_0-p_1-p_2-p_3) \\
&\quad + a_7(1-p_0-p_1-p_2-p_3) \\
&\quad + a_6(1-p_0-p_1-p_2-p_3) + a_5(1-p_0-p_1-p_2-p_3) \\
&\quad + a_4(1-p_0-p_1-p_2-p_3-p_4) + a_3(1-p_0-p_1-p_2-p_3-p_4) \\
&\quad + a_2(1-p_0-p_1-p_2-p_3-p_4) + a_1(1-p_0-p_1-p_2-p_3-p_4) \\
&\quad + a_0(1-p_0-p_1-p_2-p_3-p_4-p_5) \\
&\quad \ldots\ldots\ldots \\
&\quad \ldots\ldots\ldots \\
a_0 &= a_1p_0 + a_0p_0 \\
1 &= a_0 + \ldots + a_9 \ldots + a_{21}.
\end{aligned}
$$

Since these equations are linearly dependent it is possible to solve the equations. As these twenty two equations are linearly dependent, the first one can be eliminated. To solve the system, it is sufficient to set a_1 equal to any arbitrary value and then to find all probabilities as a function of a_1 from the next-to-last equation up. In this study, Maple has been used to solve the system and stationary distribution for different values of λ. Table II displays the stationary distribution of the drivers for the Swiss bonus-malus system.

From Table II it can be told that a driver can reach to the super bonus class (class 0) with a probability of 0.55221 (when $\lambda = 0.10141$). For higher λ this probability is 0. Moreover, for higher λ, a policy holder does not have even any significant chance to be in any class lower than 20. Likewise, there is a 60% (0.60634) chance that a driver will eventually be in the super malus class (class 21) when $\lambda = 0.94122$. As expected, when the mean number of claims is low ($\lambda = 0.10141$) the probability of being in the super malus class is also very low (0.00062).

Table II: Stationary distribution of the drivers for the Swiss BMS

s	$\lambda = 0.10141$	$\lambda = 0.34123$	$\lambda = 0.94122$
21	0.00062	0.20929	0.60634
20	0.00082	0.16568	0.23869
19	0.00107	0.13115	0.09396
18	0.00139	0.10381	0.03699
17	0.00178	0.08216	0.01456
16	0.00236	0.06505	0.00573
15	0.00309	0.05149	0.00226
14	0.00398	0.04075	0.00089
13	0.00498	0.03223	0.00035
12	0.00697	0.02557	0.00014
11	0.00906	0.02023	0.00005
10	0.01123	0.01598	0.00002
9	0.01339	0.01258	0
8	0.02199	0.01015	0
7	0.02648	0.00796	0
6	0.02991	0.00619	0
5	0.03242	0.00478	0
4	0.07989	0.00432	0
3	0.07219	0.00307	0
2	0.06523	0.00218	0
1	0.05894	0.00155	0
0	0.55221	0.00382	0

4.2. **HONG KONG BMS.** Hong Kong BMS ranked 14 in the Lemaire and Zi's "index of toughness". The Hong Kong BMS consists of a scale with 6 steps. All users enter the system in class 6. The transition rules are the following:

1. each year a one-class bonus is given
2. first claim is penalized by 2 or 3 classes. In subsequent claims all discounts are lost.
3. the maximal malus class is 6. These drivers pay the highest amount of premium.
4. the maximal bonus class is 1. Unlike the Swiss system the best class is 1. Class 1 is the best in the Hong Kong BMS.

Let $p(x)$ be the probability that a driver with average claims frequency λ causes x claims during a given year.The transition probability matrix $M_2(\lambda)$ has been obtained using the transition rule.

Table III: Hong Kong BMS transition probability matrix with class 6 at the top.

$$M_2(\lambda) = \begin{pmatrix} 1-p_0 & p_0 & 0 & 0 & 0 & 0 \\ 1-p_0 & 0 & p_0 & 0 & 0 & 0 \\ 1-p_0 & 0 & 0 & p_0 & 0 & 0 \\ 1-p_0 & 0 & 0 & 0 & p_0 & 0 \\ 1-\sum p_i & 0 & p_1 & 0 & 0 & p_0 \\ 1-\sum p_i & 0 & 0 & p_1 & 0 & p_0 \end{pmatrix}$$

Using a similar approach as in the case of Swiss BMS stationary distribution has been found .

Table IV: Stationary distribution of the drivers for the Hong Kong BMS

s	$\lambda = 0.10141$	$\lambda = 0.34123$	$\lambda = 0.94122$
6	0.01841	0.19631	0.60003
5	0.01664	0.13956	0.2341
4	0.02256	0.12604	0.09732
3	0.09088	0.15556	0.0418
2	0.08212	0.11059	0.01631
1	0.76939	0.27194	0.01043

It can be concluded from Table IV that a driver can eventually land in the super bonus class with a probability of 0.76939 when $\lambda = 0.10141$. On the other hand with the same mean claim frequency the chance of being in the super malus class is very low (0.01841).

4.3. **BRAZILIAN BMS.** The older version of Italian BMS stands in the bottom of the Lemaire and Zi's "index of toughness". In order to investigate a newer or currently used BMS in this study the Brazilian BMS has been studied. Brazil has adopted a very simple BMS. It consists of 7 classes, with premium levels 100, 90, 85, 80, 75, 70, and 65 . All users enter the system in class 7.
The transition rules are the following:

1. for claim free year a one-class bonus is given
2. each claim is penalized by 1 class.
3. the maximal malus class is 7.
4. the maximal bonus class is 1. In other words, class 1 is the super bonus class in the Brazilian BMS.

Let $p(x)$ be the probability as defined before. The transition matrix for Brazilian BMS has been shown in Table V.

Table V: Brazilian BMS transition probability matrix with class 7 at the top.

$$M_3(\lambda) = \begin{pmatrix} 1-p_0 & p_0 & 0 & 0 & 0 & 0 & 0 \\ 1-p_0 & 0 & p_0 & 0 & 0 & 0 & 0 \\ 1-\sum p_i & p_1 & 0 & p_0 & 0 & 0 & 0 \\ 1-\sum p_i & p_2 & p_1 & 0 & p_0 & 0 & 0 \\ 1-\sum p_i & p_3 & p_2 & p_1 & 0 & p_0 & 0 \\ 1-\sum p_i & p_4 & p_3 & p_2 & p_1 & 0 & p_0 \\ 1-\sum p_i & p_5 & p_4 & p_3 & p_2 & p_1 & p_0 \end{pmatrix}$$

As before Maple has been used to calculate stationary distribution for Brazilian BMS.

Table VI: Stationary distribution of the drivers for the Brazilian BMS

s	$\lambda = 0.10141$	$\lambda = 0.34123$	$\lambda = 0.94122$
7	0	0.00666	0.43738
6	0.00005	0.01633	0.25021
5	0.00034	0.03377	0.14279
4	0.00224	0.06519	0.08117
3	0.01483	0.12167	0.04583
2	0.09475	0.21761	0.02528
1	0.88778	0.53876	0.01734

The stationary distribution obtained in this system also stands for the long term probability of being in a particular class. For example, there is a 89% chance being in the super bonus class when $\lambda = 0.10141$.

If the stationary probabilties among the systems are compared an interesting results will be found which justify the index created by Lemaire and Zi. For example, with $\lambda = 0.10141$ a driver in the Swiss BMS has a 55% chance of being in the super bonus class. This probability is about 77% and 89% for the Hong Kong BMS and Brazilian BMS respectively. A similar trend could be followed for these systems across the board.

5. CONCLUSION

This study has discussed a straightforward method of calculating the stationary distribution which represents the percentage of drivers in the different classes after the BMS has been run for a long time.

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REFERENCES

[1] J. Lemaire, Bonus-Malus Systems in Automobile Insurance, vol 1, Kluwer Academic Publishers, Massachusetts, 1995.

[2] J. Lemaire, and H. Zi, A comperative Analysis of 30 Bonus-Malus Systems. *ASTIN Bulletin*,24-2 (1994), 287-309.

[3] T. Mayuzumi, A Study of the Bonus-Malus system. *ASTIN Colloquium International Actuarial Association - Brussels, Belgium, 1990.*

[4] K. Loimaranta, Some Asymptotic Properties of Bonus Systems. *ASTIN Bulletin*, 6 (1972) , 233-245.

[5] J. Walhin, and J. Paris, Using Mixed Poisson Process in Connection with Bonus-Malus Systems. *ASTIN Bulletin*, 29-1 (1999), 81-99.