Neural, Parallel, and Scientific Computations 18 (2010) 401-408

SYNTHESIS OF STOCHASTIC ATTRACTORS FOR NONLINEAR DYNAMICAL SYSTEMS AND CONTROLLING CHAOS

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ABSTRACT. We consider a nonlinear discrete-time control system with regular and chaotic dynamics forced by stochastic disturbances. The problem addressed is a design of the feedback regulator which stabilizes cycles of the closed-loop deterministic system and synthesizes a required probabilistic distribution of random attractors for corresponding stochastic system. To solve this problem, we propose a new method based on the stochastic sensitivity function technique. We study a synthesis of the stochastic sensitivity function and constructive design of regulator parameters. An effectiveness of the proposed approach is demonstrated on the stochastic Verhulst model. It is shown that this regulator forms a stochastic attractor with low level of sensitivity and suppresses chaos.

AMS (MOS) Subject Classification. 37H20

1. SENSITIVITY ANALYSIS OF STOCHASTIC CYCLES

Consider a stochastic discrete-time scalar system

$$x_{t+1} = f(x_t) + \varepsilon\sigma(x_t)\xi_t, \tag{1.1}$$

where $f(x)$ and $\sigma(x)$ are sufficiently smooth functions, ξ_t is uncorrelated random process with parameters $\mathrm{E}\xi_t = 0$, $\mathrm{E}\xi_t^2 = 1$, ε is an intensity of the noise. Suppose the deterministic system (1.1) ($\varepsilon = 0$) has n-cycle Γ. Points of the set $\Gamma = \{\bar{x}_1, \ldots, \bar{x}_n\}$ are connected by the equalities $f(\bar{x}_i) = \bar{x}_{i+1}$ $(i = 1, \ldots, n-1)$, $f(\bar{x}_n) = \bar{x}_1$. The sequence $\bar{x}_t$ is defined for all t due to the periodicity condition $\bar{x}_{t+n} = \bar{x}_t$. It is supposed that the cycle Γ is exponentially stable [1].

The necessary and sufficient condition of the exponential stability of the cycle [1] is the inequality

$$|a| < 1, \quad a = a_1 \cdot a_2 \cdot \dots \cdot a_n, \quad a_i = \frac{df}{dx}(\bar{x}_i). \tag{1.2}$$

Let x_t^ε be a solution of the stochastic system (1.1) with the initial condition $x_1^\varepsilon = \bar{x}_1 + \varepsilon\xi$, where ξ is a random value. Consider the deviations $z_t^\varepsilon = x_t^\varepsilon - \bar{x}_t$ of system (1.1) states x_t^ε from the states $\bar{x}_t$ and the relations $v_t^\varepsilon = z_t^\varepsilon/\varepsilon = (x_t^\varepsilon - \bar{x}_t)/\varepsilon$ for the following time points. For the small ε, a sensitivity of the deterministic cycle Γ states

Received April 15, 2010

$\bar{x}_t$ with respect to random disturbances of system (1.1) is defined by $v_t = \lim_{\varepsilon\to 0} v_t^\varepsilon = \dfrac{d}{d\varepsilon} x_t^\varepsilon|_{\varepsilon=0}$. The sequence v_t satisfies the linear system

$$v_{t+1} = a_t v_t + \sigma_t \xi_t, \quad a_t = \frac{df}{dx}(\bar{x}_t), \quad \sigma_t = \sigma(\bar{x}_t), \quad v_1 = \xi. \tag{1.3}$$

A dynamics of the first two moments $m_t = \mathrm{E}v_t$, $V_t = \mathrm{E}v_t^2$ for system (1.3) is governed by the equations

$$m_{t+1} = a_t m_t, \tag{1.4}$$

$$V_{t+1} = a_t^2 V_t + \sigma_t^2 \tag{1.5}$$

with initial conditions $m_1 = \mathrm{E}\xi$, $V_1 = \mathrm{E}\xi^2$.

Due to the stability criterion (1.2), the sequences m_t and V_t are stabilized: $\lim_{t\to\infty} m_t = 0$, $\lim_{t\to\infty}(V_t - w_t) = 0$. Here w_t is an unique n-periodic solution of the equation (1.5). Values $w_1, \dots, w_n$ satisfy the following linear system

$$\begin{aligned} & w_2 = a_1^2 w_1 + \sigma_1^2, \\ & \cdot\ \cdot\ \cdot \\ & w_n = a_{n-1}^2 w_{n-1} + \sigma_{n-1}^2, \\ & w_1 = a_n^2 w_n + \sigma_n^2. \end{aligned} \tag{1.6}$$

Rewrite the system (1.6) in a vector form:

(1.7)

$$Aw = s, \quad A = \begin{bmatrix} -a_1^2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -a_2^2 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \cdots & -a_{n-1}^2 & 1 \\ 1 & 0 & 0 & \cdots & 0 & -a_n^2 \end{bmatrix}, \ w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \ s = \begin{bmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_n^2 \end{bmatrix}.$$

Consider determinants $\Delta = \det A$, $\Delta_1 = \det A_1$. Here

$$A_1 = \begin{bmatrix} \sigma_1^2 & 1 & 0 & \cdots & 0 & 0 \\ \sigma_2^2 & -a_2^2 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & \\ \sigma_{n-1}^2 & 0 & 0 & \cdots & -a_{n-1}^2 & 1 \\ \sigma_n^2 & 0 & 0 & \cdots & 0 & -a_n^2 \end{bmatrix}.$$

For these determinants, we have

$$\Delta = (-1)^{n+1}(1 - a^2), \quad \Delta_1 = (-1)^{n+1}(\sigma_n^2 + \sigma_{n-1}^2 a_n^2 + \dots + \sigma_1^2 a_2^2 \cdot \dots \cdot a_n^2).$$

Due to (1.2), $\Delta \neq 0$ and system (1.7) has unique solution w. Its first component $w_1 = \Delta_1/\Delta$ can be written in an explicit form: $w_1 = (\sigma_n^2 + \sigma_{n-1}^2 a_n^2 + \dots + \sigma_1^2 a_2^2 \cdot \dots \cdot a_n^2)/(1 - a^2)$. Other components $w_2, \dots, w_n$ can be found from (1.6) recurrently: $w_i = a_{i-1}^2 w_{i-1} + \sigma_{i-1}^2$ $(i = 2, \dots, n)$. Note that $w_i \geq \sigma_{i-1}^2$.

Values $w_1, \dots, w_n$ of n-periodic function w_t characterize a response of the points $\bar{x}_1, \dots, \bar{x}_n$ of the cycle Γ to small random disturbances. The vector $w = (w_1, \dots, w_n)^\top$ is called the *stochastic sensitivity function* of a cycle [2].

For small noise, when random states x_t^ε for the steady-state regime of stochastic system (1.1) are localized near points $\bar{x}_t$ of stable deterministic cycle, the values w_t of stochastic sensitivity function give us the following approximation $D(x_t^\varepsilon) \approx \varepsilon^2 w_t$. Here $D(x_t^\varepsilon)$ is a dispersion of the random states x_t^ε.

For the case of 2-cycle ($n = 2$), the following explicit representation holds

$$w_1 = \frac{\sigma_2^2 + \sigma_1^2 a_2^2}{1 - a_1^2 a_2^2}, \quad w_2 = \frac{\sigma_1^2 + \sigma_2^2 a_1^2}{1 - a_1^2 a_2^2}. \tag{1.8}$$

2. SENSITIVITY SYNTHESIS OF STOCHASTIC CYCLES

Consider a stochastically forced controlled system

$$x_{t+1} = f(x_t, u_t) + \varepsilon\sigma(x_t, u_t)\xi_t. \tag{2.1}$$

Here $f(x, u)$ and $\sigma(x, u)$ are sufficiently smooth functions. The function $f(x, u)$ describes a deterministic part of controlled dynamics and $\sigma(x, u)$ characterizes a dependence of disturbances on the state and control, ξ_t is an uncorrelated random process with parameters $\mathrm{E}\xi_t = 0$, $\mathrm{E}\xi_t^2 = 1$, ε is a scalar parameter of noise intensity.

It is supposed that for $\varepsilon = 0$ and $u = 0$ system (2.1) has n-cycle $\Gamma = \{\bar{x}_1, \dots, \bar{x}_n\}$. A stability of Γ is not assumed.

We will select a stabilizing regulator from the class U of admissible feedbacks $u = u(x)$ satisfying conditions:

(a) $u(x)$ is sufficiently smooth and $u|_\Gamma = 0$;

(b) for the closed-loop deterministic system

$$x_{t+1} = f(x_t, u(x_t)) \tag{2.2}$$

the cycle Γ is exponentially stable.

For any $u(x) \in U$, the cycle Γ of system (2.1) has a corresponding stochastic sensitivity function $w[u]$. Varying $u(x) \in U$, one can change values of $w[u]$.

Consider the following control problem.

A problem of stochastic sensitivity synthesis. Our aim is a synthesis of the assigned stochastic sensitivity function. For the control system (2.1), it follows from (1.6) that values $w_1, \dots, w_n$ of the stochastic sensitivity function w are connected with control system parameters by equations

$$
\begin{aligned}
&(a_1 + b_1 k_1)^2 w_1 = w_2 - \sigma_1^2, \\
&\quad . \quad . \quad . \\
&(a_{n-1} + b_{n-1} k_{n-1})^2 w_{n-1} = w_n - \sigma_{n-1}^2, \\
&(a_n + b_n k_n)^2 w_n = w_1 - \sigma_n^2.
\end{aligned} \tag{2.3}
$$

As we see, a variation of the control function $u(x)$ changes only the coefficients $k_i = \dfrac{du}{dx}(\bar{x}_i)$ in system (2.3). Note that outcome of the control depends only on values of the derivative $\dfrac{du}{dx}$. Without loss of generality, it allows us to restrict a consideration by more simple regulators in the following form

$$u(x)|_{X_i} = k_i(x - \bar{x}_i), \quad i = 1, \dots, n. \tag{2.4}$$

So, the regulator (2.4) is completely determined by the vector $k = (k_1, \dots, k_n)$ of feedback parameters k_i. The condition (a) for $u(x)$ from (2.4) is fulfilled for any $k \in \mathrm{R}^n$. The condition (b) restricts the choice of k by the following set: $K = \{k \in \mathrm{R}^n \mid$ cycle Γ for system (11), (14) is exponentially stable$\}$.

This set K can be described constructively: $K = \{k \in \mathrm{R}^n \mid \prod_{i=1}^n |a_i + b_i k_i| < 1\}$. If at least one of $b_i \neq 0$, then $K \neq \emptyset$. Under these circumstances, we shall consider a problem of the stochastic sensitivity function synthesis for system (2.1) with the regulator (2.4). Consider a set $B = \{w \in R^n \mid w_i \geq \sigma_{i-1}^2 \ (i = 2, \dots, n),\ w_1 \geq \sigma_n^2\}$ of admissible stochastic sensitivity functions. Denote by $w(k)$ the solution of system (2.3) for the fixed vector $k \in K$.

Problem of a Control. *For the assigned vector $\bar{w} \in B$, it is necessary to find a vector $k \in K$ guaranteeing the equality*

$$w(k) = \bar{w}. \tag{2.5}$$

Definition 2.1. The element $\bar{w} \in B$ is said to be *attainable* in the system (2.1), (2.4) if the equality (2.5) is true for some $k \in K$.

Definition 2.2. A set of all attainable elements $W = \{\bar{w} \in B \mid \exists k \in K\ w(k) = \bar{w}\}$ is called by *attainability set* system (2.1), (2.4).

For the case $W = B$, the coefficients $k_1, \dots, k_n$ of the feedback regulator (2.4) guaranteeing required values $w_1, \dots, w_n$ of the stochastic sensitivity function can be calculated from the system (2.3). Values $w_1 = \sigma_n^2$, $w_2 = \sigma_1^2, \dots, w_n = \sigma_{n-1}^2$ are minimal elements of the attainability set W. For these values, coefficients of the feedback regulator (2.4) are unique: $k_i = -a_i/b_i$ $(i = 1, \dots, n)$. In the general case, a choice of k_i is not unique: for every k_i we have two values.

Consider a case of 2-cycle. For $n = 2$ and $b_{1,2} \neq 0$, the attainability set W is defined by inequalities $w_1 \geq \sigma_2^2$, $w_2 \geq \sigma_1^2$. Coefficients k_1, k_2 of the feedback regulator providing assigned values w_1, w_2 of stochastic sensitivity function for states $\bar{x}_1$, $\bar{x}_2$ of 2-cycle are following:

$$k_1 = \frac{1}{b_1}\left(-a_1 \pm \sqrt{\frac{w_2 - \sigma_1^2}{w_1}}\right), \quad k_2 = \frac{1}{b_2}\left(-a_2 \pm \sqrt{\frac{w_1 - \sigma_2^2}{w_2}}\right). \tag{2.6}$$

3. STOCHASTIC VERHULST SYSTEM

Consider a stochastically forced Verhulst system

$$x_{t+1} = \mu x_t(1 - x_t) + \varepsilon\xi_t, \tag{3.1}$$

where ξ_t is a sequence of the uncorrelated Gaussian random disturbances with parameters $E\xi_t = 0$, $E\xi_t^2 = 1$, ε is a parameter of noise intensity.

For $3 < \mu \le 4$, the deterministic Verhulst system ($\varepsilon = 0$) has 2-cycle with states

$$\bar{x}_1(\mu) = \frac{\mu + 1 + \sqrt{(\mu+1)(\mu-3)}}{2\mu}, \quad \bar{x}_2(\mu) = \frac{\mu + 1 - \sqrt{(\mu+1)(\mu-3)}}{2\mu}.$$

This cycle is stable on the interval $3 < \mu < \mu_* = 1 + \sqrt{6}$.

When the parameter μ passes the bifurcation value $\mu_* = 3.44949$, this 2-cycle loses stability. On the interval $(3.44949, 4]$, Verhulst system demonstrates cascades of period doubling bifurcations and transition to chaotic oscillations (see bifurcation diagram in Fig. 1a).

Consider the stochastically forced system (3.1) ($\varepsilon > 0$). Under the random disturbances, a trajectory of system (3.1) leaves a deterministic attractor and forms a stochastic attractor around it with the corresponding stationary probabilistic distribution. In Figs. 1b, c, random states of the stochastic Verhulst system are plotted for different values of the noise intensity. As one can see, noise washes out a thin structure of the deterministic bifurcation diagram [3]. For $\varepsilon = 0.01$, a dynamics of system (3.1) looks like chaotic for the whole interval $3.6 < \mu < 4$.

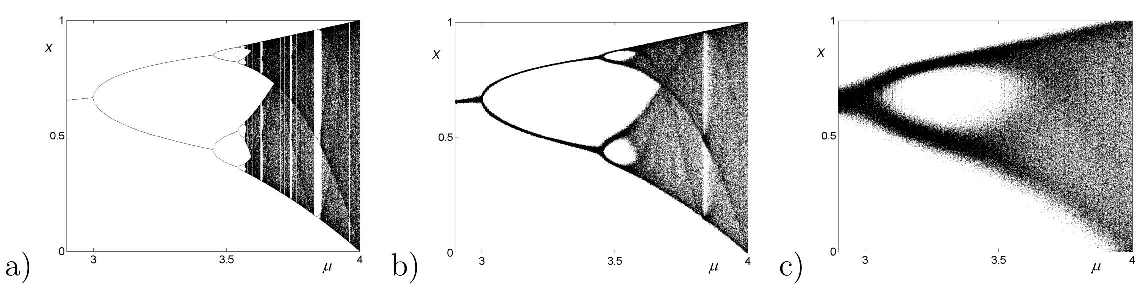

Fig. 1. Attractors of uncontrolled Verhulst system for a) $\varepsilon = 0$, b) $\varepsilon = 0.001$, c) $\varepsilon = 0.01$.

Consider in detail a zone $\mu \in (3, 3.44949)$ of the stochastic 2-cycle. A dispersion of random states around deterministic points $\bar{x}_1(\mu), \bar{x}_2(\mu)$ is non-uniform. A value of dispersion around $\bar{x}_2(\mu)$ (lower branch) is greater than one for $\bar{x}_1(\mu)$ (upper branch).

As the parameter μ approaches bifurcation points $\mu = 3$ and $\mu = 3.44949$, the dispersion of random states increases.

Our quantitative analysis of the stochastic 2-cycle is based on the stochastic sensitivity function technique. For system (3.1), due to (1.8), values of the stochastic sensitivity function are following

$$w_1 = \frac{1 + a_2^2}{1 - a_1^2 a_2^2}, \qquad w_2 = \frac{1 + a_1^2}{1 - a_1^2 a_2^2},$$
$$a_1 = -1 - \sqrt{(\mu+1)(\mu-3)}, \quad a_2 = -1 + \sqrt{(\mu+1)(\mu-3)}.$$

Consider stochastically forced Verhulst system

$$x_{t+1} = \mu x_t(1 - x_t) + u_t + \varepsilon \xi_t \tag{3.2}$$

with the control input u_t.

The aim of the control is a stabilization of 2-cycle for whole zone $3 < \mu \leq 4$ and a synthesis of the required stochastic sensitivity function $w = w(\mu)$.

We use a feedback regulator (2.4) in the following form

$$u(x)|_{X_1} = k_1(x - \bar{x}_1), \quad u(x)|_{X_2} = k_2(x - \bar{x}_2). \tag{3.3}$$

Neigborhoods X_1, X_2 of points $\bar{x}_1, \bar{x}_2$ for this example are $X_1 = ((\bar{x}_1 + \bar{x}_2)/2, 1)$, $X_2 = (0, (\bar{x}_1 + \bar{x}_2)/2)$.

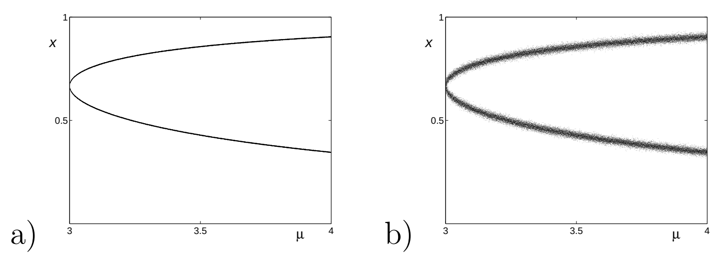

Fig. 2. Random states of controlled stochastic Verhulst system with regulator guaranteeing the stochastic sensitivity $w_1 = w_2 = 1$ for a) $\varepsilon = 0.001$, b) $\varepsilon = 0.01$.

For the considered system, an attainability set of values w_1, w_2 is restricted by inequalities $w_1 \geq 1, w_2 \geq 1$. Put $w_1(\mu) \equiv 1$, $w_2(\mu) \equiv 1$. It means our aim is to provide constant and minimal stochastic sensitivity of 2-cycle for the whole interval $3 < \mu \leq 4$. It follows from (2.6) that the regulator (3.3) is uniquely defined: $k_1 = -a_1$, $k_2 = -a_2$. In Fig. 2, random states of the corresponding closed-loop system (3.2), (3.3) are plotted for different values of noise intensity $\varepsilon = 0.001$ and $\varepsilon = 0.01$. As a result of control, random states of system (3.2), (3.3) are concentrated in small neighborhoods of the points of the deterministic 2-cycle.

Now we can summarize that our method based on stochastic sensitivity synthesis allows to stabilize the cycle $\Gamma = \{\bar{x}_1, \bar{x}_2\}$ for the unstable zone $3.44949 < \mu \leq 4$ and provide assigned small dispersions of random states near cycle points $\bar{x}_1, \bar{x}_2$ for whole zone $3 < \mu \leq 4$. One can interpret these results from the viewpoint of controlling chaos. Indeed, the interval $3.56995 < \mu \le 4$ is a well-known zone where uncontrolled system (3.1) for $\varepsilon = 0$ demonstrates a chaotic behavior. Here, as the parameter μ increases, a complicated alternation of chaos and order subintervals is observed. A set of these subintervals has a fractal structure. Thus, the uncontrolled system demonstrates a structural instability on this interval. Our method of sensitivity synthesis enables to construct a stabilizing regulator and to suppress chaos finally. In Fig. 3, an example of chaos suppression is shown. Here, for $\mu = 3.8$, $\varepsilon = 0.001$ a chaotic uncontrolled trajectory of system (3.2) with $u = 0$ is plotted for the time interval $0 \le t \le 50$. Further, for $t > 50$ we switch on our regulator (3.3) adjusted for the synthesis of $w_1 = w_2 = 1$. This regulator forms periodic oscillations with small dispersion.

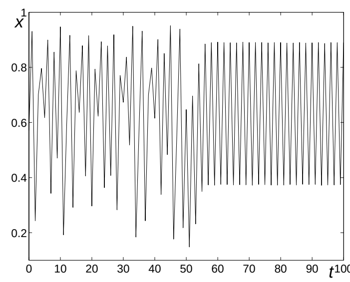

Fig. 3. Chaos suppression. A stochastic 2-cycle stabilization by regulator guaranteeing $w_1 = w_2 = 1$. Shown is a random trajectory for $\varepsilon = 0.001, \mu = 3.8$ and control is switched on at $t = 50$.

If we assign values w_1, w_2 uniformly small for the interval $3 < \mu \le 4$ then the corresponding regulator forms stable cycles with uniformly small dispersion for the whole interval too (see Fig. 2, where $w_1 \equiv w_2 \equiv 1$). It means that our regulator solves an important problem of structural stabilization.

ACKNOWLEDGEMENTS.. This work was partially supported by grants RFBR09-01-00026, 09-08-00048, 10-01-96022_ural, Federal Education Agency 2.1.1/2571, Federal Target Program N 02.740.11.0202.

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