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ANALYSIS OF STOCHASTIC ATTRACTORS FOR POPULATION DYNAMICAL SYSTEMS WITH ENVIRONMENTAL NOISE

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ABSTRACT. We study stochastic attractors of nonlinear dynamical systems modeling population dynamics. For the approximation of probabilistic characteristics of these attractors, constructive computational methods based on stochastic sensitivity functions technique are suggested. Applications of these methods for analysis of the noise-induced effects in a model of population dynamics are demonstrated. The critical value of parameter corresponding to supersensitive limit cycle is found. For this value of parameter, a new noise-induced phenomenon of qualitative changes of stochastic oscillations is shown.

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1. STOCHASTIC SENSITIVITY OF ATTRACTORS

For many dynamical processes, a basic mathematical model is the non-linear deterministic system

$$\dot{x} = f(x). \tag{1.1}$$

Here x is n-vector, $f(x)$ is n-vector function. Suppose that the system (1.1) has an exponentially stable attractor. A classical analysis of local stability of attractor is based on the linear approximation system and corresponding Lyapunov exponents.

A system of stochastic differential equations (in Ito's or Stratonovich's sense)

$$\dot{x} = f(x) + \varepsilon\sigma(x)\dot{w}, \tag{1.2}$$

is a traditional mathematical model allowing to study quantitative description of results of external disturbances. Here $w(t)$ is a n-dimensional Wiener process, $\sigma(x)$ is $n \times n$-matrix function of disturbances with intensity ε. The random trajectories of forced system (1.2) leave a deterministic attractor and form a corresponding stochastic attractor with stationary probabilistic distribution $\rho(x, \varepsilon)$.

The detailed description of random distribution $\rho(x, \varepsilon)$ is given by stationary Kolmogorov-Fokker-Planck (KFP) equation. In a common case, an analytical research of this equation is a very difficult problem. Under these circumstances asymptotics based on quasipotential $v(x) = -\lim_{\varepsilon \to 0} \varepsilon^2 \log \rho(x, \varepsilon)$ are actively used [1].

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For small noise, one can write an approximation of $\rho(x,\varepsilon)$ as follows $\rho(x,\varepsilon) \approx K \cdot \exp\left(-v(x)/\varepsilon^2\right)$. In the stability analysis, we need a good approximation of the quasipotential in a small neighborhood of a deterministic attractor.

We shall consider two types of attractors: equilibria and limit cycles.

2. STOCHASTIC EQUILIBRIUM

Let a system (1.1) has an exponentially stable equilibrium $\bar{x}$. A standard quantitative characteristics for the equilibrium stability is Lyapunov exponent $\lambda = \max\limits_i Re\lambda_i$, where λ_i are eigenvalues of the matrix $F = \frac{\partial f}{\partial x}(\bar{x})$.

Under the random disturbances, a probability distribution of the stochastic states of system (1.2) are arranged around deterministic equilibrium $\bar{x}$ and form a corresponding stochastic equilibrium. In this case, the following quadratic approximation $v(x) \approx \frac{1}{2}(x-\bar{x}, W^{-1}(x-\bar{x}))$ is used. It allows to present an asymptotics of stationary distribution in Gaussian form

$$\rho(x,\varepsilon) \approx K \cdot \exp\left(-\frac{(x-\bar{x}, W^{-1}(x-\bar{x}))}{2\varepsilon^2}\right)$$

with the covariance matrix $\varepsilon^2 W$. For the exponentially stable equilibrium $\bar{x}$, the matrix W is a unique solution of matrix equation

$$FW + WF^\top = -S, \quad F = \frac{\partial f}{\partial x}(\bar{x}), \quad S = GG^\top, \quad G = \sigma(\bar{x}). \tag{2.1}$$

This matrix W is a *stochastic sensitivity function* (SSF) of the equilibrium $\bar{x}$.

3. STOCHASTIC CYCLE

Now consider a case when the system (1.1) has a T-periodic solution $x = \xi(t)$ with an exponentially stable phase curve γ (limit cycle). Under the random disturbances, stochastic trajectories of system (1.2) form a stochastic cycle around the deterministic curve γ.

Let Π_t be a hyperplane that is orthogonal to cycle γ at the point $\xi(t)$ $(0 \leqslant t \leqslant T)$. Consider a random variable X_t. Random values X_t are points of intersection of random trajectories with the hyperplane Π_t in a neighborhood of $\xi(t)$. For the stochastic cycle, a random value X_t has a probabilistic distribution $\rho_t(x,\varepsilon)$. For small noise, with the help of corresponding quadratic approximation of quasipotential, the following Gaussian asymptotics can be obtained [2]

$$\rho_t(x,\varepsilon) = K \cdot \exp\left(-\frac{(x-\xi(t))^\top W^+(t)(x-\xi(t))}{2\varepsilon^2}\right).$$

Here $m_t = \xi(t)$ is a mean value and $D(t,\varepsilon) = \varepsilon^2 W(t)$ is a covariance matrix.

The matrix function $W(t)$ is a solution of boundary value problem

$$\dot{W} = F(t)W + WF^\top(t) + P(t)S(t)P(t), \qquad W(0) = W(T), \quad W(t)r(t) \equiv 0. \tag{3.1}$$

Here $S(t) = \sigma(\xi(t))\sigma^\top(\xi(t))$, $F(t) = \dfrac{\partial f}{\partial x}(\xi(t))$, $r(t) = f(\xi(t))$, $P(t) = P_{r(t)}$, where $P_r = I - rr^\top / r^\top r$ is a projection matrix onto the subspace orthogonal to the vector $r \neq 0$.

This matrix function $W(t)$ is a stochastic sensitivity function of the cycle γ.

4. SENSITIVITY ANALYSIS OF 2D-CYCLES

For the case $n = 2$, the projection matrix is given by $P(t) = p(t)p^\top(t)$, where $p(t)$ is a normalized vector orthogonal to $f(\xi(t))$. As a result the matrix $W(t)$ can be written [3] as $W(t) = \mu(t)P(t)$. Here $\mu(t) > 0$ is T-periodic scalar stochastic sensitivity function (SSF). This scalar function is governed by the equation

$$\dot{\mu} = a(t)\mu + b(t)$$

with T-periodic coefficients $a(t) = p^\top(t)(F^\top(t) + F(t))p(t)$, $b(t) = p^\top(t)S(t)p(t)$. The explicit formula for solution $\mu(t)$ is given by following:

$$\mu(t) = g(t)(c+h(t)), \quad g(t) = \exp\left(\int_0^t a(s)ds\right), \quad h(t) = \int_0^t \frac{b(s)}{g(s)}ds, \quad c = \frac{g(T)h(T)}{1 - g(T)}.$$

The value $M = \max \mu(t)$, $t \in [0, T]$ plays an important role in the analysis of stochastic cycle. We shall consider M as a *sensitivity factor* of a cycle γ response to random disturbances.

5. STOCHASTIC MODEL OF POPULATION DYNAMICS

Consider a stochastically forced system

$$\begin{aligned} \dot{x} &= \frac{r}{\varepsilon}x(1-x) - \frac{a^2x^2}{\varepsilon(1+b^2x^2)}y + \sigma\dot{w}_1 \\ \dot{y} &= \frac{a^2x^2}{1+b^2x^2}y - my + \sigma\dot{w}_2, \end{aligned} \tag{5.1}$$

where $w_i(t)$ are independent Wiener processes, σ is a noise intensity. For $\sigma = 0$, this system is a well-known deterministic Truscott-Brindley model [4], [5] for the prey-predator dynamics of phytoplankton x and zooplankton y.

For $a^2 > m(1+b^2)$, unforced system ($\sigma = 0$) has a nontrivial equilibrium $\bar{x} = \sqrt{\dfrac{m}{a^2 - b^2 m}}$, $\bar{y} = \dfrac{r}{m}\bar{x}(1-\bar{x})$. We study stochastic system (5.1) for the fixed set of parameters $\varepsilon = 0.01$, $r = m = 1$, $b = 7$ and $a \in [7.1, 10]$. For $7.1 \leq a \leq 10$, this system demonstrates Hopf bifurcation at the points $a_1 = 7.345$ and $a_2 = 8.975$. In the intervals $[7.1, a_1)$ and $(a_2, 10]$ the equilibrium $\bar{x}, \bar{y}$ is stable. In the interval (a_1, a_2) this equilibrium is unstable and deterministic system has a stable limit cycle. Note that a transition zone from stable equilibria to limit cycles of maximal amplitudes is very narrow. One can see it in Fig. 1, where extremal values of variable x (Fig. 1a) and a variable y (Fig. 1b) are plotted for considered attractors. Lyapunov exponents for limit cycles are plotted in Fig. 5.

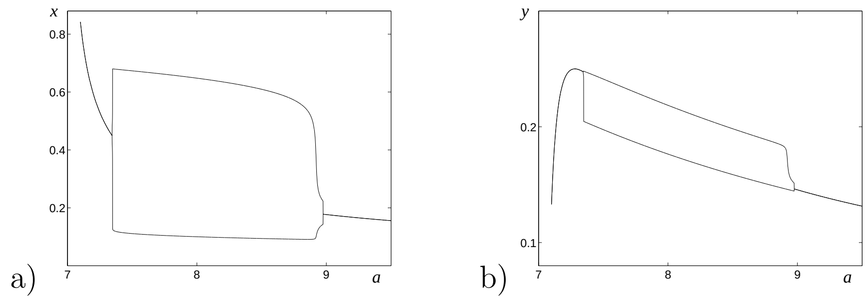

Fig. 1. Attractors of deterministic system

The random trajectories of stochastically forced model leave the deterministic attractor and form corresponding stochastic attractor around it. Results of a direct numerical simulation of the stochastic system (5.1) are plotted in Fig. 2.

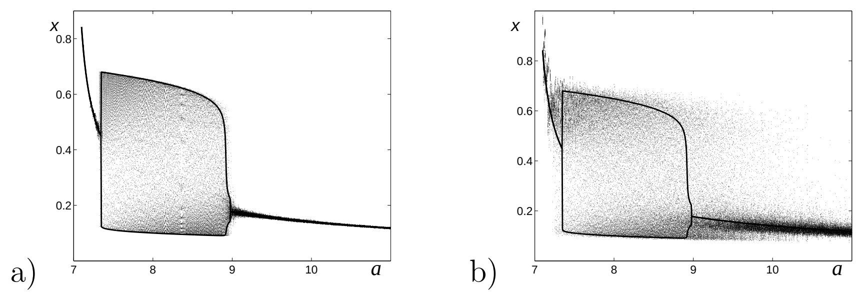

Fig. 2. Stochastic attractors for a) $\sigma = 0.001$, b) $\sigma = 0.01$

Here random states of stochastic attractors (grey color) and deterministic attractors (black color) are presented for two values of noise intensity $\sigma = 0.001$ (Fig. 2a) and $\sigma = 0.01$ (Fig. 2b).

Consider stochastic attractors for $a \in (a_2, 10)$. A dispersion of random states on this interval is non-uniform. As a parameter a tends to a_2, the dispersion grows. This feature can be explained with the help of SSF. In Fig. 3, values $w_{11}(a)$ and $w_{22}(a)$ of the SSF matrix $W(a)$ for equilibria $\bar{x}(a), \bar{y}(a)$ are shown. As one can see, these values essentially grow near bifurcation point a_2.

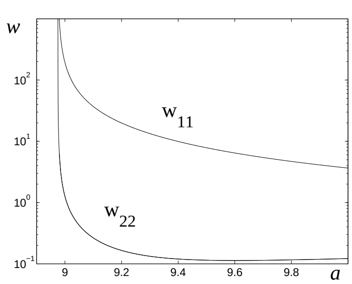

Fig. 3. Stochastic sensitivity of equilibria

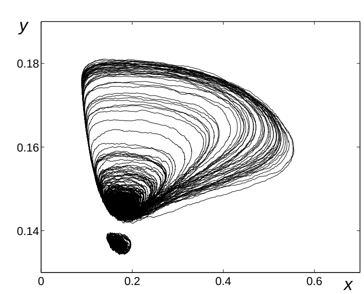

Fig. 4. Noise-induced oscillations. Stochastic attractors for $\sigma = 0.002$: $a = 9$ (upper), $a = 9.3$ (lower)

High level of stochastic sensitivity can cause noise-induced transitions from stochastic equilibria to stochastic oscillations with high amplitude. This phenomenon is observed in model (5.1). In Fig. 4, stochastic attractors generated by noise with intensity $\sigma = 0.002$ for values $a = 9.0$ and $a = 9.3$ are compared. For $a = 9.3$, random states are concentrated near deterministic equilibrium. For $a = 9.0$, one can see stochastic oscillations of high amplitude. Similar noise-induced oscillations are observed for sufficiently large zone from the right of a_2 (see Fig. 2).

Consider a zone of limit cycles $a_1 < a < a_2$. In Fig. 6, the sensitivity factor $M(a)$ is plotted. We can see an essential overfall of stochastic sensitivity values. Maximum value of sensitivity factor $M = 1.7 \cdot 10^{10}$ corresponds to $a_* = 7.3486135$. Consider in detail an interval $7.3486 < a < 7.34863$. We can see from Fig. 5 that Lyapunov exponent λ monotonically decreases with growth a. It means an increase of a stability degree of a cycle to disturbances of initial data. One should think it should be accompanied by the appropriate decrease in the sensitivity of a cycle to random disturbances. However, here the converse is observed. The value M behaves absolutely otherwise (see Fig. 6). Here we have typical example of "sensitive dependence to noise without sensitive dependence to initial conditions" [6].

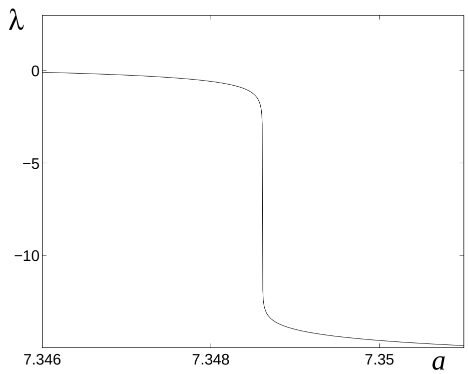

Fig. 5. Lyapunov exponent for deterministic cycles

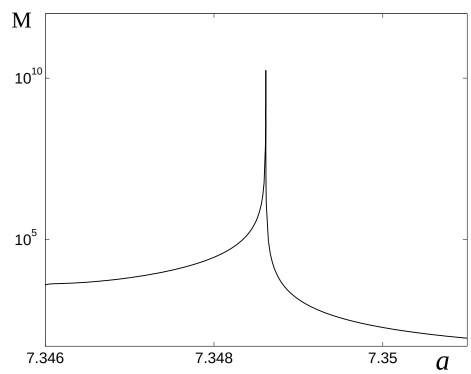

Fig. 6. Sensitivity factor for stochastic cycles

On the considered interval the function $M(a)$ is not monotonic. Its graph has sharp high peak. As a result the function $M(a)$ has an essential overfall of values. We compare the stochastic cycles of model (5.1) for values a close to a_*. For three values of parameter, $a = 7.3486$, $a_* = 7.3486135$, $a = 7.34863$ in Fig. 7 our results for $\varepsilon = 10^{-8}$ are demonstrated. As we see the model (5.1) with a_* (Fig. 7b) is supersensitive. For small background stochastic disturbances the burst of response amplitude is shown.

Unexpected noise-induced effects for the supersensitive cycle with parameter a_* are demonstrated in Fig. 8. Here stochastic attractors are plotted for different values of noise intensity. For $\sigma = 10^{-9}$ we observe a stochastic cycle with small dispersion (see Fig. 8a). For $\sigma = 10^{-8}$ a bottom part of the stochastic cycle is essentially washed out (see Fig. 8b). In Figs. 8c,d,e $\sigma = 10^{-7}, 10^{-6}, 10^{-5}$ we observe a splitting of a single stochastic cycle into two coexisting stochastic cycles of different amplitude. As

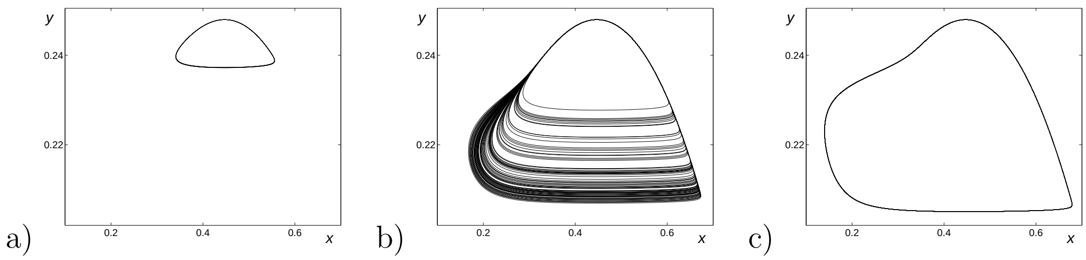

Fig. 7. Stochastic attractors for $\sigma = 10^{-8}$: a) $a = 7.3486$, b) $a = 7.3486135$, c) $a = 7.34863$

parameter σ grows, a smaller cycle vanishes (see Figs. 8f,g,h). In Fig. 8h for $\sigma = 10^{-2}$ we observe a single stochastic cycle.

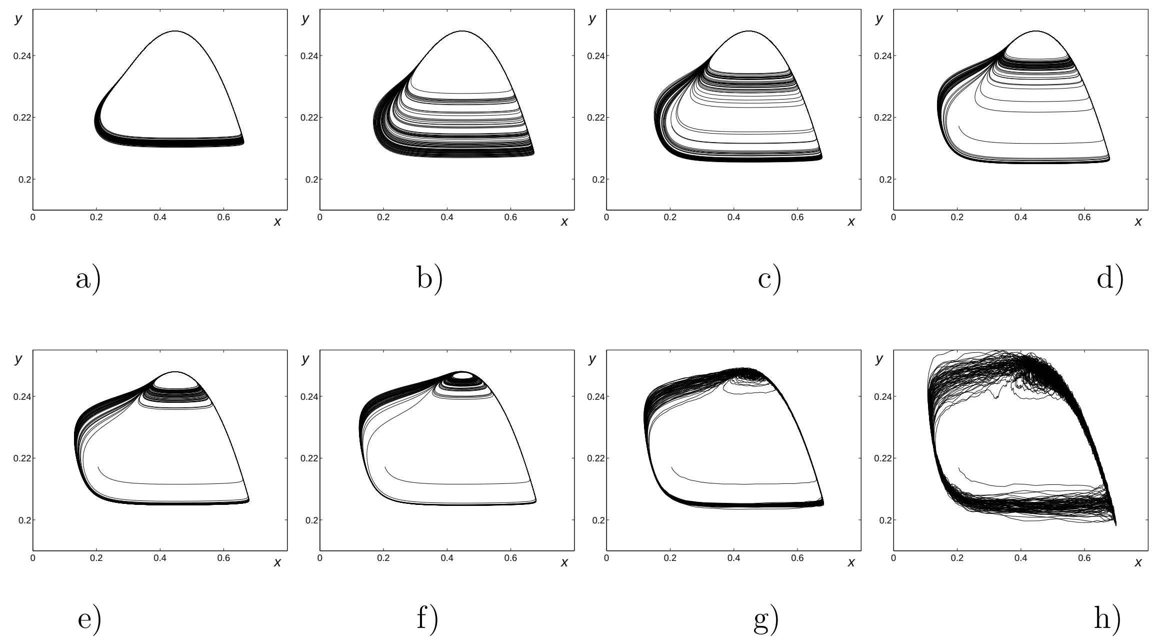

Fig. 8. Noise-induced transitions:
$\sigma = 10^{-9}$ (a), 10^{-8} (b), 10^{-7} (c), 10^{-6} (d), 10^{-5} (e), 10^{-4} (f), 10^{-3} (g), 10^{-2} (h).

Corresponding noise-induced transitions between two regimes of oscillations for solutions $x(t)$ are presented in Fig. 9. For noise intensity $\sigma = 10^{-6}$ oscillations of two fixed amplitudes are clearly observed. Here we have an example of noise-induced intermittency.

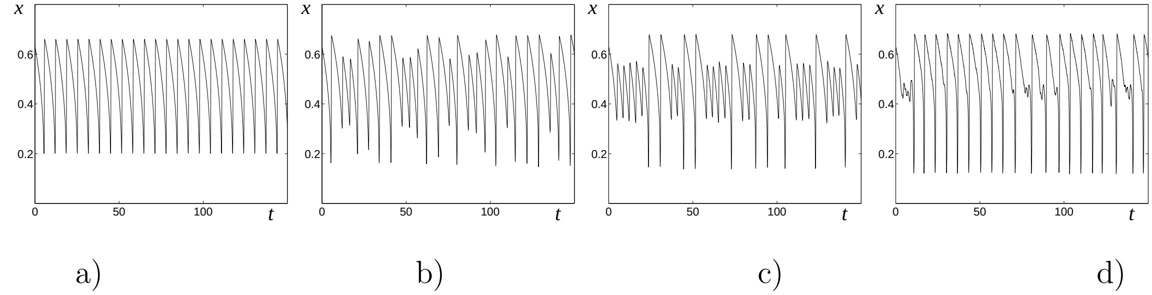

Fig. 9. Noise-induced intermittency: a) $\sigma = 0$, b) $\sigma = 10^{-7}$, c) $\sigma = 10^{-6}$, d) $\sigma = 10^{-3}$.

Large value of sensitivity factor imply new noise-induced phenomena for this stochastic population model.

Thus, the stochastic sensitivity function technique is a useful analytical tool for the prediction of new noise-induced response for stochastically forced nonlinear dynamical models.

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