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NUMERICAL ALGORITHM FOR SINGULARLY PERTURBED DELAY DIFFERENTIAL EQUATIONS WITH LAYER AND OSCILLATORY BEHAVIOR

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ABSTRACT. We consider the numerical approximation of singularly perturbed linear second order reaction-diffusion boundary value problems with a small shift(δ) in the undifferentiated term and the shift depends on the small parameter(ϵ). The presence of small parameter induces twin boundary layers. The problem is discretized using standard finite difference scheme on an uniform mesh and the retarded arguments are interpolated/extrapolated using the known computational grid points. We present a new algorithm to interpolate/exptrapolate the retarded term in terms of its neighbouring points. The scheme is proved to be stable and the error estimate is also given. It is shown that the shift has significant effect on the behavior of the solution. Numerical experiments are performed to support both the theoretical results as well as the existing results in the literature.

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1. INTRODUCTION

We consider the boundary value problems(BVPs) for the class of singularly perturbed differential difference equations(DDEs) given by

$$Lu(x) = \epsilon^2 u''(x) + \alpha(x)u(x-\delta(\epsilon)) + \beta(x)u(x) = f(x), \quad x \in \Omega = (0,1) \tag{1.1}$$

subject to the interval and boundary conditions:

$$u(x) = \phi(x) \;\; for \;\; -\delta(\epsilon) \le x \le 0 \tag{1.2a}$$

$$u(1) = u_1 \tag{1.2b}$$

where $\epsilon \to 0$ is the perturbation parameter, $\delta(\epsilon)(0 < \delta < 1)$ is the delay parameter and $\alpha(x)$, $\beta(x)$, f(x) and $\phi(x)$ are smooth functions of x and are assumed, for simplicity, to be independent of ϵ. Furthermore, when $\delta = 0$ the solution of the corresponding ordinary differential equation (ODE) has layers at x=0 and x=1 when

$$\alpha(x) + \beta(x) \le -\theta < 0 \quad \forall \quad x \in \bar{\Omega} \tag{1.3}$$

The layers are maintained when $\delta \neq 0$ but remain small [1, 2]. The solution $u(x)$ must be continuous on $[0,1]$, continuously differentiable on $(0,1)$ and also statisfies (1.1) and (1.2).

Boundary value problems involving DDEs arise naturally in many real life situations [3] like in studying variational problems in control theory where the problem is complicated by the effect of time delays in signal transmission [4], and also in the mathematical model (for the stochastic activity of neurons) of Stein [5, 6]. Stein's

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model is for the calculation of first exit time for the generation of action potentials in nerve cells by random synaptic inputs distributed as poisson process with exponential decay between the inputs in the dentrites. So, DDEs are a very important class of dynamical systems and often arise in either natural or technological control problems and hence ignoring them is to ingnore the reality. These applications motivate the approximation of DDEs.

Our goal in this paper is to elucidate the effect of the delay in a simpler class of DDEs (1.1), (1.2) which arise from Stein's model of neuronal variability. For a neurophysiological demonstration of the Stein's model one can refer to [5, 6, 7]. In our previous papers [8, 9, 10] we have considered delay to be of $\circ(\epsilon)$ for different classes of DDEs from Stein's model in which we used Taylors series approximation to the retarded term. In [10] we have studied the same class of DDEs (1.1), (1.2) using Taylors series approximation to the retarded terms and showed the effect of δ and η on the solution structure. The idea of using Taylors series approximation to the retarded terms was first given by Feldstein [11] although he didn't publish his work, it has become a landmark work to most of the researchers in numerical DDEs.

Others who approximated the Stein's model (DDE) are Tuckwell and Richter [6], Tuckwell and Cope [12], Wilbur and Rinzel [7], Lange and Miura [2], Kadalbajoo and Sharma [13], Patidar and Sharma [14]. Most of the above literature uses Taylors series approximation to the retarded arguments. Lange and Miura gave a series of papers (to list a few [2], [15]) on singularly perturbed differential difference equations by extending the method of matched asymptotic expansions developed for ODEs. Lange and Miura have analysed the same class of DDEs (1.1), (1.2) with small shifts and concluded that the small shift δ can change the character of the layers and can even destroy the layers when the shift increase but remain small. When δ is $\bigcirc(\epsilon)$ the approach of expanding the retarded term using truncated Taylors series will lead to misleading results [1, 2]. This is the motivation to this work. In this we are interested in a direct discretization of our model (1.1), (1.2) without a priori estimate to the retarded arguments.

Due to presence of δ in the reaction term of (1.1), a direct discretization of our

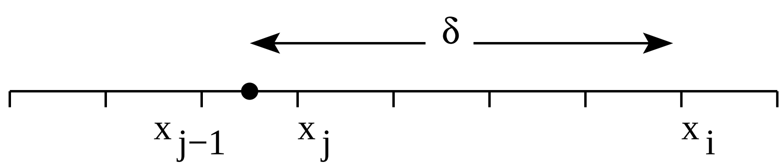

FIGURE 1. Important ostacle

model is not possible, because $x_i - \delta$ may not coincide with the computational grid points generated for a given N (see Fig. 1). The approximation of retarded term is one of the most important obstacles to overcome in approximating the solutions of DDEs. A new approach to the retarded term is given in section 2.1 and we have introduced our discrete difference equations in section 2.2. The error estimates are given in section 3 and the numerical experiments are given in section 4. Throughout this paper we assumed that $\delta = \tau\epsilon$ and we have analysed the behavior for both $\tau = \circ(1)$ and $\tau = \bigcirc(1)$. We investigate the effect of this small shift on the layer structure of the solution.

Our algorithm can easily be extended to the more general class of DDEs

$$\epsilon^2 u''(x) + \alpha(x)u(x - \delta(\epsilon)) + \beta(x)u(x) + \omega(x)u(x + \eta(\epsilon)) = f(x), \quad x \in \Omega = (0, 1)$$

subject to the interval conditions:

$$u(x) = \phi(x) \text{ for } -\delta(\epsilon) \le x \le 0$$

$$u(x) = \chi(x) \text{ for } 1 \le x \le 1 + \eta(\epsilon)$$

For simplicity we have restricted to (1.1), (1.2). Throughout this paper, C and θ denote positive constant independent of ϵ and in the case of discrete problem they are also independent of the mesh parameter N. $\|\cdot\|_\infty$ denotes the discrete maximum norm over the appropriate domain

$$\|f\|_\infty = \max_{x \in \bar{\Omega}^N} |f(x)|$$

2. THE DISCRETIZATION

We shall discretize our model (1.1), (1.2) on an uniform mesh $\bar{\Omega}^N$. We have proposed a new approach to treat the retarded term and is as follows.

2.1. A new approach to the Retarded term.

In this we propose a new algorithm to locate the Interpolation/Extrapolation points in order to express $x_i - \delta$ interms of the computational grid points and thereby to compute $u(x - \delta)$.

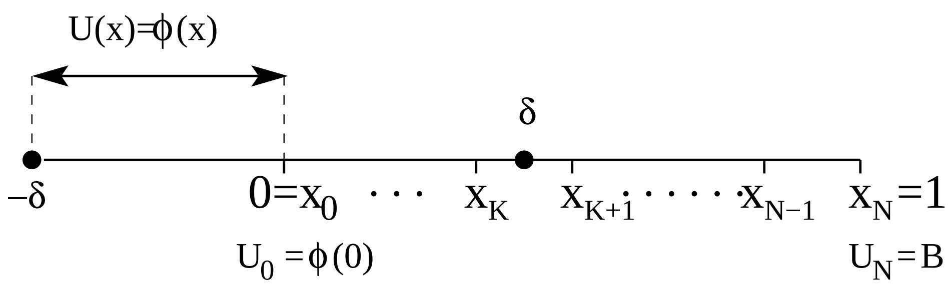

FIGURE 2. Location of extraplotion points

The algorithm involves the following steps

- For a fixed N, compute $\bar{\Omega}^N = \{x_i : x_j = jh, j = 0, 1, \ldots, N\}$
- $u(x) = \phi(x)$ for $-\delta \le x \le 0$ and $U_0 = u(x_0) = u(0) = \phi(0)$ & $U_N = u(x_N) = u(1) = B$ (from interval (1.2a) and boundary (1.2b) conditions)
- Fix the value of δ. Say $\delta = \tau\epsilon$, where τ may be $\circ(1)$ or $\bigcirc(1)$, but remain small
- Compute the non-negative integer K by

$$K = \left[\frac{\delta}{h}\right] = [\delta N] = \text{ integer part of } \delta * N \tag{2.1}$$

- For any x_i in $\{x_j\}_{j=1}^{K} \subset \{x : 0 < x \le \delta\}$ it is easy to verify that $x_i - \delta \in (-\delta, 0]$ (see Fig. 2.1) and hence $u(x_i - \delta)$ can be replaced by $\phi(x_i - \delta)$ (from the interval condition (1.2a))
- Now, for any $x_i \in \{x_j\}_{j=K+1}^{N-1} \subset \{x : \delta < x < 1\}$, it is easy to check that $x_i - \delta \in [x_{i-K-1}, x_{i-K}]$

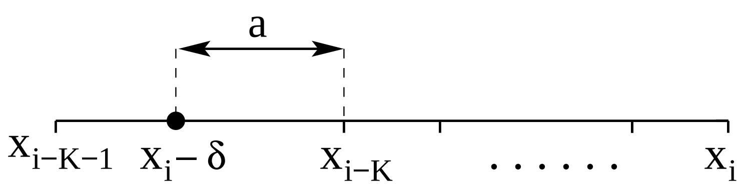

FIGURE 3

- Now the retarded term $x_i - \delta$ can be written in terms of x_{i-K-1} and x_{i-K} and is given by (see Fig. 3)

$$x_i - \delta = a x_{i-K-1} + (1-a) x_{i-K} \ \forall\ K+1 \le i < N \tag{2.2}$$

where $a = \frac{x_{i-K} - x_i + \delta}{h}$=$(x_{i-K} - x_i + \delta)N = (\delta N - K) \ge 0$.

- Now, without affecting any of our conclusions one can write

$$\begin{aligned} U(x_i - \delta) &= U(a x_{i-K-1} + (1-a) x_{i-K}) \\ &= a U_{i-K-1} + (1-a) U_{i-K} \quad \forall\ K+1 \le i < N \end{aligned}$$

This idea of writing $U(a x_{i-K-1} + (1-a) x_{i-K}) = a U_{i-K-1} + (1-a) U_{i-K}$ has also been done by Abrahamsson [16] in his scheme refered to as Abrahamsson's scheme or modification of the Abrahamsson-Keller-Kreiss box scheme [17], where $u(\frac{x_i + x_{i+1}}{2})$ is replaced by $\frac{u(x_i) + u(x_{i+1})}{2}$. In Abrahamsson's scheme the solution at the average of two points is replaced by the average of the solution at those two points. In our case the solution at the weighted average of two neighbouring points of $x_i - \delta$ is replaced by the weighted average of the solution at those points.

Remark 1. If $\delta < h$ then K = $[\delta N] <$ [h N] = 1 $\rightarrow K = 0$, hence $x_i - \delta \in (x_{i-1}, x_i)$. In this case we just have to interpolate the nodes x_i & x_{i-1} to get $x_i - \delta$.

Remark 2. For some $\delta > 0$, if $x_i - \delta \in \bar{\Omega}^N \rightarrow$ there exist K such that $x_i - \delta = x_{i-K}$ and hence a=0. Hence no interpolation/extrapolation is needed, but in practice $x_i - \delta \notin \bar{\Omega}^N$ mostly.

2.2. **Difference equations.** Let U=$\{U_i\}_{i=0}^N$ be any given function defined on the computational grid, we shall approximate the second-order derivatives and the retarded term at the grid point x_i as follows: The second order derivative $D_+D_-U_i$ is

$$D_+D_-U_i = \frac{(U_{i-1} - 2U_i + U_{i+1})}{h^2}$$

We introduce our difference operator for (1.1), (1.2) by

$$L^h U_i = \hat{f}_i \tag{2.3}$$

subject to

$$U_0 = \phi(0) \tag{2.4a}$$

$$U_N = B \tag{2.4b}$$

where
(2.5)

$$L^hU_i = \begin{cases} \epsilon^2 D_+D_-U_i + \beta_i U_i & \text{for } 1 \le i \le K \\ \epsilon^2 D_+D_-U_i + \alpha_i(aU_{i-K-1} + (1-a)U_{i-K}) + \beta_i U_i & \text{for } K+1 \le i \le N-1 \end{cases}$$

(2.6)
$$\hat{f}_i = \begin{cases} f_i + \beta_i \phi(x_i - \delta) & \text{for } 1 \le i \le K \\ f_i & \text{for } K+1 \le i \le N-1 \end{cases}$$

where $\phi(x)$ is the interval condition (1.2a), K and a are as given in section 2.1. The above equations will get reduced to the following tridiagonal system of $N-1$ equations with $N+1$ unknowns.

(2.7) $$a\alpha_i U_{i-K-1} + (1-a)\alpha_i U_{i-K} + E_i U_{i-1} - F_i U_i + G_i U_{i+1} = H_i \quad \forall i = 1, 2, \ldots, N-1$$

where

$$\begin{aligned} E_i &= \frac{\epsilon^2}{h^2} \\ F_i &= \frac{2\epsilon^2}{h^2} - \beta_i \\ G_i &= \frac{\epsilon^2}{h^2} \\ H_i &= \hat{f}_i \end{aligned}$$

$\beta_i = \beta(x_i)$, $\alpha_i = \alpha(x_i)$ and $f_i = f(x_i)$ for all $x_i \in \bar{\Omega}$.

3. ERROR ANALYSIS

Lemma 3.1 (Discrete Minimum Principle)**.** *Suppose* $\phi_0 \ge 0$ *and* $\phi_N \ge 0$. *Then* $L^h\phi_i \le 0$ *for all* $i = 1, 2, \ldots, N-1$ *implies* $\phi_i \ge 0 \; \forall \; i = 0, 1, \ldots, N$.

Proof. Let k be such that

$$\phi_k = \min_{0 \le i \le N} \phi_i$$

and assume that $\chi_k < 0$. Then we have $\phi_k - \phi_{k-1} \le 0$, $\phi_{k+1} - \phi_k \ge 0$ and for $1 \le k \le m$

$$\begin{aligned} L^h\phi_k &= \epsilon^2 \frac{(\phi_{k-1} - 2\phi_k + \phi_{k+1})}{h^2} + \beta_k \phi_k \\ &= \epsilon^2 \frac{([\phi_{k+1} - \phi_k] - [\phi_k - \phi_{k-1}])}{h^2} + \beta_k \phi_k \\ &> 0 \end{aligned}$$

provided $\beta_k \le 0$, which contradicts our assumption that $L^h\phi_i \le 0$ for all $i = 1, 2, \ldots, N-1$. Hence $\phi_k < 0$ is not true, which implies $\phi_k \ge 0$. Since k is arbitrary it follows that $\phi_k \ge 0$ for all $0 \le i \le N$. □

Theorem 3.2. *Under the assumptions* (1.3)*, the solution of our difference equations* (2.3), (2.4) *exist and is unique. It also satisfies the following bound*

$$\|U\|_\infty \le \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)$$

Proof. Lets prove the existence and uniqueness of the solution of the difference equations (2.3), (2.4). Let $\{U_i\}_{i=0}^N$ and $\{V_i\}_{i=0}^N$ be two solutions of (2.3), (2.4). Let us define a mesh function $Z_i = U_i - V_i$ for all $i = 0, 1, \dots, N$. It is easy to verify that $Z_0 = 0 = Z_N$ which follows from the equation (2.4). Now,

$$L^h Z_i = L^h U_i - L^h V_i = 0$$

U_i and V_i are solutions of (2.3). By an application of Lemma 3.1 we get

$$Z_i = U_i - V_i \geq 0 \quad \forall i \tag{3.1}$$

Now, let us define another mesh function $W_i = V_i - U_i$ for all $i = 0, 1, \dots, N$. It is easy to verify that $W_0 = 0 = W_N$ which follows from the equation (2.4). Now,

$$L^h W_i = L^h V_i - L^h U_i = 0$$

U_i and V_i are solutions of (2.3). By an application of Lemma 3.1 we get

$$W_i = V_i - U_i \geq 0 \quad \forall i \tag{3.2}$$

From equations (3.1) and (3.2), we get $U_i = V_i$ for all i. Hence uniqueness is proved. For linear equations, the existence is implied by uniqueness.

Now, lets prove the estimate. Lets consider the barrier functions given by

$$\chi_i^{\pm} = \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm U_i, \quad 0 \leq i \leq N$$

where C is a positive constant.

$$\begin{aligned}
\chi_0^{\pm} &= \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm U_0 \\
&= \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm \phi_0 \\
&= \theta^{-1}\|f\|_\infty + \left(C\|\phi\|_\infty \pm \phi_0\right) + C|B| \\
&\geq 0
\end{aligned}$$

Similarly,

$$\begin{aligned}
\chi_N^{\pm} &= \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm U_N \\
&= \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm B \\
&= \theta^{-1}\|f\|_\infty + C\|\phi\|_\infty + \left(C|B| \pm B\right) \\
&\geq 0
\end{aligned}$$

Case 1: For $1 \leq i \leq K$

$$\begin{aligned}
L^h\chi_i^{\pm} &= \epsilon^2 D_+ D_i \chi_i^{\pm} + \beta_i \chi_i^{\pm} \\
&= \beta_i\left(\theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)\right) \pm L^h U_i \\
&= \beta_i\left(\theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)\right) \pm \left(f_i + \beta_i\phi(x_i - \delta)\right) \\
&= -\left(\|f\|_\infty \pm f_i\right) - C\theta\left(\|\phi\|_\infty + |B|\right) \mp \beta_i\phi(x_i - \delta) \quad \text{using (1.3)}
\end{aligned}$$

Now chose C such that

$$L^h\chi_i^{\pm} < 0, \quad i = 1, 2, \dots, K \tag{3.3}$$

Case 2: For $K+1 \leq i \leq N-1$

$$\begin{aligned} L^h\chi_i^{\pm} &= \epsilon^2 D_+D_i\chi_i^{\pm} + \alpha_i\left(a\chi_{i-K-1}^{\pm} + (1-a)\chi_{i-K}^{\pm}\right) + \beta_i\chi_i^{\pm} \\ &= (\alpha_i a + \alpha_i(1-a) + \beta_i)\left(\theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)\right) \pm L^hU_i \\ &= (\alpha_i a + \alpha_i(1-a) + \beta_i)\left(\theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)\right) \pm f_i \\ &= (\alpha_i + \beta_i)\left(\theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right)\right) \pm f_i \\ &= (-\|f\|_\infty \pm f_i) + (\alpha_i + \beta_i)C\left(\|\phi\|_\infty + |B|\right) \\ &< 0 \quad using\ (1.3) \end{aligned}$$

Hence

$$L^h\chi_i^{\pm} < 0, \quad i = K+1, K+2, \ldots, N-1 \tag{3.4}$$

From (3.3) and (3.4) we get

$$L^h\chi_i^{\pm} < 0, \quad i = 1, 2, \ldots, N-1 \tag{3.5}$$

Using Lemma 3.1, the above inequality gives

$$\begin{aligned} \chi_i^{\pm} &= \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \pm U_i \\ &\geq 0, \quad 0 \leq i \leq N \end{aligned}$$

which gives

$$\|U\|_\infty \leq \theta^{-1}\|f\|_\infty + C\left(\|\phi\|_\infty + |B|\right) \tag{3.6}$$

Hence we have proved that the solution of the difference equations (2.3), (2.4) is uniformly bounded and is independent of the parameters h,ϵ and δ. □

Corollary 1. *Assume* (1.3) *and* $\delta > 0$. *The unique solution* $\{U_i\}_{i=0}^N$ *of our discrete problem satisfies*

$$\|U - u\|_\infty \leq \frac{1}{\theta}\|\tau\|_\infty \tag{3.7}$$

where the truncation error τ_i *satisfies,*

$$\tau_i \leq \frac{h^2\epsilon^2}{12}|u^{iv}| \tag{3.8}$$

Proof. It is easy to verify (3.8) which follows from the Taylors series. Now,

$$L^hU_i - L^hu(x_i) = \tau_i \quad \forall\ i$$

It is obvious that $\tau_0 = 0 = \tau_N$. Now by an application of theorem 3.2 one can get (3.7). □

4. NUMERICAL RESULTS

In this section we present numerical results to show the efficiency of the algorithms discussed in the previous section with a comparison to other results in literature. Here we take two test problems, first with contant coefficients (page 263, [2]) and second with variable coefficients with small delay

Example 4.1 (Lange & Miura: [2], page 263)**.**

$$\epsilon^2 u''(x) - 2u(x-\delta) - u(x) = 1,$$

under the interval and boundary conditions

$$u(x) = 1 \text{ for } -\delta \leq x \leq 0, \; u(1) = 0$$

Example 4.2.

$$\epsilon^2 u''(x) - (2 + x^2)u(x-\delta) + (1 - \cos(x))u(x) = x^{4.5} + \sin(x),$$

under the interval and boundary conditions

$$u(x) = 1 \; for \; -\delta \leq x \leq 0, \; u(1) = 1$$

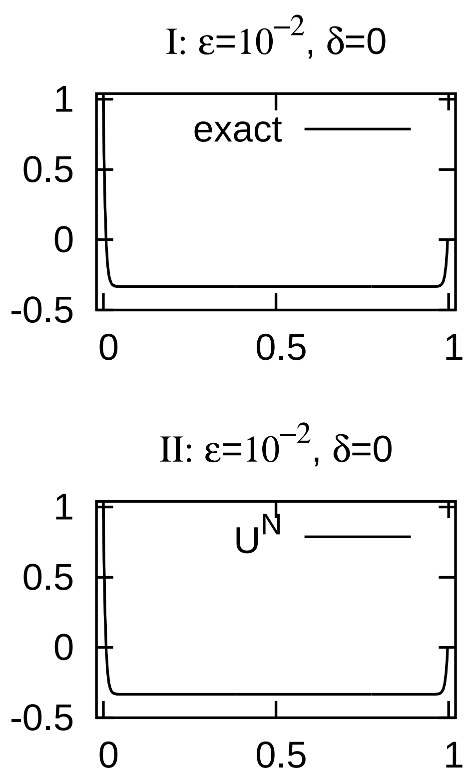

FIGURE 4. Graphs of exact (I), numerical (II) solution of Example 4.1 for $\epsilon = 10^{-2}$, $\delta = 0$

Since the exact solution is not known when $\delta \neq 0$, the maximum pointwise error is estimated using the double mesh principle [18] defined by

$$E^N_{\epsilon,\delta} = \max_{1 \leq i \leq N-1} |U^N_i - U^{2N}_{2i}|.$$

where U^{2N}_{2i} is the solution obtained on a mesh containing $2N$ number of mesh points.

Example 4.1 is taken from one of the landmark papers for numerical DDEs, namely Lange & Miura page 263 [2]. Table 1 shows the computed maximum pointwise errors of example 4.1 for $\delta = 0.03$ and for various values of $\epsilon(\frac{1}{2} \leq \epsilon^2 \leq \frac{1}{2^{10}})$ and Table 2 shows the computed maximum pointwise errors for a fixed $\epsilon = 0.1$ and for different

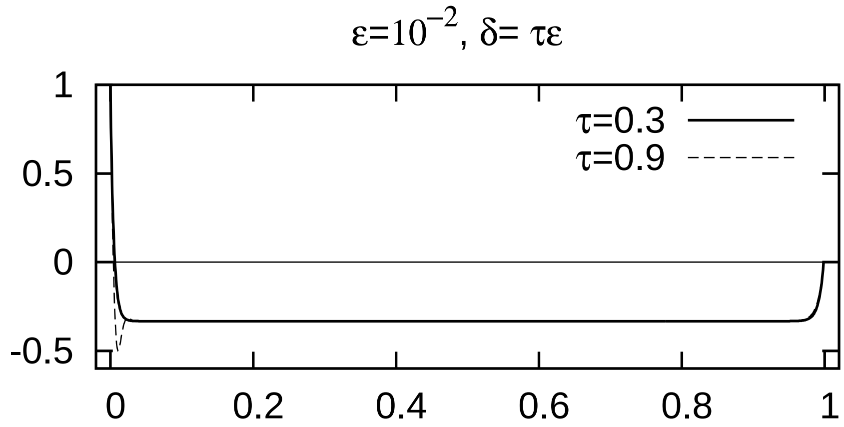

FIGURE 5. Graphs of numerical solution of example 4.1 for $\epsilon = 10^{-2}$ and $\delta = 0.3, 0.9$

$\delta = \{0.03(0.02)0.13\}$. Similarly, Tables 3 & 4 show the maximum errors of example 4.2.

Figures 4(I) and 4(II) show the exact and numerical solution of Example 4.1 respectively, showing the efficiency of the difference scheme in capturing the layer region. Figure 5 shows the computed numerical solution of Example 4.1 for $\epsilon = 0.01$ and for various values of $\delta = 0.3, 0.9$. Figures 6(I), 6(II) and 6(III) correspond to the numerical solution of Example 4.1 for $\delta = \tau\epsilon$ with τ=1.5, 3 & 5 respectively. With subgraph in Figure 6(III) one can conclude that the solution oscillates through out the domain. A comparison of Figure 6(I–III) with Figures 5(b), 5(c), 5(d) of Lange & Miura (page 263, [2]) proves the efficiency of our algorithm in modelling the solution for $\delta = \bigcirc(\epsilon)$.

Figure 7(I) shows the computed numerical solution of example 4.2 with $\delta = \circ(\epsilon)$. Figure 7(II) shows the computed numerical solution of Example 4.2 for $\delta = 1.5\epsilon$. One can observe that the solution agrees the twin layers at $x = 0$ and $x = 1$ but oscillates near the left boundary surprisingly. Figures 7(III-VI) shows the computed solution of Example 4.2 with different δ. From Figures 6(I-III) & 7(III-VI) we observe that when the delay increases to $\bigcirc(\epsilon)$ layer structure of the solution changes to oscillatory behavior.

Table 1. Maximum Pointwise error($E^N_{\epsilon,\delta}$) for Example 4.1 with $\delta = 0.03$

ϵ^2	Number of grid points N					
	100	200	300	400	500	600
2^{-1}	0.00000803	0.00000201	0.00000089	0.00000050	0.00000032	0.00000022
2^{-2}	0.00001732	0.00000433	0.00000192	0.00000108	0.00000069	0.00000048
2^{-3}	0.00003183	0.00000796	0.00000354	0.00000199	0.00000127	0.00000088
2^{-4}	0.00005854	0.00001465	0.00000651	0.00000366	0.00000234	0.00000163
2^{-5}	0.00015633	0.00003913	0.00001740	0.00000979	0.00000626	0.00000435
2^{-6}	0.00040861	0.00010240	0.00004553	0.00002562	0.00001640	0.00001139
2^{-7}	0.00104474	0.00026242	0.00011673	0.00006568	0.00004204	0.00002920
2^{-8}	0.00260987	0.00065831	0.00029307	0.00016495	0.00010559	0.00007334
2^{-9}	0.00637779	0.00162133	0.00072284	0.00040704	0.00026064	0.00018105
2^{-10}	0.01531354	0.00394921	0.00176542	0.00099507	0.00063744	0.00044290

Table 2. Maximum Pointwise error($E^N_{\epsilon,\delta}$) for Example 4.1 with $\epsilon^2 = 0.01$

δ	Number of grid points N					
	100	200	300	400	500	600
0.03	0.00074981	0.00018816	0.00008368	0.00004708	0.00003014	0.00002093
0.05	0.00105355	0.00026430	0.00011754	0.00006613	0.00004233	0.00002940
0.07	0.00129141	0.00032391	0.00014405	0.00008104	0.00005187	0.00003602
0.09	0.00149532	0.00037499	0.00016676	0.00009382	0.00006005	0.00004170
0.11	0.00168105	0.00042152	0.00018744	0.00010546	0.00006750	0.00004688
0.13	0.00185621	0.00046539	0.00020695	0.00011643	0.00007452	0.00005175

TABLE 3. Maximum Pointwise error($E^N_{\epsilon,\delta}$) for Example 4.2 with $\delta = 0.05$

ϵ^2	Number of grid points N					
	100	200	300	400	500	600
2^{-1}	0.00001203	0.00000301	0.00000134	0.00000075	0.00000048	0.00000033
2^{-2}	0.00002410	0.00000603	0.00000268	0.00000151	0.00000096	0.00000067
2^{-3}	0.00004098	0.00001025	0.00000455	0.00000256	0.00000164	0.00000114
2^{-4}	0.00007868	0.00001967	0.00000874	0.00000492	0.00000315	0.00000219
2^{-5}	0.00021053	0.00005264	0.00002340	0.00001316	0.00000842	0.00000585
2^{-6}	0.00054523	0.00013636	0.00006061	0.00003409	0.00002182	0.00001515
2^{-7}	0.00138943	0.00034757	0.00015450	0.00008691	0.00005562	0.00003863
2^{-8}	0.00349115	0.00087366	0.00038836	0.00021847	0.00013982	0.00009710
2^{-9}	0.00863816	0.00216286	0.00096155	0.00054092	0.00034621	0.00024043
2^{-10}	0.02103134	0.00526983	0.00234314	0.00131821	0.00084371	0.00058593

TABLE 4. Maximum Pointwise error($E^N_{\epsilon,\delta}$) for Example 4.2 with $\epsilon^2 = 0.01$

δ	Number of grid points N					
	100	200	300	400	500	600
0.03	0.00066716	0.00016690	0.00007419	0.00004173	0.00002671	0.00001855
0.05	0.00099732	0.00024946	0.00011088	0.00006237	0.00003992	0.00002772
0.07	0.00127532	0.00031896	0.00014177	0.00007975	0.00005104	0.00003544
0.09	0.00151715	0.00037942	0.00016864	0.00009486	0.00006071	0.00004216
0.11	0.00173239	0.00043322	0.00019255	0.00010831	0.00006932	0.00004814
0.13	0.00192692	0.00048185	0.00021416	0.00012047	0.00007710	0.00005354

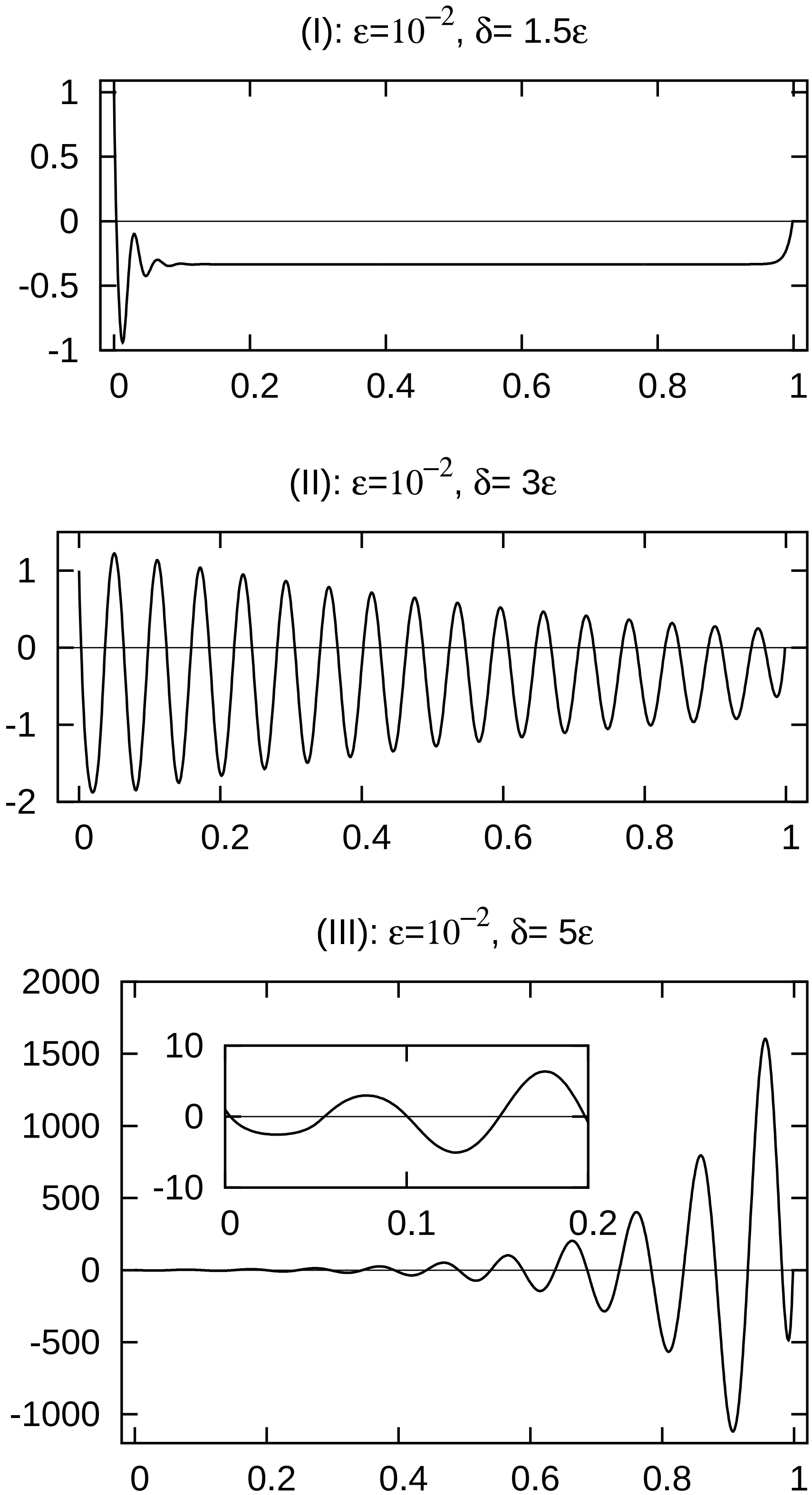

FIGURE 6. Graphs (I)–(III) correspond to the numerical solutions of Example 4.1 for $\epsilon = 10^{-2}$ and for $\delta = 1.5, 3, 5$ respectively. The subgraph in III shows the oscillating behavior of the solution from $x = 0$ to $x = 0.2$

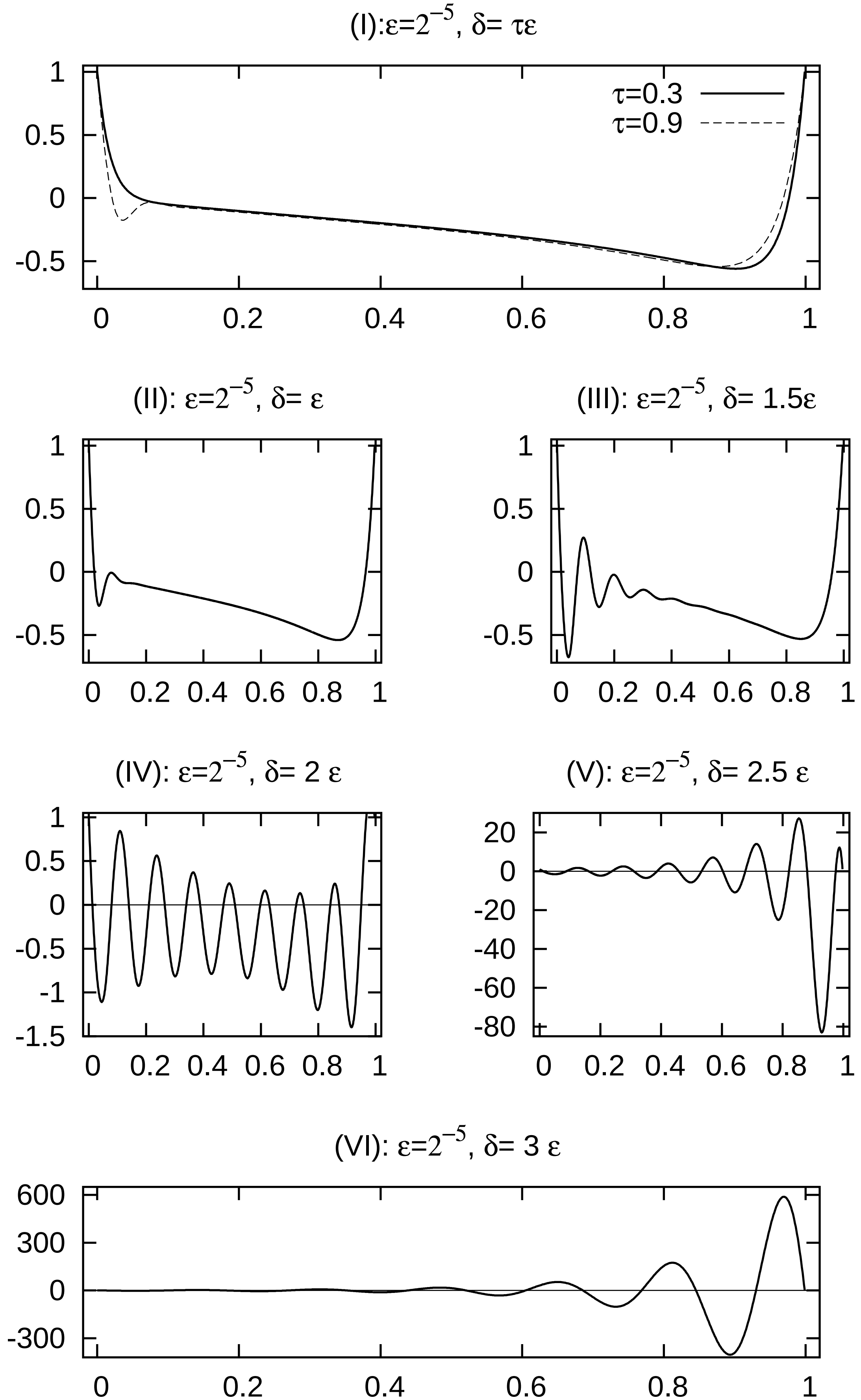
(I):ε=2^−5, δ= τε
τ=0.3
τ=0.9
1
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1
(II): ε=2^−5, δ= ε
(III): ε=2^−5, δ= 1.5ε
(IV): ε=2^−5, δ= 2 ε
-1
-1.5
(V): ε=2^−5, δ= 2.5 ε
20
-20
-40
-60
-80
(VI): ε=2^−5, δ= 3 ε
600
300
-300

FIGURE 7

5. DISCUSSION

The main objective of this paper is to to elucidate the effect of the delay on the solution of BVPs of DDEs with twin layers. Our numerical algorithm addresses the retarded term in a new way without any a-priori estimates for the retarded argument. To this end, we have shown the efficiency of our alogrithm by comparing the results obtained with one of the landmark papers by Lange and Miura [2]. This new approach to the retarded term can easily be extended to numerical schemes on shishkin mesh (piecewise uniform). We have observed that when the delay is $\bigcirc(\epsilon)$ the layer structures of the solutions changes to oscillatory behavior.

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