Neural, Parallel, and Scientific Computations 19 (2011) 51-64

STABILITY OF A CONTINUOUS TIME MODEL

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ABSTRACT. A qualitative analysis of an SIV (susceptible-infected-vaccinated) model of the spread of *gonorrhea* in a homosexually active population is performed. A basic reproduction number $\mathbb{R}_o$ is identified and a threshold nature of the disease is established; it is shown that if $\mathbb{R}_o < 1$, the disease free equilibrium is globally asymptotically stable implying the disappearance of the disease in the population; if $\mathbb{R}_o > 1$ then it is shown that the endemic equilibrium is globally asymptotically stable implying the invasion of the population by the disease. These results are established by using the theory of the asymptotically autonomous differential equations.

AMS (MOS) Subject Classification. 92D30, 34D23

1. INTRODUCTION

Systems of differential equations are naturally generated from many real world situations that can be mathematically modelled by observable phenomena likes many measurable behaviors of physical sciences, engineering, population dynamics, and human epidermis. Such systems are normally parametric dependent. In the sense that the variation of the magnitudes of those parameters will trigger the stability of equilibria of the systems. In the other words, the stability characteristics of equilibria are sensitive to the changes of magnitudes of the parameters. It is of great interest to study the dynamic characteristics, like stability and bifurcation of equilibria of such systems, while the analysis of the stability of equilibria will lead to an initial understanding of how the solution behaves. In an epidermis system, for example, it will be a system of three or four non linear differential equations. The stability, component wise, is represented by the steady states of the equations. There is always a reference value, called a reproduction number, which is also parametric dependent, that governs the stability characteristics of equilibria of such system. The changes of stability can then be manipulated by proper management of the ranges of magnitude of such reproduction number.

The purpose of this article is to formulate and study the stability characteristics of a continuous-time model proposed by Lima and Torres [9]

Received November 17, 2010

$$\left.\begin{aligned} \frac{dS(t)}{dt} &= (1-p)\rho - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \gamma I(t) + \delta V(t) \\ \frac{dI(t)}{dt} &= \frac{\beta S(t)I(t)}{N(t)} - (\gamma+\mu)I(t) \\ \frac{dV(t)}{dt} &= -(\delta+\mu)V(t) + p\rho \\ N(t) &= S(t) + I(t) + V(t) \end{aligned}\right\}, \qquad t > 0 \tag{1.1}$$

where parameters ρ, μ, β, γ, δ are all positive numbers.

2. MODEL FORMULATION AND REDUCTION

The model (1.1) is properly formulated in the sense that non-negative initial values lead to solutions which remain non-negative for all $t > 0$.

We have from (1.1) that

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dV}{dt}$$

and hence

$$\frac{dN(t)}{dt} = \rho - \mu N(t), \qquad t > 0. \tag{2.1}$$

It can be found from (1.1) and (2.1) that $S(0) > 0 \Longrightarrow S(t) > 0$, $I(0) > 0 \Longrightarrow I(t) > 0$ and

$$\left.\begin{aligned} V(t) &= V(0)e^{-(\delta+\mu)t} + \frac{p\rho}{\delta+\mu}\left[1 - e^{-(\delta+\mu)t}\right] \\ N(t) &= N(0)e^{-\mu t} + \frac{\rho}{\mu}\left(1 - e^{-\mu t}\right) \end{aligned}\right\}, \qquad t > 0. \tag{2.2}$$

A consequence of (2.2) is that

$$V(0) > 0 \Longrightarrow V(t) > 0 \quad \text{and} \quad N(0) > 0 \Longrightarrow N(t) > 0 \quad \text{for} \quad t > 0$$

and furthermore we have from the above that

$$\left.\begin{aligned} \lim_{t\to\infty} V(t) &= \frac{p\rho}{\delta+\mu} \\ \lim_{t\to\infty} N(t) &= \frac{\rho}{\mu} \end{aligned}\right\}. \tag{2.3}$$

In view of the above analysis, the autonomous system (1.1) can be reformulated in the form of the following non-autonomous differential algebraic system:

$$\left.\begin{aligned} \frac{dS(t)}{dt} &= (1-p)\rho - \mu S(t) - \frac{\beta S(t)I(t)}{N(t)} + \gamma I(t) + \delta V(t) \\ \frac{dI(t)}{dt} &= I(t)\left[\frac{\beta S(t)}{N(t)} - (\gamma+\mu)\right] \\ V(t) &= V(0)e^{-(\delta+\mu)t} + \frac{p\rho}{\delta+\mu}\left[1 - e^{-(\delta+\mu)t}\right] \\ N(t) &= N(0)e^{-\mu t} + \frac{\rho}{\mu}\left(1 - e^{-\mu t}\right) \end{aligned}\right\}, \qquad t > 0. \tag{2.4}$$

System (2.4) is a special kind of a non-autonomous differential algebraic system, in which some of the dependent variables converge as $t \to \infty$ to certain limit values. Systems of the form (2.4) together with (2.3) are known as asymptotically autonomous type (see for instance Castillo-Chavez and Thieme [2], Markus [10], Thieme and Castillo-Chavez [11], Thieme [13]) having an associated autonomous limit system given by

$$\left.\begin{aligned} \frac{dS}{dt} &= (1-p)\rho - \mu S - \beta\frac{\mu}{\rho}SI + \gamma I + \frac{\delta p\rho}{\delta+\mu} \\ \frac{dI}{dt} &= I\left[\frac{\beta\mu}{\rho}S - (\gamma+\mu)\right] \end{aligned}\right\}, \quad t > 0 \tag{2.5}$$

which provides a considerable simplification of (1.1), resulting in a reduction of the three dimensional system to a two dimensional system providing a reduction in the dimensionality of the problem. Analysis of non-autonomous systems are usually more complicated in comparison with that of autonomous systems. In many cases approximations are made wherever possible to remove the explicit time dependence . One of the techniques proposed for this purpose is to consider non-autonomous systems which have the property of being asymptotically autonomous in the sense that a non-autonomous system approaches as $t \to \infty$ to an autonomous one. Such simplifications have been exploited in the study of population dynamics modelled by chemostat systems (see for instance Butler and Wolkowicz [1] and Waltman [15], Han et al. [5, 6], Thieme [12, 13]) In our analysis of system (1.1), we will use the following format of a result related to the application of asymptotically autonomous systems to simplify (1.1) to an autonomous one dimensional system which is much easier to analyze.

Lemma 2.1 (Han et al. [6])**.** *Consider a non-autonomous differential system given by*

$$\frac{dx(t)}{dt} = f(t, x(t)), \qquad t > 0 \tag{2.6}$$

and an associated autonomous sytem

$$\frac{dx}{dt} = g(x), \qquad t > 0 \tag{2.7}$$

where f and g are continuous and locally Lipschitzian for $x \in R^n$ where

$$\lim_{t\to\infty} f(t, x) = g(x) \tag{2.8}$$

uniformly for $x \in R^n$ and solutions of the above nonautonomous and autonomous limiting system exist for $t > 0$. If solutions of the nonautonomous system are bounded and an equilibrium x^* of the limiting system is globally asymptotically stable then any solution $x(t)$ of the above nonautonomous system satisfies

$$\lim_{t\to\infty} x(t) = x^*. \tag{2.9}$$

The non-autonomous system

$$\frac{dx(t)}{dt} = f(t, x(t)), \qquad t > 0$$

is said to be asymptotically autonomous and the associated limit system is given by

$$\frac{dx}{dt} = g(x), \qquad t > 0.$$

Thus we are motivated to study the asymptotically autonomous system

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)\rho - \mu S - \frac{\beta SI}{N(t)} + \gamma I + \delta V(t)\\
\frac{dI(t)}{dt} &= I\left[\frac{\beta S}{N(t)} - (\gamma+\mu)\right]\\
N(t) &= N(0)e^{-\mu t} + \frac{\rho}{\mu}(1-e^{-\mu t})\\
V(t) &= V(0)e^{-(\mu+\delta)t} + \frac{p\rho}{\mu+\delta}(1-e^{-(\mu+\delta)t})
\end{aligned}\right\}, \qquad t>0. \tag{2.10}$$

By an application of the Lemma, we can simplify the original system to a two dimensional system of the form.

$$\left.\begin{aligned}
\frac{dS(t)}{dt} &= (1-p)\rho - \frac{\beta\mu}{\rho}S(t)I(t) - \mu S(t) + \gamma I(t) + p\rho\frac{\delta}{\delta+\mu}\\
&= \left[(1-p)\rho + p\rho\frac{\delta}{\delta+\mu}\right] - \frac{\beta\mu}{\rho}S(t)I(t) - \mu S(t) + \gamma I(t)\\
&= \alpha - \frac{\beta\mu}{\rho}S(t)I(t) - \mu S(t) + \gamma I(t)\\
\frac{dI(t)}{dt} &= \frac{\beta\mu}{\rho}S(t)I(t) - (\mu+\gamma)I(t)
\end{aligned}\right\} \tag{2.11}$$

where

$$\begin{aligned}
\alpha &= (1-p)\rho + p\rho\frac{\delta}{\delta+\mu}\\
&= \rho - p\rho(1-\frac{\delta}{\delta+\mu})\\
&= \rho - p\rho\frac{\mu}{\delta+\mu}\\
&= \rho\left(1-\frac{p\mu}{\delta+\mu}\right)
\end{aligned}$$

and note that $\alpha > 0$ due to the facts that $0 < p < 1$ and $\mu > 0$, $\delta > 0$.

3. EQUILIBRIA AND LOCAL STABILITY

In modelling the dynamics of epidemic systems, the concept a *basic reproduction number* is fundamental. There are many discussions about this number in the literature and we refer to Diekmann and Heestterbeek [4], Heffernen et al [7], Van den Driessche and Wartmough [14]. In Epidemiological models, the basic reproduction number denoted by $\mathbb{R}_o$ denotes the number of susceptible individuals infected by a single infected individual during the infectious duration of the individual in a population which consists of susceptibles. It is clear from this definition that when $\mathbb{R}_o < 1$, each infectious individual produces on average less than one infected individual and hence it will follow that the infectious disease will eventually disappear; if $\mathbb{R}_o > 1$, the infection will spread through the susceptible population and establish itself. Such a threshold nature of $\mathbb{R}_o$ is a very meaningful concept in the dynamical nature of epidemic models. $\mathbb{R}_o$ is usually dependent on the various parameters of the epidemic

model and such a dependence can be exploited to assess the benefits of plausible control strategies such as vaccination.

In our model (2.11), in the absence of any infectious individual, the susceptible population is governed by

$$\frac{dS}{dt} = \alpha - \mu S, \qquad t > 0 \tag{3.1}$$

and hence $S(t) \to \frac{\alpha}{\mu}$ which implies that the susceptible population will reach the size $\frac{\alpha}{\mu}$ and stay at this level in the absence of any infectious individuals. In such a population, infection spread is governed by

$$\left.\begin{aligned} \frac{dI}{dt} &= \left[\frac{\beta\mu}{\rho}\frac{\alpha}{\mu} - (\mu+\gamma)\right] I \\ &= (\mu+\gamma)\left[\frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right) - 1\right] I \end{aligned}\right\}, \qquad t > 0 \tag{3.2}$$

from which it follows that

$$\frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right) > 1 \Longrightarrow \frac{dI}{dt} > 0$$

$$\frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right) < 1 \Longrightarrow \frac{dI}{dt} < 0.$$

Thus in our analysis in the following we choose the basic reproduction number $\mathbb{R}_o$ for the model (1.1) to be given by

$$\mathbb{R}_o = \frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right). \tag{3.3}$$

There are several different ways of the derivation of an expression for the basic reproduction number (van den Driessche and Watmough [14], Diekmann and Heesterbeek [4]). We also note that $\mathbb{R}_o$ from (3.3) can be expressed in the following form:

$$\mathbb{R}_o = \left(\frac{\beta\mu}{\rho}\right)\left(\frac{1}{\mu+\gamma}\right)\left[\frac{\rho}{\mu}\left(1 - \frac{p\mu}{\delta+\mu}\right)\right]$$

whose right side is the product of *contact rate, average duration of infection and final size of the susceptibles under no infectives*. One can also derive $\mathbb{R}_o$ as the spectral radius of the linear variational matrix from (2.11) corresponding to a disease free steady state.

The equilibria of (2.5) are the non-negative solutions (x, y) of the system of equations

$$\left.\begin{aligned} \left[(1-p)\rho + \frac{p\rho\delta}{\delta+\mu}\right] - \mu x - \frac{\mu}{\rho}\beta xy + \gamma y = 0 \\ y\left[\frac{\beta\mu}{\rho}x - (\gamma+\mu)\right] = 0 \end{aligned}\right\}. \tag{3.4}$$

One of the solutions of (3.4) is given by E_o where

$$E_o = (x_o^*, y_o^*) = \left(\frac{\rho}{\mu}\left[1 - \frac{p\mu}{\delta+\mu}\right], 0\right) \tag{3.5}$$

which is a disease free equilibrium and the endemic equilibrium is given by E_1 given by

$$E_1 = (x_1^*, y_1^*) = \left.\begin{aligned} &\left(\frac{\rho}{\mu}\left[\frac{\mu+\gamma}{\beta}\right], \frac{\rho}{\mu}\left[1 - \frac{p\mu}{\delta+\mu} - \frac{\mu+\gamma}{\beta}\right]\right) \\ &= \left(\frac{\rho}{\mu}\left[\frac{\mu+\gamma}{\beta}\right], \frac{\rho}{\mu}\left[\frac{\mu+\gamma}{\beta}\left[R_o - 1\right]\right]\right) \end{aligned}\right\}. \tag{3.6}$$

We note that when $\mathbb{R}_o \leq 1$, (2.11) has only the disease free equilibrium while when $\mathbb{R}_o > 1$, the system (2.11) has two non-negative equilibria corresponding to both disease free equilibrium and the endemic equilibrium respectively.

To determine the local stability of the equilibria, we consider the linear variational system corresponding to (2.5) given by

$$\begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \end{bmatrix} = [J] \begin{bmatrix} X \\ Y \end{bmatrix}$$

where

$$[J] = \begin{bmatrix} -\frac{\beta\mu}{\rho}Y - \mu & -\frac{\beta\mu}{\rho}X + \gamma \\ \frac{\beta\mu}{\rho}Y & \frac{\beta\mu}{\rho}X - (\mu+\gamma) \end{bmatrix}. \tag{3.7}$$

The eigenvalues of the linear variational system corresponding to the equilibrium E_o are the roots of the characteristic equation

$$J|_{E_o} = det \begin{bmatrix} -\mu - \lambda & \gamma - \frac{\beta\mu}{\rho}\frac{\rho}{\mu}\left(1 - \frac{p\mu}{\delta+\mu}\right) \\ 0 & \frac{\beta\mu}{\rho}\frac{\rho}{\mu}\left(1 - \frac{p\mu}{\delta+\mu}\right) - (\mu+\gamma) - \lambda \end{bmatrix}$$

which is equivalent to the quadratic equation

$$\lambda^2 + \lambda\left[\mu - (\mu+\gamma)(\mathbb{R}_o - 1)\right] - \mu(\mu+\gamma)(\mathbb{R}_o - 1) = 0. \tag{3.8}$$

When $\mathbb{R}_o < 1$, the roots of (3.8) have negative real parts and when $\mathbb{R}_o > 1$, one of the roots of (3.8) is positive while the other is negative. It will follow from this that the disease free equilibrium E_o is locally asymptotically stable when $\mathbb{R}_o < 1$ and unstable when $\mathbb{R}_o > 1$.

The variational matrix corresponding to the endemic equilibrium E_1 is given by

$$J|_{E_1} = det \begin{bmatrix} -\frac{\beta\mu}{\rho}y_1^* - \mu & -\frac{\beta\mu}{\rho}x_1^* + \gamma \\ \frac{\beta\mu}{\rho}y_1^* & \frac{\beta\mu}{\rho}x_1^* - (\mu+\gamma) \end{bmatrix} = \begin{bmatrix} -\frac{\beta\mu}{\rho}y_1^* - \mu & -\mu \\ \frac{\beta\mu}{\rho}y_1^* & 0 \end{bmatrix} \tag{3.9}$$

whose eigenvalues are the roots of the quadratic equation

$$\lambda^2 + \lambda\left[\mu + \frac{\beta\mu}{\rho}y_1^*\right] + \frac{\beta\mu}{\rho}y_1^*\mu = 0 \tag{3.10}$$

and the roots of this equation have negative real parts when $y_1^* > 0$ and this is the case when $\mathbb{R}_o > 1$. Thus $\mathbb{R}_o > 1$ implies that the endemic equilibrium is locally asymptotically stable.

4. GLOBAL STABILITY OF EQUILIBRIA

We consider the two dimensional autonomous system

$$\left.\begin{aligned}\frac{dS}{dt} &= \alpha - \mu S - \frac{\beta\mu}{\rho}SI + \gamma I \\ \frac{dI}{dt} &= I\left[\frac{\beta\mu}{\rho}S - (\gamma+\mu)\right]\end{aligned}\right\}, \qquad t>0 \tag{4.1}$$

and proceed to establish sufficient conditions for the global asymptotic stability of both disease free and endemic equilibria of (4.1). By global asymptotic stability we mean global attractivity of the equilibria augmented with their local asymptotic stability. We first convert the system (4.1) into an asymptotically non-autonomous differential algebraic system. For this purpose we let

$$X(t) = S(t) + I(t), \qquad t \geq 0. \tag{4.2}$$

It will then follow from (4.1) and (4.2) that

$$\frac{dX}{dt} = \frac{dS}{dt} + \frac{dI}{dt} = \alpha - \mu X, \qquad t>0 \tag{4.3}$$

and hence we have

$$X(t) = X(0)e^{-\mu t} + \frac{\alpha}{\mu}(1-e^{-\mu t}), \qquad t>0 \tag{4.4}$$

If we let

$$S(t) = X(t) - I(t), \qquad t \geq 0 \tag{4.5}$$

then the autonomous differential system (4.1) is equivalent to the asymptotically non-autonomous differential algebraic system

$$\left.\begin{aligned}\frac{dI(t)}{dt} &= I(t)\left[\frac{\beta\mu}{\rho}S(t) - (\gamma+\mu)\right] \\ S(t) &= X(0)e^{-\mu t} + \frac{\alpha}{\mu}(1-e^{-\mu t}) - I(t)\end{aligned}\right\}, \qquad t>0. \tag{4.6}$$

whose limit system is given by

$$\left.\begin{aligned}\frac{dI(t)}{dt} &= I(t)\left[\frac{\beta\mu}{\rho}\left(\frac{\alpha}{\mu} - I(t)\right) - (\gamma+\mu)\right] \\ S(t) &= \frac{\alpha}{\mu} - I(t)\end{aligned}\right\}, \quad t>0. \tag{4.7}$$

The system (4.7) is uncoupled and we can rewrite the first of (4.7) in the following form:

$$\begin{aligned}\frac{dI(t)}{dt} &= I(t)\left[\frac{\beta\mu}{\rho}\frac{\alpha}{\mu} - (\gamma+\mu) - \frac{\beta\mu}{\rho}I(t)\right] \\ &= \frac{\beta\mu}{\rho}I(t)\left[\frac{\rho}{\mu}\left\{(1 - \frac{p\mu}{\delta+\mu} - \left(\frac{\gamma+\mu}{\beta}\right)\right\} - I(t)\right] \\ &= \frac{\beta\mu}{\rho}I(t)\left[\frac{\rho}{\mu}\left(\frac{\mu+\gamma}{\beta}\right)\left\{\frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right) - 1\right\} - I(t)\right] \\ &= \frac{\beta\mu}{\rho}I(t)\left[\frac{\rho}{\mu}\left(\frac{\mu+\gamma}{\beta}\right)\{\mathbb{R}_o - 1\} - I(t)\right], \quad t>0. \end{aligned} \tag{4.8}$$

Equation (4.8) is a well known logistic equation which has been a prototype for the development of numerous models in population dynamics and has one equilibrium I_o^* which is the trivial one when $\mathbb{R}_o \leq 1$. When $\mathbb{R}_o > 1$, the equation (4.8) has a positive equilibrium I_1^* given by

$$I_1^* = \frac{\rho}{\mu}\left(\frac{\mu+\gamma}{\beta}\right)(\mathbb{R}_o - 1).$$

It is not difficult to solve (4.8) explicitly by separation of variables or by a simple change of variable leading to a linear equation in the new variable. From the properties of the logistic equation one can easily derive the following:

$$\left.\begin{array}{lll} \mathbb{R}_o \leq 1, & I(0) > 0 \Longrightarrow I(t) \to 0 \quad \text{as} \quad t \to \infty & \\ \mathbb{R}_o > 1, & I(0) > 0 \Longrightarrow I(t) \to \dfrac{\rho}{\mu}\left(\dfrac{\gamma+\mu}{\beta}\right)(\mathbb{R}_o - 1) \quad \text{as} \quad t \to \infty & \end{array}\right\}. \tag{4.9}$$

We have from the above and $S(t) = X(t) - I(t)$ that

$$\left.\begin{array}{l} \mathbb{R}_o \leq 1 \Longrightarrow S(t) \to \dfrac{\alpha}{\mu} \quad \text{as} \quad t \to \infty \\ \mathbb{R}_o > 1 \Longrightarrow S(t) \to \dfrac{\alpha}{\mu} - \dfrac{\rho}{\mu}\dfrac{\gamma+\mu}{\beta}(\mathbb{R}_o - 1) \\ \qquad = \dfrac{\rho}{\mu}\left(1 - \dfrac{p\mu}{\delta+\mu}\right) - \dfrac{\rho}{\mu}\left(1 - \dfrac{p\mu}{\delta+\mu}\right) + \dfrac{\rho}{\mu}\left(\dfrac{\gamma+\mu}{\beta}\right) \\ \qquad = \dfrac{\rho}{\mu}\left(\dfrac{\gamma+\mu}{\beta}\right) \end{array}\right\}. \tag{4.10}$$

We are now ready to summarize our analysis of the system (1.1) in the form of the following:

Theorem 4.1. *Consider the system of differential equations* (1.1) *as defined in the above and define the basic reproduction number* $\mathbb{R}_o$ *as follows:*

$$\mathbb{R}_o = \frac{\beta}{\mu+\gamma}\left(1 - \frac{p\mu}{\delta+\mu}\right).$$

If $\mathbb{R}_o < 1$ *then the system* (1.1) *has the disease free equilibrium* $(S_o^*, I_o^*, V_o^*, N_o^*)$ *where*

$$\begin{aligned} S_o^* &= \frac{\rho}{\mu}\left(1 - \frac{p\mu}{\delta+\mu}\right) \\ I_o^* &= 0 \\ V_o^* &= \frac{p\rho}{\delta+\mu} \\ N_o^* &= \frac{\rho}{\mu}. \end{aligned}$$

If $\mathbb{R}_o > 1$, *then the system* (1.1) *has in addition to the disease free equilibrium an endemic equilibrium* $(S_1^*, I_1^*, V_1^*, N_1^*)$ *where*

$$\begin{aligned} S_1^* &= \frac{\rho}{\mu}\left(\frac{\gamma+\mu}{\beta}\right) \\ I_1^* &= \frac{\rho}{\mu}\left(\frac{\mu+\gamma}{\beta}\right)(\mathbb{R}_o - 1) \\ V_1^* &= \frac{p\rho}{\delta+\mu} \\ N_1^* &= \frac{\rho}{\mu} \end{aligned}$$

Furthermore if $\mathbb{R}_o < 1$, *then the disease free equilibrium is globally asymptotically stable and if* $\mathbb{R}_o > 1$ *then the disease free equilibrium is unstable while the endemic equilibrium is globally asymptotically stable.*

Proof. Details of proof will follow from the above analysis together with (4.8), (4.9) and (4.10) and we omit the arguments to avoid repetition.

We conclude with the remark that the model (1.1) can be further generalized to include more general contact rate functions, time delays, age dependence and multiple strains of the disease. If we want to eradicate the disease, we have to organize a control mechanism so as to reduce the basic reproduction rate to a value less than one. We note from our analysis, that an increase in the value of δ can result in a decrease in the value of the basic reproduction rate. Thus an increase in the vaccination rate has the potential to reduce the value of the basic reproduction rate and consequently the disease can eventually disappear. We have established a threshold behaviour of the model in the sense that $\mathbb{R}_o < 1$ implies the global stability of the disease free equilibrium while $\mathbb{R}_o > 1$ leads to the existence of an endemic equilibrium and its global stability. The results are illustrated by the examples below.

5. **EXAMPLES**

The subject of mathematical epidemiology is concerned with the formulation and analysis of mathematical models to describe the spreading and control of infectious diseases. Such an analysis can be useful for the development of health policies by national and international organizations such as WHO so as to control or eradicate infectious diseases. Mathematical analysis of epidemic models predominantly deal with the stability of disease free equilibria corresponding to the absence of infectious individuals in a population of susceptible to a disease and endemic equilibria where the disease has established itself in the population. Some models include vaccination strategies which when implemented can lead to the stability of a disease free equilibrium corresponding to a possible elimination of a disease.

One of the epidemic models of infectious diseases is the transmission of gonorrhea among a homosexual population. Gonorrhea is one of the sexually transmitted diseases caused by 'Neisseria Gonorrhoea', a bacterium that can grow and multiply in mucus membranes of the human body. This disease is infectious and if left untreated can lead to blindness, sterility, heart failure and even death. In males, this disease causes itching during urination and some discharge from the penis while in

females the disease is asymptomatic and males appear to seek treatment more than the infected females. Gonorrhea has a short period of incubation and does not confer immunity to those who have recovered from treatment by antibiotics. One can refer to Heathcote and Yorke [8] for more details regarding gonorrhea transmission models. By using the theory of asymptotically autonomous differential equations we establish a threshold nature, which has been discussed in Section 2 and 3, of the spread of the disease.

We consider the system of ordinary differential equations (2.1) and (2.2) in the form of an SIV epidemic model (susceptible-infected-vaccinated); note that in the case of gonorrhea, infected individuals treated with antibiotics get temporary protection from the disease and become susceptible again. Our model is based on the following system of differential algebraic system proposed by Lima and Torres [9].

As the first example, we consider the system (2.1) and (2.2) with the following sets of parameters in which the following notation is used:

$S(t)$: susceptible population of homosexuals at time t
$I(t)$: infected population of infectious homosexuals at time t
$V(t)$: those infected and treated at time t
$N(t)$: the total of the above three compartments; i.e. the size of the homosexuals
ρ: a positive constant denoting the recruitment rate of homosexual population
p: proportion of homosexuals successfully vaccinated at recruitment $(0 < p < 1)$
μ: natural death rate of the population
γ: recovery rate of the infected and becoming susceptible
δ: vaccination rate and becoming susceptible
β: incidence rate of the infection
where the corresponding parameters are given by

$$\gamma = 0.006, \qquad \delta = 0.001, \qquad \beta = 0.10, \qquad \mu = 0.05, \qquad p = 0.005, \qquad \rho = 0.30$$

with initial values

$$S(0) = 1, \qquad I(0) = 2, \qquad V(0) = 3, \qquad N(0) = 4$$

For this case one can compute the following:

$$\mathbb{R}_0 = 1.7770, \quad S_1^* = 3.3600, \quad I_1^* = 2.6106, \quad V_1^* = 0.0294, \quad N_1^* = 6.0000\,.$$

Our numerical simulation has provided the following final values at the end our simulation:

$$S^* = 3.3600, \qquad I^* = 2.6106, \qquad V^* = 0.0294, \qquad N^* = 6.0000.$$

A Runge-Kutta method of order 4 has been implemented in Matlab and the numerical solution is graphically displayed in Figure 1 below.

In the second example we consider the global stability of the disease free equilibrium case and use the following parameter values:

$$\gamma = 0.006, \quad \delta = 0.001, \quad \beta = 0.100, \quad \mu = 0.50, \quad p = 0.100, \quad \rho = 0.300$$

with the initial value vector given by [0.1, 0.2, 0.3, 0.4]. For this case, one can calculate the following:

$$\mathbb{R}_0 = 0.1779, \quad S_o = 0.5401, \quad I_o = 0, \quad V_o = 0.0599, \quad N_o = 0.6000\,.$$

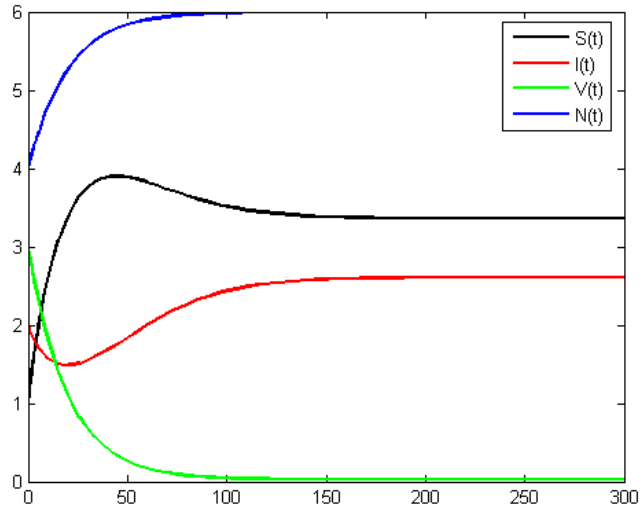

FIGURE 1. Stability of the endemic Equilibrium

The result of the Matlab simulation of this case is displayed in the Figure 2 below. For the purpose of completeness, we also show two more examples for $R_0 > 3$ (Figure 3) and $R_0 < 0.1$ (Figure 4).

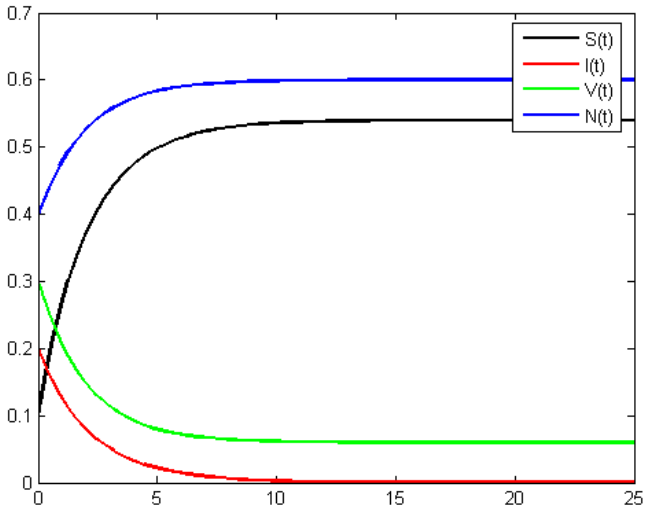

FIGURE 2. Stability of the Disease Free Equilibrium

In summary, we tabulate (Table 1) the results of four situations (figures 1 to 4) with various initial conditions, where the value of $\mathbb{R}_0$ ranges from $\mathbb{R}_0 < 1$ to $\mathbb{R}_0 > 1$.

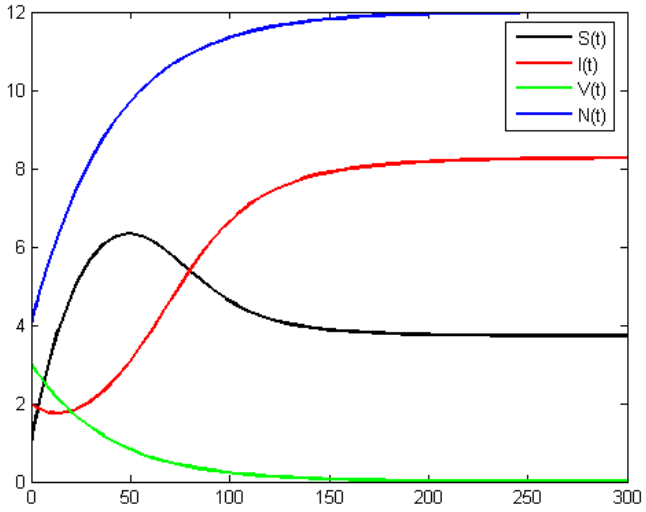

FIGURE 3. Stability of the endemic Equilibrium

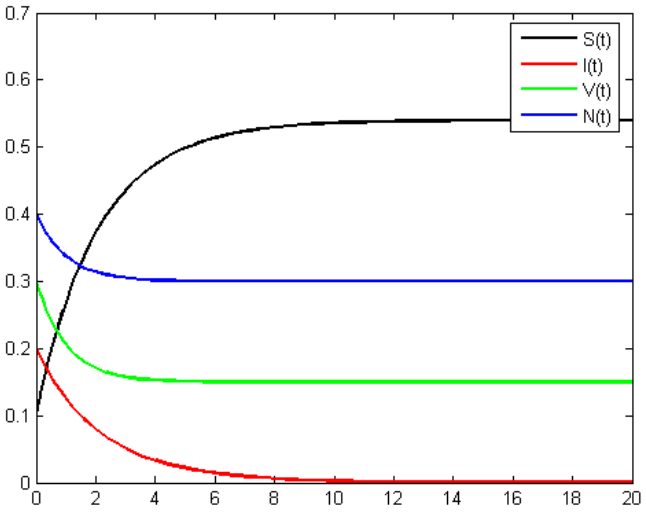

FIGURE 4. Stability of the Disease Free Equilibrium

Figure	γ	δ	β	μ	p	ρ	$\mathbb{R}_0$	S(0)	I(0)	V(0)	N(0)
3	0.006	0.001	0.1	0.025	0.0005	0.3	3.2243	1	2	3	4
1	0.006	0.001	0.1	0.05	0.005	0.3	1.7770	1	2	3	4
2	0.006	0.001	0.1	0.5	0.1	0.3	0.1779	0.1	0.2	0.3	0.4
4	0.006	0.001	0.1	1	0.5	0.3	0.0450	0.1	0.2	0.3	0.4

TABLE 1. Summary of Nonlinear Gonorrhea Models for various values of $\mathbb{R}_0$

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