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GLOBAL STABILITY OF TWO-SCALE NETWORK HUMAN EPIDEMIC DYNAMIC MODEL

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ABSTRACT. The recent high rate of globalization of new disease strains and infectious agents at non-endemic zones is closely associated with complex human population structure and the large-scale inter-patch connections human transportations. The complexities in the human population structure create heterogeneities with respect to patch dwelling populations as well as endemic population structure.We present a multi-group SIRS dynamic epidemic process in the context of scale structured population. For simplicity we limit the scale to the value of two and define a multi-scale extension algorithm for the epidemic process. We investigate the global uniform asymptotic stability of the disease free equilibrium of the scale structured epidemic dynamic process and its impact on the emergence, propagation and resurgence of the disease. The presented results are demonstrated by numerical simulation results.

Key Words and Phrases: Disease-free steady state, Global asymptotic stability, Threshold value, Positively invariant set, Lypunov function, Lypunov direct method

1. INTRODUCTION

The rapid global dispersion of new disease strains and infectious agents of the human speicies from disease outbreak zones is closely interrelated with the recent advent of high technology in the areas such as communication and transportation [4]. For instance, the 2009 H1N1 flu pandemic [23] is a result of the many inter-patch connections facilitated human transportation of disease. A significant number of mathematical models describing the dynamics of infectious diseases of humans have been studied. Models describing the dynamics of insect vector born diseases [10, 36], influenza [5], HIV [32, 33, 35] and AIDS [34] are studied.

There are studies [5, 6, 8, 9, 10, 15, 16, 11, 12, 26, 37, 38, 14] describing the dynamics of human mobility and disease in meta-populations. Generally, these models can be called multi-group models as they describe the dynamics of diseases in a network of the patches of a meta-population. These models can be further categorized into two general classes based on the modeling approach, namely: Langrangian [37, 38, 14, 10, 15, 16] and Eulerian [11, 12, 7, 8, 9, 5, 6] models. In addition, individuals in the population based on their residence and also their current location. In Langrangian models, individuals do not change their residence, but are allowed to visit other patches in the meta-population. The Eulerian models on the other hand label individuals in the population based only on the current location. Moreover, this

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model can be considered to be migration models because only the present location of individuals is important.

Many authors have investigated the dynamics of diseases described with SIRS models. A significant portion of SIRS models study the dynamics of the disease under variant incident rates [24, 25, 26, 27, 28, 29, 30]. Using Lypunov functions, the local nonlinear and global stability of the equilibria is established [24]. By constructing a Lypunov function based on the structure of the biological system [27, 17, 18], the existence, uniqueness and global stability of the endemic equilibrium are investigated. Furthermore, the bifurcation and stability analysis of the disease free and endemic equilibria are investigated in [26, 29, 30]. SIRS epidemic models have also been described and studied using complex network of human contacts [31]. In [42], a special SIRS epidemic model is formulated with a proportional direct transfer from the infectious state to the susceptible state immediately after the infectious period.

In more complex meta-population structures, the understanding of the dynamics of infectious diseases is still in the infancy level. This is due to the high degree of heterogeneities and complexity of spatial human population structures. Recently, Wanduku and Ladde [1] characterized the dynamics of human mobility process in scale structured meta-populations. They formulated a Langrangian type dynamic model for the human mobility process.

In this paper we incorporate the human mobility process and the resulting heterogeneities described in [1] into a human epidemic model. We extend and expand the single-scale network SIR epidemic model [14], into a two-scale network SIRS human epidemic model for a metapopulation. The SIRS epidemic process is motivated by the work [42]. The presented model of two-scale network human mobility process is described by a large-scale system of ordinary differential equations.

The work is organized as follows. In Section 2 we describe the general SIRS epidemic process under the influence of mobility process [1]. In Section 3, the model validation is exhibited. The existence and asymptotic stability of the disease free equilibrium is shown in Section 4. We present simulation results in Section 5. Finally a few conclusions are drawn in Section 6.

2. LARGE SCALE TWO LEVEL SIRS EPIDEMIC PROCESS

We assume all assumptions and notations defined in [1] and make a few definitions, subsequently.

Definition 2.1 (Endemic population decomposition and Aggregation). For each $r \in I(1, M)$, let $i \in I^r_i(1, n_r)$. The total population N^{rr}_{i0} of residents of site s^r_i at time t is distributed among the sites in their intra and inter regional domain $C(s^r_i)$, and it is partitioned into three general disease compartments namely, susceptible (S), infectious (I) and removals (R) (those who were previously sick and have acquire permanent immunity from the disease). That is, A^{rq}_{il} is the number of residents of site s^r_i whose disease status is of type $A, A \in \{S, I, R\}$, and are visiting to site $s^q_l, l \in I^q_i(1, n_q)$ in region C_q, where $q \in I^r(1, M)$. Furthermore, when $r = q$, A^{rr}_{ik} is the number of residents of site s^r_i with disease status $A \in \{S, I, R\}$, and are visiting to site $s^r_k, k \in I^r_i(1, n_r)$ in their home region C_r. Moreover, when $k = i$, A^{rr}_{ii} is the number of residents of site s_i^r who have disease status of type $A, A \in \{S, I, R\}$ and remain as permanent residents at their home site. Hence N_i^{rr} is given by

$$N_{i0}^{rr} = S_{i0}^{rr} + I_{i0}^{rr} + R_{i0}^{rr}, \tag{2.1}$$

where

$$S_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} S_{ik}^{rq}, \quad I_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} I_{ik}^{rq}, \quad \text{and} \quad R_{i0}^{rr} = \sum_{q=1}^{M}\sum_{k=1}^{n_q} R_{ik}^{rq}. \tag{2.2}$$

Remark 2.1. We note that the effective population $eff(N_{i0}^{rr})$ present at the site s_i^r at anytime is different from the census population or the total number of residents N_{i0}^{rr} (2.1) with permanent residence site s_i^r. At anytime t, the effective community size of site s_i^r is made up of the permanent residents of site s_i^r and all visitors of to site s_i^r. This is as given below

$$eff(N_i^{rr}) = \sum_{q=1}^{M}\sum_{k=1}^{n_q} S_{ki}^{qr} + \sum_{q=1}^{M}\sum_{k=1}^{n_q} I_{ki}^{qr} + \sum_{q=1}^{M}\sum_{k=1}^{n_q} R_{ki}^{qr}. \tag{2.3}$$

$eff(N_i^{rr})$ represents the population that is at risk for infection at site s_i^r and it is the population size resulted by the mobility process in the two-scale network structure.

Definition 2.2 (Disease Transmission Process). The disease transmission process in any site s_i^r in region C_r in a mobile population necessitates: (1) a susceptible person to travel from site s_k^u in region C_u to site s_i^r, ($u = r$ and $k = i$ if there is no traveling), (2) an infectious person traveling from site s_l^q in region $C_q, q \neq r$ to site s_i^r, (3) the susceptible and infectious persons meeting at a contact zone z (which may be the home, market place or recreational facility etc) in site s_i^r with a probability p of a person being at a zone z at anytime t, and (4) β is the probability of the infectious agent being transmitted from the infectious person to the susceptible person knowing that the contact between the susceptible and the infectious individual took place.

Let n_{ri} be the number of contact zones denoted by $z_{i_b}^r, b \in \{1, 2, \ldots, n_{r_i}\} \equiv I(1, n_{r_i})$ at each site s_i^r.Furthermore, let $p_{i_b}^r$ be the probability that a member of the effective population would be in a zone $z_{i_b}^r$ at a time t; in addition, we assume that the events of visiting contact zones are independent, and the probability $p_{i_b}^r$ of being in a given zone $z_{i_b}^r$ is independent of the permanent residence of the individual. In each zone $z_{i_b}^r$, there is random mixing and transmission of the infectious agent from an infectious person to a susceptible person via a direct contact between the two individuals. Moreover, let β_{ikj}^{ruv*} be the probability that the transmission takes place given that the contact occurs in any zone $z_{i_b}^r, \forall b \in I(1, n_{r_i})$ in site s_i^r between a susceptible S_{ki}^{ur} from site s_k^u in region C_u and an infectious individual I_{mi}^{vr} from site s_m^v in region C_v. Then the infectious rate (average number of contacts per individual per unit time required to transmit the disease), $\beta_{i_bkm}^{ruv*}$, in zone $z_{i_b}^r$ between S_{ki}^{ur} and I_{mi}^{vr} is given by

$$\beta_{i_bkm}^{ruv*} = (p_{i_b}^r)^2 \beta_{ikm}^{ruv*}, \tag{2.4}$$

whenever $v, u \in I(1, M)$, and $v \neq u$. The infection process in zone $z_{i_b}^r$ is illustrated by the following transition.

$$S_{ki}^{ur} + I_{mi}^{vr} \xrightarrow{\beta_{i_bkm}^{ruv*}} I_{ki}^{ur} + I_{mi}^{vr}. \tag{2.5}$$

Hence, the net transmission rate of the infection process at the site s_i^r in region C_r of the meta-population with M regions is given by

$$\sum_{v=1}^{M}\sum_{u=1}^{M}\sum_{m=1}^{n_v}\sum_{k=1}^{n_u}\sum_{b=1}^{n_{r_i}}\beta_{i_b km}^{ruv*} I_{mi}^{vr} S_{ki}^{ur} \tag{2.6}$$

We set

$$\beta_{ikm}^{ruv} = \sum_{b=1}^{n_{r_i}}\beta_{i_b km}^{ruv*} \tag{2.7}$$

We further assume that the disease status of an individual in the population does not affect travel rates and the mobility pattern.

A diagram illustrating the disease transmission and mobility processes in the two scale dynamic structure described in Definition 2.2 is exhibited in Figure 1.

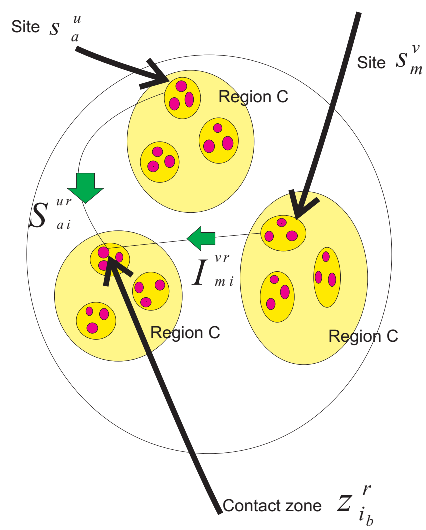

FIGURE 1. Shows the movement of susceptible (S_{ai}^{ur}) and infective (I_{mi}^{vr}) from arbitrary home site s_a^u in region C_u and from site s_m^v in region C_v, to visit an arbitrary contact zone $z_{i_b}^r$ in site s_i^r, which is in region C_r. Disease transmission takes place in zone $z_{i_b}^r$.

Definition 2.3 (Acquisition and Loss of Permanent Immunity Process). In each site s_i^r, let $\frac{1}{\varrho_i^r}$ be the average active infectious period of infected individual (I) who recovered from the disease and acquired permanent immunity (R),immediately after the infectious period. Also, let $\frac{1}{\eta_i^r}$ be the average infectious period of infected person in site s_i^r, who is recovered from the disease and become susceptible (S), immediately, after the infectious period. Furthermore, let $\frac{1}{\alpha_i^r}$ be the average immunity period of removal person (R) in site s_i^r, who has lost his/her their immunity and become susceptible (S) again immediately after the immunity period. The recovery process of an infected person in site s_i^r as well as the loss of immunity of a removal person is illustrated in the following disease transition processes:

$$I_{ki}^{ur} \xrightarrow{\varrho_i^r} R_{ki}^{ur}, \quad I_{ki}^{ur} \xrightarrow{\eta_i^r} S_{ki}^{ur}, \quad R_{ki}^{ur} \xrightarrow{\alpha_i^r} S_{ki}^{ur}, \tag{2.8}$$

for $u \in I(1, M)$ and $k \in I(1, n_u)$.

Definition 2.4 (Population Demography). Let B_i^r be a constant birthrate of the human population at site s_i^r and at time t. We assume that every new born is a susceptible and becomes a resident of the site of birth. Let δ_i^r be the per capita natural mortality rate, and let d_i^r be the per capita disease related mortality rate of all members of the effective population at site s_i^r.

A compartmental framework illustrating the different process and stages in the SIRS epidemic described above is exhibited in Figure 2. From Definition 2.1, the

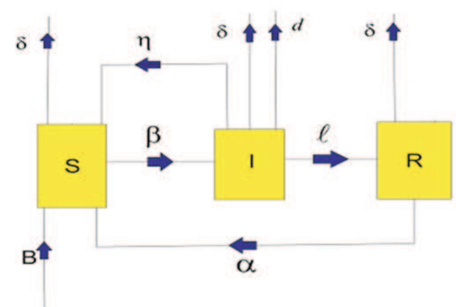

FIGURE 2. Compartmental framework summarizing the transition stages in the SIRS epidemic process. All the parameters presented in this figure are define in Section 2 for particular sites and regions.

derivation of the rate of change in the number of individuals of type A_{il}^{rq}, $A \in \{S, I, R\}$, under the mobility process [1] are described by the following:

$$
\begin{aligned}
\frac{dA_{il}^{rq}}{dt} = & [\texttt{no. of new arrivals of residents of site } s_i^r\texttt{, at site } s_l^q \\
& \quad \texttt{in region } C_q] \\
& + [\texttt{no. of births at site } s_l^q \texttt{ and no. of new conversions to} \\
& \quad \texttt{disease class } A_{il}^{rq}] \\
& - [\texttt{no. of residents returning home at site } s_i^r] \\
& - [\texttt{no. of deaths and conversions from disease class } A_{il}^{rq}] \\
& \pm [\texttt{new transmissions}].
\end{aligned} \tag{2.9}
$$

where $r, q \in I(1, M)$, $i \in I(1, n_r)$ and $l \in I_i^r(1, n_q)$. The complete SIRS epidemic model under the influence of a large scale two-level population mobility process [1] is described by:

(2.10)

$$\frac{dS^{rq}_{il}}{dt}=\begin{cases}[B^r_i+\sum_{k=1}^{n_r}\rho^{rr}_{ik}S^{rr}_{ik}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho^{rq}_{ia}S^{rq}_{ia}+\eta^r_iI^{rr}_{ii}\\ +\alpha^r_iR^{rr}_{ii}-(\gamma^r_i+\sigma^r_i+\delta^r_i)S^{rr}_{ii}-\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{iia}S^{rr}_{ii}I^{ur}_{ai}], & \text{for } q=r, l=i\\ [\sigma^{rr}_{ij}S^{rr}_{ii}+\eta^r_jI^{rr}_{ij}+\alpha^r_jR^{rr}_{ij}-(\rho^{rr}_{ij}+\delta^r_j)S^{rr}_{ij}\\ -\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{jia}S^{rr}_{ij}I^{ur}_{aj}], & \text{for } q=r, l=j, j\neq i,\\ [\gamma^{rq}_{il}S^{rr}_{ii}+\eta^q_lI^{rq}_{il}+\alpha^q_lR^{rq}_{il}-(\rho^{rq}_{il}+\delta^q_l)S^{rq}_{il}\\ -\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{qru}_{lia}S^{rq}_{il}I^{uq}_{al}], & \text{for } q\neq r,\end{cases}$$

(2.11)

$$\frac{dI^{rq}_{il}}{dt}=\begin{cases}[\sum_{k=1}^{n_r}\rho^{rr}_{ik}I^{rr}_{ik}+\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho^{rq}_{ia}I^{rq}_{ia}-\eta^r_iI^{rr}_{ii}-\varrho^r_iI^{rr}_{ii}\\ -(\gamma^r_i+\sigma^r_i+\delta^r_i+d^r_i)I^{rr}_{ii}+\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{iia}S^{rr}_{ii}I^{ur}_{ai}], & \text{for } q=r, l=i\\ [\sigma^{rr}_{ij}I^{rr}_{ii}-\eta^r_jI^{rr}_{ij}-\varrho^r_jI^{rr}_{ij}-(\rho^{rr}_{ij}+\delta^r_j+d^r_j)I^{rr}_{ij}\\ +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{jia}S^{rr}_{ij}I^{ur}_{aj}], & \text{for } q=r, l=j, i\neq j,\\ [\gamma^{rq}_{il}I^{rr}_{ii}-\eta^q_lI^{rq}_{il}-\varrho^q_lI^{rq}_{il}-(\rho^{rq}_{il}+\delta^q_l+d^q_l)I^{rq}_{il}\\ +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{qru}_{lia}S^{rq}_{il}I^{uq}_{al}], & \text{for } q\neq r,\end{cases}$$

$$\text{(2.12)}\quad \frac{dR^{rq}_{il}}{dt}=\begin{cases}[\sum_{k=1}^{n_r}\rho^{rr}_{ik}R^{rr}_{ik}+\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho^{rq}_{il}R^{rq}_{il}+\varrho^r_iI^{rr}_{ii}\\ \quad -(\gamma^r_i+\sigma^r_i+\alpha^r_i+\delta^r_i)R^{rr}_{ii}], & \text{for } q=r, l=i\\ [\sigma^{rr}_{ij}R^{rr}_{ii}+\varrho^r_jI^{rr}_{ij}-(\rho^{rr}_{ij}+\alpha^r_j+\delta^r_j)R^{rr}_{ij}], & \text{for } q=r, l=j, i\neq j,\\ [\gamma^{rq}_{il}R^{rr}_{ii}+\varrho^q_lI^{rq}_{il}-(\rho^{rq}_{il}+\alpha^q_l+\delta^q_l)R^{rq}_{il}], & \text{for } q\neq r,\end{cases}$$

where $i\in I(1,n_r), l\in I^r_i(1,n_q); r\in I(1,M), q\in I^r(1,M)$. Furthermore, the parameters B^r_i, η^u_a, α^u_a, δ^u_a and d^u_a are nonnegative, and ϱ^u_a is positive for $r,u\in I(1,M)$, $i\in I(1,n_r)$, and $a\in I(1,n_u)$. Also, at time $t=t_0$, and for each $r\in I(1,M)$, and $i\in I(1,n_r)$, $(S^{rr}_{ii}(t_0),S^{rr}_{ij}(t_0),S^{rq}_{il}(t_0))=(S^{rr}_{ii0},S^{rr}_{ij0},S^{rq}_{il0})$, $(I^{rr}_{ii}(t_0),I^{rr}_{ij}(t_0),I^{rq}_{il}(t_0))=(I^{rr}_{ii0},I^{rr}_{ij0},I^{rq}_{il0})$, $(R^{rr}_{ii}(t_0),R^{rr}_{ij}(t_0),R^{rq}_{il}(t_0))=(R^{rr}_{ii0},R^{rr}_{ij0},R^{rq}_{il0})$, whenever $j\in I^r_i(1,n_r)$ and $l\in I^r_i(1,n_q)$. Furthermore, we denote $n=\sum_{u=1}^{M}n_u$.

We express the state of system (2.10)–(2.12) in vector form and use it, subsequently. We denote

$$\begin{aligned}x^{ru}_{ia}&=(S^{ru}_{ia},I^{ru}_{ia},R^{ru}_{ia})^T\in\mathbb{R}^3\\ x^{ru}_{i0}&=(x^{ruT}_{i1},x^{ruT}_{i2},\ldots,x^{ruT}_{i,nu})^T\in\mathbb{R}^{3n_u},\\ x^{ru}_{00}&=(x^{ruT}_{10},x^{ruT}_{20},\ldots,x^{ruT}_{n_r0})^T\in\mathbb{R}^{3n_rn_u},\\ x^{r0}_{00}&=(x^{r1T}_{00},x^{r2T}_{00},\ldots,x^{rMT}_{00})^T\in\mathbb{R}^{3n_r\sum_{u=1}^{M}n_u},\\ x^{00}_{00}&=(x^{10}_{00},x^{20}_{00},\ldots,x^{M0}_{00})^T\in\mathbb{R}^{3\left(\sum_{r=1}^{M}n_r\right)\left(\sum_{u=1}^{M}n_u\right)},\end{aligned}\tag{2.13}$$

where $r,u\in I(1,M)$, $i\in I(1,n_r)$, $a\in I^r_i(1,n_u)$. We set $n=\sum_{u=1}^{M}n_u$.

Definition 2.5.

1. $\mathbf{p-norm\ in}\ \mathbb{R}^{3n^2}$: Let $z^{00}_{00}\in\mathbb{R}^{3n^2}$ be an arbitrary vector defined in (2.13), where $z^{ru}_{ia}=(z^{ru0}_{ia1},z^{ru0}_{ia2},z^{ru0}_{ia3})^T$ whenever $r,u\in I(1,M)$, $i\in I(1,n_r)$, $a\in I^r_i(1,n_u)$. The

$p-norm$ on $\mathbb{R}^{3n^2}$ is defined as follows

$$\|z_{00}^{00}\|_p = \left(\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\sum_{j=1}^{3}|z_{iaj}^{ru0}|^p\right)^{\frac{1}{p}} \tag{2.14}$$

whenever $1 \leq p < \infty$, and

$$\bar{z} \equiv \|z_{00}^{00}\|_p = \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u, 1\leq j\leq 3} |z_{iaj}^{ru0}|, \tag{2.15}$$

whenever $p = \infty$. Let

$$\underline{\mathrm{k}} \equiv k_{00min}^{00} = \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} |k_{ia}^{ru}|. \tag{2.16}$$

2. **Closed Ball in $\mathbb{R}^{3n^2}$**: Let $z_{00}^{*00} \in \mathbb{R}^{3n^2}$ be fixed. The closed ball in $\mathbb{R}^{3n^2}$ with center at z_{00}^{*00} and radius $r > 0$ denoted $\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z_{00}^{*00}; r)$ is the set

$$\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z_{00}^{*00}; r) = \{z_{00}^{00} \in \mathbb{R}^{3n^2} : \|z_{00}^{00} - z_{00}^{*00}\|_p \leq r\} \tag{2.17}$$

3. MODEL VALIDATION RESULTS

In this section, we show that the initial value problem associated with the system (2.10)–(2.12) has a unique solution. Furthermore, we also establish non-negativity of solution process. These results would validate the correctness of the model. From (2.10)–(2.12), define the vector $y_{00}^{00} \in \mathbb{R}^{n^2}$ as follows: For $i \in I(1, n_r), l \in I_i^r(1, n_q), r \in I(1, M)$ and $q \in I^r(1, M)$,

$$\begin{aligned}
y_{ia}^{ru} &= S_{ia}^{ru} + I_{ia}^{ru} + R_{ia}^{ru} \in \mathbb{R}_+ = [0, \infty) \\
y_{i0}^{ru} &= (y_{i1}^{ru}, y_{i2}^{ru}, \ldots, y_{i,nu}^{ru})^T \in \mathbb{R}_+^{n_u}, \\
y_{00}^{ru} &= (y_{10}^{ruT}, y_{20}^{ruT}, \ldots, y_{n_r0}^{ruT})^T \in \mathbb{R}_+^{n_r n_u}, \\
y_{00}^{r0} &= (y_{00}^{r1T}, y_{00}^{r2T}, \ldots, y_{00}^{rMT})^T \in \mathbb{R}_+^{n_r \sum_{u=1}^{M} n_u}, \\
y_{00}^{00} &= (y_{00}^{10T}, y_{00}^{20T}, \ldots, y_{00}^{M0T})^T \in \mathbb{R}_+^{\left(\sum_{r=1}^{M} n_r\right)\left(\sum_{u=1}^{M} n_u\right)},
\end{aligned} \tag{3.1}$$

and obtain

$$dy_{il}^{rq} = \begin{cases} \left[B_i^r + \sum_{k\neq i}^{n_r} \rho_{ik}^{rr} y_{ik}^{rr} + \sum_{q\neq r}^{M}\sum_{a=1}^{n_q} \rho_{ia}^{rq} y_{ia}^{rq} \right. & \\ \quad \left. -(\gamma_i^r + \sigma_i^r + \delta_i^r) y_{ii}^{rr} - d_i^r I_{ii}^{rr}\right] dt, & \text{for } q = r, l = i \\ \left[\sigma_{ij}^{rr} y_{ii}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) y_{ij}^{rr} - d_j^r I_{ij}^{rr}\right] dt, & \text{for } q = r, a = j \text{ and } i \neq j, \\ \left[\gamma_{il}^{rq} y_{ii}^{rr} - (\rho_{il}^{rq} + \delta_l^q) y_{il}^{rq} - d_l^q I_{il}^{rq}\right] dt, & \text{for } q \neq r, y_{il}^{rq}(t_0) \geq 0, \end{cases} \tag{3.2}$$

We now show the existence of a unique nonnegative solution of the system (2.10)–(2.12) in the following theorem.

Theorem 3.1. *Given any initial condition $x_{00}^{00}(t_0) \in \mathbb{R}_+^{3n^2}$, there is a unique nonnegative solution of the system* (2.10)–(2.12) *in $\mathbb{R}_+^{3n^2}$, for $t \geq t_0$.*

Proof. We observe that the rate functions of the system are nonlinear, continuous in their argument variables, and locally Lipschitz continuous with respect to x_{00}^{00}. This implies the existence of a unique local solution. The unique global existence follows from the extension results about local solutions in [44]. The nonnegativity of the solution follows from Lemma 3.1.

In the following, we exhibit that the solution of the initial value problem (3.2) is nonnegative. That is for all $t \geq 0$, $y^{ru}_{ia}(t) \geq 0$ is nonnegative, whenever $y^{ru}_{ia}(t_0) \geq 0$. □

Lemma 3.1. *Let $r, u \in I(1, M)$, $i \in I^r(1, n_r)$ and $a \in I^r_i(1, n_u)$. For all $t \geq t_0$, from (3.1), if $y^{ru}_{ia}(t_0) \geq 0$, then $y^{ru}_{ia}(t) \geq 0$.*

Proof. It follows from (3.1) and (2.10)–(2.12) that the system (3.2) is of the form $u' = A(t, u)w(t, u), u(t_0) \geq 0$, in [19, equation (8)] and satisfies the quasimonotonicity condition. Furthermore, from Remark 4 in [19], we assert that this system (3.2) has nonnegative solutions whenever $y^{rq}_{il}(0) \geq 0$, $\forall i \in I(1, n_r), l \in I^r_i(1, n_q), r \in I(1, M)$, and $q \in I^r(1, M)$. □

Remark 3.1. From the decomposition described in (2.1), we observe that $y^{ru}_{ia}(t) = N^{ru}_{ia} = S^{ru}_{ia}(t) + I^{ru}_{ia}(t) + R^{ru}_{ia}(t)$. Furthermore, that $N^{rr}_{i0} = \sum_{u=1}^{M} \sum_{a=1}^{n_u} y^{ru}_{ia}$. Therefore, Lemma 3.1 established that for any nonnegative initial endemic population, the number of residents of site s^r_i present at home, y^{rr}_{ii}, or visiting any given site s^{ru}_{ia} in any other region C_u, y^{ru}_{ia}, is nonnegative. This implies that the total population of residents of site s^r_i present at home and also visiting sites in regions in their intra and intraregional accessible domains, $N^{rr}_{i0}(t)$, is nonnegative. Moreover, Lemma 3.1 exibits that the effective population at any site in any region given by (2.3) is nonnegative at all time $t \geq t_0$. Furthermore, $R^{n^2}_{+} = \{y \in R^{n^2} : y \geq 0\}$ is a self-invariant set with respect to (3.2).

In the following lemma, we use Lemma 3.1 to find an upper bound for the solution of (2.10)–(2.12)

Lemma 3.2. *Let $\mu = \min_{1 \leq u \leq M, 1 \leq a \leq n_u}(\delta^u_a)$. If*

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} y^{ru}_{ia}(t_0) \leq \frac{1}{\mu} \sum_{r=1}^{M} \sum_{i=1}^{n_r} B^r_i, \tag{3.3}$$

then

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} y^{ru}_{ia}(t) \leq \frac{1}{\mu} \sum_{r=1}^{M} \sum_{i=1}^{n_r} B^r_i, \quad for \quad t \geq 0, \tag{3.4}$$

Proof. From (3.1), define

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} dy^{ru}_{ia} = \sum_{r=1}^{M} \sum_{i=1}^{n_r} \left[dy^{rr}_{ii} + \sum_{a \neq i}^{n_r} dy^{rr}_{ia} + \sum_{u \neq r}^{M} \sum_{a=1}^{n_r} dy^{ru}_{ia} \right] \tag{3.5}$$

From (2.10)–(2.12) and (3.5), one can see that

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} dy^{ru}_{ia} = \left[\sum_{r=1}^{M} \sum_{i=1}^{n_r} B^r_i - \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{u=1}^{M} \sum_{a=1}^{n_u} (\delta^u_a y^{ru}_{ia} + d^u_a I^{ru}_{ia}) \right] dt \tag{3.6}$$

From Lemma 3.1, and (3.6), we have

$$d \left\{ \sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} y^{ru}_{ia} \right\} \leq \left[\sum_{r=1}^{M} \sum_{i=1}^{n_r} B^r_i - \mu \sum_{r=1}^{M} \sum_{i=1}^{n_r} \sum_{u=1}^{M} \sum_{a=1}^{n_u} y^{ru}_{ia} \right] dt \tag{3.7}$$

for a nonnegative differential of t. We note that (3.7) is a first order deterministic differential inequality [43], and its solution is given by

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0)\right] e^{-\mu t} \tag{3.8}$$

Therefore, (3.4) is satisfied provided (3.3) is valid. □

Remark 3.2. From Lemma 3.2, we conclude that a closed ball in R^{3n^2} under the sum norm with radius $r = \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$ is self-invariant with regard to a two-scale network dynamic of human epidemic process that is under the influence of human mobility process [1].

4. ASYMPTOTIC BEHAVIOR OF DISEASE FREE EQUILIBRIUM

In this section, we study the asymptotic behavior of the disease free equilibrium state of the system (2.10)–(2.12). The disease free equilibrium is obtained by solving the system of algebraic equations obtained by setting the rate functions of the system of ordinary differential equations to zero. In addition, conditions that $I = R = 0$ in the event when there is no disease in the population. We summarize the results as follows.

For any $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$, let

$$D_i^r = \gamma_i^r + \sigma_i^r + \delta_i^r - \sum_{a=1}^{n_r} \frac{\rho_{ia}^{rr}\sigma_{ia}^{rr}}{\rho_{ia}^{rr} + \delta_a^r} - \sum_{u\neq r}^{M}\sum_{a=1}^{n_u} \frac{\rho_{ia}^{rr}\gamma_{ia}^{ru}}{\rho_{ia}^{ru} + \delta_a^u} > 0. \tag{4.1}$$

Furthermore, let $(S_{ia}^{ru*}, I_{ia}^{ru*}, R_{ia}^{ru*})$, be the equilibrium state of the system (2.10)–(2.12). One can see that the disease free equilibrium state is given by $E_{ia}^{ru} = (S_{ia}^{ru*}, 0, 0)$, where

$$S_{ia}^{ru*} = \begin{cases} \dfrac{B_i^r}{D_i^r}, & \text{for } u = r, a = i, \\ \dfrac{B_i^r}{D_i^r}\dfrac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr} + \delta_j^r}, & \text{for } u = r, a \neq i, \\ \dfrac{B_i^r}{D_i^r}\dfrac{\gamma_{ia}^{ru}}{\rho_{ia}^{ru} + \delta_a^u}, & \text{for } u \neq r. \end{cases} \tag{4.2}$$

The asymptotic stability property of E_{ia}^{ru} will be established by verifying the conditions of the Lyapunov second method given in [45]. In order to study the qualitative properties of (2.10)–(2.12) with respect to the equilibrium state $(S_{ia}^{ru*}, 0, 0)$, first, we use the change of variable. For this purpose, we use the following transformation:

$$\begin{cases} U_{ia}^{ru} = S_{ia}^{ru} - S_{ia}^{ru*} \\ V_{ia}^{ru} = I_{ia}^{ru} \\ W_{ia}^{ru} = R_{ia}^{ru}. \end{cases} \tag{4.3}$$

By employing this transformation, system (2.10)–(2.12) is transformed into the following forms

(4.4)
$$dU^{rq}_{il}=\begin{cases}\big[\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho^{rq}_{ia}U^{rq}_{ia}+\eta^r_iV^{rr}_{ii}+\alpha^r_iW^{rr}_{ii}\\ -(\gamma^r_i+\sigma^r_i+\delta^r_i)U^{rr}_{ii}-\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{iia}(S^{rr*}_{ii}+U^{rr}_{ii})V^{ur}_{ai}\big]dt, & \text{for } q=r, l=i\\ \big[\sigma^{rr}_{ij}U^{rr}_{ii}+\eta^r_jV^{rr}_{ij}+\alpha^r_jW^{rr}_{ij}-(\rho^{rr}_{ij}+\delta^r_j)U^{rr}_{ij}\\ \quad-\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{jia}(S^{rr*}_{ij}+U^{rr}_{ij})V^{ur}_{aj}\big]dt, & \text{for } q=r, l=j, j\neq i,\\ \big[\gamma^{rq}_{il}U^{rr}_{ii}+\eta^q_lV^{rq}_{il}+\alpha^q_lW^{rq}_{il}-(\rho^{rq}_{il}+\delta^q_l)U^{rq}_{il}\\ \quad-\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{qru}_{lia}S^{rq}_{il}I^{uq}_{al}\big]dt, & \text{for } q\neq r,\end{cases}$$

(4.5)
$$dV^{rq}_{il}=\begin{cases}\big[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho^{rq}_{ia}V^{rq}_{ia}-(\eta^r_i+\varrho^r_i\quad+\gamma^r_i+\sigma^r_i+\delta^r_i+d^r_i)W^{rr}_{ii}\\ +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{iia}(S^{rr*}_{ii}+U^{rr}_{ii})V^{ur}_{ai}\big]dt, & \text{for } q=r, l=i\\ \big[\sigma^{rr}_{ij}V^{rr}_{ii}-(\eta^r_j+\varrho^r_j+\rho^{rr}_{ij}+\delta^r_j+d^r_j)V^{rr}_{ij}\\ \quad+\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{rru}_{jia}(S^{rr*}_{ij}+U^{rr}_{ij})V^{ur}_{aj}\big]dt, & \text{for } q=r, l=j, j\neq i,\\ \big[\gamma^{rq}_{il}V^{rr}_{ii}-(\eta^q_l+\varrho^q_l+\rho^{rq}_{il}+\delta^q_l+d^q_l)V^{rq}_{il}\\ \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta^{qru}_{lia}(S^{rq*}_{il}+U^{rq}_{il})V^{uq}_{al}\big]dt, & \text{for } q\neq r,\end{cases}$$

and

(4.6)
$$dW^{rq}_{il}=\begin{cases}\big[\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho^{rq}_{il}W^{rq}_{il}+\varrho^r_iV^{rr}_{ii}-(\gamma^r_i+\sigma^r_i+\alpha^r_i+\delta^r_i)W^{rr}_{ii}\big]dt, & \text{for } q=r, l=i\\ \big[\sigma^{rr}_{ij}W^{rr}_{ii}+\varrho^r_jV^{rr}_{ij}-(\rho^{rr}_{ij}+\alpha^r_j+\delta^r_j)W^{rr}_{ij}\big]dt, & \text{for } q=r, l=j, j\neq i\\ \big[\gamma^{rq}_{il}W^{rr}_{ii}+\varrho^q_lV^{rq}_{il}-(\rho^{rq}_{il}+\alpha^q_l+\delta^q_l)W^{rq}_{il}\big]dt, & \text{for } q\neq r\end{cases}$$

We state and prove the following lemmas that would be useful in the proofs of the stability results.

Lemma 4.1. *Let $V:\mathbb{R}^{3n^2}\times\mathbb{R}_+\to\mathbb{R}_+$ be a function defined by*

$$V(\tilde{x}^{00}_{00})=\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}V(\tilde{x}^{ru}_{ia}), \tag{4.7}$$

where,

$$\begin{aligned}V(\tilde{x}^{ru}_{ia}) &= (S^{ru}_{ia}-S^{ru*}_{ia}+I^{ru}_{ia})^2+c^{ru}_{ia}(I^{ru}_{ia})^2+(R^{ru}_{ia})^2\\ \tilde{x}^{00}_{00} &= (U^{ru}_{ia},V^{ru}_{ia},W^{ru}_{ia})^T \quad \text{and} \quad c^{ru}_{ia}\geq 0.\end{aligned} \tag{4.8}$$

Then $V\in\mathcal{C}^{2,1}(\mathbb{R}^{3n^2}\times\mathbb{R}_+,\mathbb{R}_+)$, and it satisfies

$$b(\|\tilde{x}^{00}_{00}\|)\leq V(\tilde{x}^{00}_{00}(t))\leq a(\|\tilde{x}^{00}_{00}\|) \tag{4.9}$$

where

$$b(\|\tilde{x}^{00}_{00}\|)=\min_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\left\{\frac{c^{ru}_{ia}}{2+c^{ru}_{ia}}\right\}\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U^{ru}_{ia})^2+(V^{ru}_{ia})^2+(W^{ru}_{ia})^2\right]$$

(4.10)
$$a(\|\tilde{x}^{00}_{00}\|)=\max_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\{c^{ru}_{ia}+2\}\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\left[(U^{ru}_{ia})^2+(V^{ru}_{ia})^2+(W^{ru}_{ia})^2\right].$$

Proof. From (4.3), (4.7) can be written as

$$\begin{aligned}
V(x_{ia}^{ru}) &= (U_{ia}^{ru}+V_{ia}^{ru})^2 + c_{ia}^{ru}(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\\
&= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\\
&= (U_{ia}^{ru})^2 + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + 2\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right)\left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right) + (W_{ia}^{ru})^2\\
&= \left(-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}+1\right)(U_{ia}^{ru})^2 + \left(-\left(1+\frac{c_{ia}^{ru}}{2}\right)+c_{ia}^{ru}+1\right)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\\
&\quad + \left[\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right) + \left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right)\right]^2
\end{aligned}$$

Therefore, by nothing the fact that $\min\{1-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}, \frac{c_{ia}^{ru}}{2}, 1\}$, we have

$$V(x_{ia}^{ru}) \geq \frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right] \tag{4.11}$$

Hence from (4.11) we have

$$\begin{aligned}
V(\tilde{x}_{00}^{00}) &\geq \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2)\right]\\
&\geq b(\|\tilde{x}_{00}^{00}\|).
\end{aligned} \tag{4.12}$$

On the other hand, it follows from (4.7) that

$$\begin{aligned}
V(x_{ia}^{ru}) &= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\\
&\leq 2(U_{ia}^{ru})^2 + (c_{ia}^{ru}+2)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\\
&\leq (c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right]
\end{aligned} \tag{4.13}$$

Thus, from (4.11) and (4.13) we have

$$\begin{aligned}
V(x_{00}^{00}(t)) &\leq \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}(c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right]\\
&\leq a(\|\tilde{x}_{00}^{00}\|)
\end{aligned} \tag{4.14}$$

Therefore from (4.7), (4.12) and (4.14), we establish the desired inequality. □

Remark 4.1. Lemma 4.1 shows that the Lyapunov function V defined in (4.7) is positive definite (4.12), decrescent ((4.14)) function [45].

We now state the following lemma

Lemma 4.2. *Assume that the hypotheses of Lemma* 4.1 *are satisfied. Let*

$$d_{ii}^{rr} = \beta_{iib}^{rrv}(\frac{S_{ii}^{rr}}{\mu_{ii}^{rr}} + \frac{\bar{B}^2}{\mu_{ii}^{rr}})$$

$$d_{ia}^{rr} = \beta_{aib}^{rrv}(\frac{S_{ia}^{rr*}}{\mu_{ia}^{rr}} + \frac{\bar{B}^2}{\mu_{ia}^{rr}})$$

$$d^{ru}_{ia} = \beta^{urv}_{aib}(\frac{S^{ru*}_{ia}}{\mu^{ru}_{ia}} + \frac{\bar{B}^2}{\mu^{ru}_{ia}})$$

$$c^{u}_{a} = \sum_{v=1}^{M}\sum_{b=1}^{n_v} c^{vu}_{ba} \tag{4.15}$$

for some positive numbers c^{ru}_{ia}, for all $r, u \in I^r(1, M)$, $i \in I(1, n)$ and $a \in I^r_i(1, n_r)$. Furthermore, let

(4.16)

$$\mathfrak{U}^{ru}_{ia} = \begin{cases} \dfrac{\left[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu^{ru}_{ia} + \dfrac{3}{2}\mu^{rr}_{ii} + \sum_{a\neq i}^{n_r}\dfrac{(\sigma^{rr}_{ia})^2}{\mu^{rr}_{ii}} + \sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\dfrac{(\gamma^{ru}_{ia})^2}{\mu^{rr}_{ii}}\right]}{(\gamma^r_i + \sigma^r_i + \delta^r_i)} & \text{for } u = r, i = a \\ \dfrac{\left[\dfrac{(\rho^{rr}_{ia})^2}{\mu^{rr}_{ia}} + \mu^{rr}_{ii} + \mu^{rr}_{ia}\right]}{(\rho^{rr}_{ia} + \delta^r_a)}, & \text{for } u = r, a \neq i \\ \dfrac{\left[\dfrac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}} + \mu^{rr}_{ii} + \mu^{ru}_{ia}\right]}{(\rho^{ru}_{ia} + \delta^u_a)}, & \text{for } u \neq r, \end{cases}$$

(4.17)

$$\mathfrak{V}^{ru}_{ia} = \begin{cases} \dfrac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu^{ru}_{ia} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{rrv}_{iib}(S^{rr*}_{ii} + \mu^{rr}_{ii}) + \frac{1}{2}d^{rr}_{ii}}{\eta^r_i + \varrho^r_i + \gamma^r_i + \sigma^r_i + \delta^r_i + d^r_i}, & \text{for } a = i, u = r \\ \dfrac{\frac{1}{2}\mu^{rr}_{ii} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{rrv}_{aib}(S^{rr*}_{ia} + \mu^{rr}_{ia}) + \frac{1}{2}d^{rr}_{ia}}{\eta^r_a + \varrho^r_a + \rho^{rr}_{ia} + \delta^r_a + d^r_a}, & \text{for } a \neq i, u = r \\ \dfrac{\frac{1}{2}\mu^{rr}_{ii} + \sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{urv}_{aib}(S^{ru*}_{ii} + \mu^{ru}_{ia}) + \frac{1}{2}d^{ru}_{ia}}{\eta^u_a + \varrho^u_a + \rho^{ru}_{ia} + \delta^u_a + d^u_a}, & \text{for } u \neq r. \end{cases}$$

and

(4.18)

$$\mathfrak{W}^{ru}_{ia} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu^{ru}_{ia} + \dfrac{(\alpha^r_i)^2}{\mu^{rr}_{ii}} + \frac{1}{2}\mu^{rr}_{ii} + \frac{1}{2}\sum_{a\neq i}^{n_r}\dfrac{(\sigma^{rr}_{ia})^2}{\mu^{rr}_{ii}} + \frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\dfrac{(\gamma^{ru}_{ia})^2}{\mu^{rr}_{ii}}\right]}{(\gamma^r_i + \sigma^r_i + \alpha^r_i + \delta^r_i)}, & \text{for } u = r, a = i, \\ \dfrac{\left[\dfrac{(\rho^{rr}_{ia})^2}{2\mu^{rr}_{ia}} + \frac{1}{2}\mu^{rr}_{ii} + \frac{1}{2}\mu^{rr}_{ia} + \dfrac{(\alpha^r_a)^2}{\mu^{rr}_{ia}}\right]}{(\rho^{rr}_{ia} + \alpha^r_a + \delta^r_a)}, & \text{for } u = r, a \neq i, \\ \dfrac{\left[\frac{1}{2}\dfrac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}} + \frac{1}{2}\mu^{rr}_{ii} + \frac{1}{2}\mu^{ru}_{ia} + \frac{(\alpha^u_a)^2}{\mu^{ru}_{ia}}\right]}{(\rho^{ru}_{ia} + \alpha^u_a + \delta^u_a)}, & \text{for } u \neq r \end{cases}$$

for some suitably defined positive number μ^{ru}_{ia}, depending on δ^u_a, for all $r, u \in I^r(1, M)$, $i \in I(1, n)$ and $a \in I^r_i(1, n_r)$. Assume that $\mathfrak{U}^{ru}_{ia} \leq 1$, $\mathfrak{V}^{ru}_{ia} < 1$ and $\mathfrak{W}^{ru}_{ia} \leq 1$. There exist positive numbers ϕ^{ru}_{ia}, ψ^{ru}_{ia} and φ^{ru}_{ia} such that the differential operator $\dot{V}$ associated *with the system* (4.4)–(4.6) *satisfies the following inequality*

$$\dot V(\tilde x^{00}_{00}) \le \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Big[-[\phi^{rr}_{ii}(U^{rr}_{ii})^2+\psi^{rr}_{ii}(V^{rr}_{ii})^2+\varphi^{rr}_{ii}(W^{rr}_{ii})^2] - \sum_{a\neq i}^{n_r}[\phi^{rr}_{ia}(U^{rr}_{ia})^2+\psi^{rr}_{ia}(V^{rr}_{ia})^2+\varphi^{rr}_{ia}(W^{rr}_{ia})^2] - \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi^{ru}_{ia}(U^{rr}_{ia})^2+\psi^{ru}_{ia}(V^{ru}_{ia})^2+\varphi^{ru}_{ia}(W^{ru}_{ia})^2]\Big]. \tag{4.19}$$

Moreover,

$$\dot V(\tilde x^{00}_{00}) \le -cV(\tilde x^{00}_{00}) \tag{4.20}$$

where a positive constant c *is defined by*

$$c=\frac{\min_{1\le r,u\le M,1\le i\le n_r,1\le a\le n_u}(\phi^{ru}_{ia},\psi^{ru}_{ia},\varphi^{ru}_{ia})}{\max_{1\le r,u\le M,1\le i\le n_r,1\le a\le n_u}\{C^{ru}_{ia}+2\}} \tag{4.21}$$

Proof. The computation of differential operator [45, 46] applied to the Lyapunov function V in (4.7) with respect to the large-scale system of ordinary differential equation (2.10)–(2.12) is as follows:

$$\begin{aligned}\dot V(\tilde x^{rr}_{ii}) &= 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+C^{rr}_{ii})\rho^{ru}_{ia}V^{ru}_{ia}V^{rr}_{ii}+\rho^{ru}_{ia}U^{ru}_{ia}U^{rr}_{ii}+\rho^{ru}_{ia}V^{ru}_{ia}U^{rr}_{ii}+\rho^{ru}_{ia}U^{ru}_{ia}V^{rr}_{ii}\\ &+\rho^{ru}_{ia}W^{ru}_{ia}W^{rr}_{ii}]+2\alpha^r_iU^{rr}_{ii}W^{rr}_{ii}+2(\alpha^r_i+\varrho^r_i)V^{rr}_{ii}W^{rr}_{ii}\\ &-2[\varrho^r_i+d^r_i+2(\gamma^r_i+\sigma^r_i+\delta^r_i)]V^{rr}_{ii}U^{rr}_{ii}-2(\gamma^r_i+\sigma^r_i+\delta^r_i)(U^{rr}_{ii})^2\\ &-2[c^{rr}_{ii}\eta^r_i+2(c^{rr}_{ii}+1)(\varrho^r_i+\gamma^r_i+\sigma^r_i+\delta^r_i+d^r_i)](V^{rr}_{ii})^2\\ &-2(\gamma^r_i+\sigma^r_i+\alpha^r_i+\delta^r_i)(W^{rr}_{ii})^2\\ &+2\sum_{u=1}^{M}\sum_{a=1}^{n_u}c^{rr}_{ii}\beta^{rru}_{iia}(S^{rr*}_{ii}+U^{rr}_{ii})V^{ur}_{ai}V^{rr}_{ii},\quad \text{for } u=r, a=i\end{aligned} \tag{4.22}$$

$$\begin{aligned}\sum_{a\neq i}^{n_r}\dot V(\tilde x^{rr}_{ia}) &= \sum_{a\neq r}^{n_r}\{2(1+c^{rr}_{ia})\sigma^{rr}_{ia}V^{rr}_{ia}V^{rr}_{ii}+2\sigma^{rr}_{ia}U^{rr}_{ia}U^{rr}_{ii}\\ &+2\sigma^{rr}_{ia}V^{rr}_{ia}U^{rr}_{ii}+2\sigma^{rr}_{ia}U^{rr}_{ia}V^{rr}_{ii}+2\sigma^{rr}_{ia}W^{rr}_{ia}W^{rr}_{ii}\\ &-2[c^{rr}_{ia}\eta^r_a+2(c^{rr}_{ia}+1)(\varrho^r_a+\rho^{rr}_{ia}+\delta^r_a)](V^{rr}_{ia})^2-2(\rho^{rr}_{ia}+\delta^r_a)(U^{rr}_{ia})^2\\ &-2(\rho^{rr}_{ia}+\alpha^r_a+\delta^r_a)(W^{rr}_{ia})^2+2\alpha^r_aW^{rr}_{ia}U^{rr}_{ia}+2(\alpha^r_a+\varrho^r_a)V^{rr}_{ia}W^{rr}_{ia}\\ &-2[\varrho^r_a+d^r_a+2(\rho^{rr}_{ia}+\delta^r_a)]V^{rr}_{ia}U^{rr}_{ia}\}\\ &+2\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ia}\beta^{rrv}_{aib}(S^{rr*}_{ia}+U^{rr}_{ia})V^{vr}_{ba}V^{rr}_{ia},\\ &\qquad \text{for } u=r,\quad a\neq i\end{aligned} \tag{4.23}$$

$$\sum_{u=1}^{M}\sum_{a=1}^{n_r}\dot V(\tilde x^{ru}_{ia}) = \sum_{u=1}^{M}\sum_{a=1}^{n_u}\{2(1+c^{ru}_{ia})\gamma^{ru}_{ia}V^{ru}_{ia}V^{rr}_{ii}+2\gamma^{ru}_{ia}U^{ru}_{ia}U^{rr}_{ii}$$

$$
\begin{aligned}
&+2\gamma^{ru}_{ia}V^{ru}_{ia}U^{rr}_{ii}+2\gamma^{ru}_{ia}U^{ru}_{ia}V^{rr}_{ii}\\
&+2\gamma^{ru}_{ia}W^{ru}_{ia}W^{rr}_{ii}-2[c^{ru}_{ia}\eta^{u}_{a}+2(c^{ru}_{ia}+1)(\varrho^{u}_{a}+\rho^{ru}_{ia}+\delta^{u}_{a}+d^{u}_{a})](V^{ru}_{ia})^2\\
&-2(\rho^{ru}_{ia}+\delta^{u}_{a})(U^{ru}_{ia})^2\\
&-2(\rho^{ru}_{ia}+\alpha^{u}_{a}+\delta^{u}_{a})(W^{rr}_{ia})^2+2\alpha^{u}_{a}W^{ru}_{ia}U^{ru}_{ia}+2(\alpha^{u}_{a}+\varrho^{u}_{a})V^{ru}_{ia}W^{ru}_{ia}\\
&-2[\varrho^{u}_{a}+d^{u}_{a}+2(\rho^{ru}_{ia}+\delta^{u}_{a})]V^{ru}_{ia}U^{ru}_{ia}\}\\
&+2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{ru}_{ia}\beta^{urv}_{aib}(S^{ru*}_{ia}+U^{ru}_{ia})V^{vu}_{ba}V^{ru}_{ia},\\
&\qquad \text{for } u\neq r
\end{aligned}
\tag{4.24}
$$

By using Remark 3.2 and the algebraic inequality

$$
2ab\leq\frac{a^2}{g(c)}+b^2g(c)
\tag{4.25}
$$

where $a,b,c\in\mathbb{R}$, and the function g is such that $g(c)\geq 0$. The sixth term in (4.22), (4.23) and (4.24) is estimated as follows:

$$
\begin{aligned}
&2\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ii}\beta^{rrv}_{iib}(S^{rr*}_{ii}+U^{rr}_{ii})V^{vr}_{bi}V^{rr}_{ii}\\
&\qquad\leq\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ii}\beta^{rrv}_{iib}(S^{rr*}_{ii}g^{r}_{i}(\delta^{r}_{i})+g^{r}_{i}(\delta^{r}_{i}))(V^{rr}_{ii})^2\\
&\qquad\quad+\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ii}\beta^{rrv}_{iib}\left(\frac{S^{rr*}_{ii}}{g^{r}_{i}(\delta^{r}_{i}})+\frac{\bar{B}^2}{g^{r}_{i}(\delta^{r}_{i})}\right)(V^{vr}_{bi})^2\\
&\sum_{a\neq r}^{n_r}2\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ia}\beta^{rrv}_{aib}(S^{rr*}_{ia}+U^{rr}_{ia})V^{vr}_{ba}V^{rr}_{ia}\\
&\qquad\leq\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ia}\beta^{rrv}_{aib}(S^{rr*}_{ia}g^{r}_{i}(\delta^{r}_{a})+g^{r}_{i}(\delta^{r}_{a}))(V^{rr}_{ia})^2\\
&\qquad\quad+\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{rr}_{ia}\beta^{rrv}_{aib}\left(\frac{S^{rr*}_{ia}}{g^{r}_{i}(\delta^{r}_{a}})+\frac{\bar{B}^2}{g^{r}_{i}(\delta^{r}_{a})}\right)(V^{vr}_{bi})^2
\end{aligned}
$$

and

$$
\begin{aligned}
&2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{ru}_{ia}\beta^{urv}_{aib}(S^{ru*}_{ia}+U^{ru}_{ia})V^{vu}_{ba}V^{ru}_{ia}\\
&\qquad\leq\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{ru}_{ia}\beta^{urv}_{aib}(S^{ru*}_{ia}g^{r}_{i}(\delta^{u}_{a})+g^{r}_{i}(\delta^{u}_{a}))(V^{ru}_{ia})^2\\
&\qquad\quad+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}c^{ru}_{ia}\beta^{urv}_{aib}\left(\frac{S^{ru*}_{ia}}{g^{r}_{i}(\delta^{u}_{a}})+\frac{\bar{B}^2}{g^{r}_{i}(\delta^{u}_{a})}\right)(V^{vu}_{ba})^2
\end{aligned}
\tag{4.26}
$$

From (4.22), (4.23) and repeated usage of inequality (4.25) and (4.26) coupled with algebraic manipulations and simplifications, we have the following inequality

$$
\begin{aligned}
\dot{V}(\tilde{x}_{00}^{00}) \leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[&4\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + 3\mu_{ii}^{rr} - 2(\gamma_i^r + \sigma_i^r) + 2\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} \\
&+ 2\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}\Bigg](U_{ii}^{rr})^2 \\
&+ \Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+c_{ii}^{rr})\mu_{ia}^{ru} + \mu_{ia}^{ru}] + \mu_{ii}^{rr} + 2\frac{(\varrho_i^r + d_i^r)^2}{\mu_{ii}^{rr}} + 4\frac{(\gamma_i^r + \sigma_i^r + \delta_i^r)^2}{\mu_{ii}^{rr}} \\
&- 2[c_{ii}^{rr}\eta_i^r + (c_{ii}^{rr}+1)(\varrho_i^r + \gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r)] \\
&+ \sum_{a\neq r}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} + \sum_{a\neq r}^{n_r}\frac{(1+c_{ia}^{rr})(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} \\
&+ \sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(1+c_{ia}^{ru})(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}} \\
&+ c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr} + \mu_{ii}^{rr})\Bigg](V_{ii}^{rr})^2 \\
&+ \Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + 2\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}} + \mu_{ii}^{rr} - 2(\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r) \\
&+ \sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} + \sum_{a\neq r}^{M}\sum_{a=1i}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}\Bigg](W_{ii}^{rr})^2 \\
&+ \sum_{a\neq i}^{n_r}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + 2\mu_{ii}^{rr} + 2\mu_{ia}^{rr} - 2(\rho_{ia}^{rr} + \delta_a^r)\Bigg](U_{ia}^{rr})^2 \\
&+ \Bigg[(1+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \mu_{ii}^{rr} \\
&(1+c_{ia}^{rr})\mu_{ii}^{rr} - 2[c_{ia}^{rr}\eta_a^r + (1+c_{ia}^{rr})(\eta_a^r + \rho_{ia}^{rr} + \delta_a^r + d_a^r)] \\
&+ 2\frac{(\varrho_a^r + d_a^r)^2}{\mu_{ia}^{rr}} + 4\frac{(\rho_{ia}^{rr} + \delta_a^r)^2}{\mu_{ia}^{rr}} + \mu_{ia}^{rr} \\
&+ c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(S_{ia}^{rr}\mu_{ia}^{rr} + \mu_{ia}^{rr})\Bigg](V_{ia}^{rr})^2 + \Bigg[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \mu_{ii}^{rr} + \mu_{ia}^{rr} + \frac{2(\alpha_a^r)^2}{\mu_{ia}^{rr}} \\
&- 2(\rho_{ia}^{rr} + \alpha_a^r + \delta_a^r)\Bigg](W_{ia}^{rr})^2\Bigg\} \\
&+ \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + 2\mu_{ii}^{rr} + 2\mu_{ia}^{ru} - 2(\rho_{ia}^{ru} + \delta_a^u)\Bigg](U_{ia}^{ru})^2 \\
&+ \Bigg[(1+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \mu_{ii}^{rr} + (1+c_{ia}^{ru})\mu_{ii}^{rr}
\end{aligned}
$$

$$
\begin{aligned}
&- 2[c^{ru}_{ia}\eta^{u}_{a} + (1 + c^{ru}_{ia})(\eta^{u}_{a} + \rho^{ru}_{ia} + \delta^{u}_{a} + d^{u}_{a})] \\
&+ 2\frac{(\varrho^{u}_{a} + d^{u}_{a})^2}{\mu^{ru}_{ia}} + 4\frac{(\rho^{ru}_{ia} + \delta^{u}_{a})^2}{\mu^{ru}_{ia}} + \mu^{ru}_{ia} + c^{ru}_{ia}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta^{urv}_{aib}(S^{ru*}_{ia}\mu^{ru}_{ia} + \mu^{ru}_{ia})\Bigg](V^{ru}_{ia})^2 \\
&+ \left[\frac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}} + \mu^{rr}_{ii} + \mu^{ru}_{ia} + \frac{2(\alpha^{u}_{a})^2}{\mu^{ru}_{ia}} - 2(\rho^{ru}_{ia}\alpha^{u}_{a} + \delta^{u}_{a})\right](W^{ru}_{ia})^2\Bigg\}\Bigg\} \\
&+ \sum_{r=1}^{M}\sum_{i=1}^{n_r} c^{rr}_{ii}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta^{rrv}_{iib}(\frac{S^{rr*}_{ii}}{\mu^{rr}_{ii}} + \frac{\bar{B}^2}{\mu^{rr}_{ii}})\right](V^{vr}_{bi})^2 \\
&+ \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} c^{rr}_{ia}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta^{rrv}_{aib}(\frac{S^{rr*}_{ia}}{\mu^{rr}_{ia}} + \frac{\bar{B}^2}{\mu^{rr}_{ia}})\right](V^{vr}_{ba})^2 \\
&+ \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r} c^{ru}_{ia}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta^{urv}_{aib}(\frac{S^{ru*}_{ia}}{\mu^{ru}_{ia}} + \frac{\bar{B}^2}{\mu^{ru}_{ia}})\right](V^{vu}_{ba})^2,
\end{aligned}
\tag{4.27}
$$

where $\mu^{ru}_{ia} = g^{r}_{i}(\delta^{u}_{a})$, g^{r}_{i} is appropriately defined by (4.25).

For each $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$, using algebraic manipulations and (4.16), (4.17) and (4.18), the coefficients of $(U^{ru}_{ia})^2$, $(V^{ru}_{ia})^2$ and $(W^{ru}_{ia})^2$ in (4.27) defined by ϕ^{ru}_{ia}, ψ^{ru}_{ia} and φ^{ru}_{ia} respectively:

$$
\phi^{ru}_{ia} = \begin{cases} 2(\gamma^{r}_{i} + \sigma^{r}_{i} + d^{r}_{i})(1 - \mathfrak{U}^{ru}_{ia}), & \text{for } u = r, a = i \\ 2(\rho^{rr}_{ia} + \delta^{r}_{a} + d^{r}_{a})(1 - \mathfrak{U}^{ru}_{ia}), & \text{for } u = r, a \neq i \\ 2(\rho^{ru}_{ia} + \delta^{u}_{a} + d^{u}_{a})(1 - \mathfrak{U}^{ru}_{ia}), & \text{for } u \neq r, \end{cases}
\tag{4.28}
$$

$$
\psi^{ru}_{ia} = \begin{cases} [2c^{rr}_{ii}(1 - \mathfrak{V}^{rr}_{ii})(\eta^{r}_{i} + \varrho^{r}_{i} + \gamma^{r}_{i} + \sigma^{r}_{i} + \delta^{r}_{i} + d^{r}_{i}) - \mathfrak{E}^{rr}_{ii}] & \\ \quad +2(\varrho^{r}_{i} + \gamma^{r}_{i} + \sigma^{r}_{i} + \delta^{r}_{i} + d^{r}_{i}), & \text{for } u = r, a = i \\ [2c^{rr}_{ia}(1 - \mathfrak{V}^{rr}_{ia})(\eta^{r}_{a} + \varrho^{r}_{a} + \rho^{rr}_{ia} + \delta^{r}_{a} + d^{r}_{a}) - \mathfrak{E}^{rr}_{ia}] & \\ \quad +2(\varrho^{r}_{a} + \rho^{rr}_{ia} + \delta^{r}_{a} + d^{r}_{a}), & \text{for } u = r, a \neq i \\ [2c^{ru}_{ia}(1 - \mathfrak{V}^{ru}_{ia})(\eta^{u}_{a} + \varrho^{u}_{a} + \rho^{ru}_{ia} + \delta^{u}_{a} + d^{u}_{a}) - \mathfrak{E}^{ru}_{ia}] & \\ \quad +2(\varrho^{u}_{a} + \rho^{ru}_{ia} + \delta^{u}_{a} + d^{u}_{a}), & \text{for } u \neq r \end{cases}
$$

and

$$
\varphi^{ru}_{ia} = \begin{cases} 2(\gamma^{r}_{i} + \sigma^{r}_{i} + \alpha^{r}_{i} + \delta^{r}_{i})(1 - \mathfrak{W}^{ru}_{ia}), & \text{for } u = r, a = i, \\ 2(\rho^{rr}_{ia} + \delta^{r}_{a})(1 - \mathfrak{W}^{ru}_{ia}), & \text{for } u = r, a \neq i, \\ 2(\rho^{ru}_{ia} + \delta^{u}_{a})(1 - \mathfrak{W}^{ru}_{ia}), & \text{for } u \neq r \end{cases}
\tag{4.29}
$$

where

$$\mathfrak{E}^{ru}_{ia} = \begin{cases} \Big[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu^{ru}_{ia} + \mu^{rr}_{ii} + 2\frac{(\varrho^r_i+d^r_i)^2}{\mu^{rr}_{ii}} + 4\frac{(\gamma^r_i+\sigma^r_i+\delta^r_i}{\mu^{rr}_{ii}} + \sum_{a\neq r}^{n_r}\frac{(2+c^{rr}_{ia})(\sigma^{rr}_{ia})^2}{\mu^{rr}_{ii}} \\ + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\left(\frac{(\gamma^{ru}_{ia})^2}{\mu^{rr}_{ii}} + \frac{(1+c^{ru}_{ia})(\gamma^{ru}_{ia})^2}{\mu^{rr}_{ii}}\right) + \sum_{b\neq i}^{n_r} c^{rr}_{bi}d^{rr}_{ii} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c^{vr}_{bi}d^{rr}_{ii}\Big], \\ \qquad\qquad \text{for } u=r, a=i, \\ (1+c^{rr}_{ii})\frac{(\rho^{rr}_{ia})^2}{\mu^{rr}_{ia}} + \frac{(\rho^{rr}_{ia})^2}{\mu^{rr}_{ia}} + 2\mu^{rr}_{ii} + 2\frac{(\varrho^r_a+d^r_a)^2}{\mu^{rr}_{ia}} \\ +4\frac{(\rho^{rr}_{ia}+\delta^r_a)^2}{\mu^{rr}_{ia}} + \mu^{rr}_{ia} + \sum_{b\neq i}^{n_r} c^{rr}_{ba}d^{rr}_{ia} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c^{vr}_{ba}d^{rr}_{ia}, \qquad \text{for } u=r, a\neq i, \\ (1+c^{rr}_{ii})\frac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}} + \frac{(\rho^{ru}_{ia})^2}{\mu^{ru}_{ia}} + 2\mu^{rr}_{ii} + 2\frac{(\varrho^u_a+d^u_a)^2}{\mu^{ru}_{ia}} \\ +4\frac{(\rho^{ru}_{ia}+\delta^u_a)^2}{\mu^{ru}_{ia}} + \mu^{ru}_{ia} + \sum_{b\neq i}^{n_r} c^{ru}_{ba}d^{ru}_{ia} + \sum_{v\neq r}^{M} + \sum_{b=1}^{n_r} c^{vu}_{ba}d^{ru}_{ia}, u\neq r \end{cases}$$

Under the assumptions on $\mathfrak{U}^{ru}_{ia}$, $\mathfrak{V}^{ru}_{ia}$ and $\mathfrak{W}^{ru}_{ia}$, it is clear that $\phi^{ru}_{ia}, \psi^{ru}_{ia}$ and φ^{ru}_{ia} are positive for suitable choice of c^{ru}_{ia} defined in (4.8). We substitute (4.15), (4.28), (4.29) and (4.30) into (4.27). Thus inequality (4.27) can be rewritten as

$$\begin{aligned} \dot{V}(\tilde{x}^{00}_{00}) \le \sum_{r=1}^{M}\sum_{i=1}^{n_r} - \big\{[\phi^{rr}_{ii}(U^{rr}_{ii})^2 + \psi^{rr}_{ii})(V^{rr}_{ii})^2 \\ \varphi^{rr}_{ii}(W^{rr}_{ii})^2] + \sum_{a\neq r}^{n_r}[\phi^{rr}_{ia}(U^{rr}_{ia})^2 + \psi^{rr}_{ia}(V^{rr}_{ia})^2 \\ + \varphi^{rr}_{ia}(W^{rr}_{ia})^2] + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi^{ru}_{ia}(U^{ru}_{ia})^2 + \psi^{ru}_{ia}(V^{ru}_{ia})^2 \\ +\varphi^{ru}_{ia}(W^{ru}_{ia})^2]\big\} \end{aligned} \tag{4.30}$$

This proves the inequality (4.19). Now, the validity of (4.20) follows from (4.19), that is,

$$\dot{V}(\tilde{x}^{00}_{00}) \le -cV(\tilde{x}^{00}_{00}),$$

where $c = \frac{\min_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u}(\phi^{ru}_{ia},\psi^{ru}_{ia},\varphi^{ru}_{ia})}{\max_{1\le r,u\le M, 1\le i\le n_r, 1\le a\le n_u}\{C^{ru}_{ia}+2\}}$. This establishes the result. □

We now formally state the asymptotic stability result for the disease free equilibria.

Theorem 4.1. *Given $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$. Let us assume that the hypotheses of Lemma 4.2 are satisfied. Then the disease free solutions E^{ru}_{ia}, are globally, uniformly, asymptotically stable. Moreover, the solutions E^{ru}_{ia} are exponentially stable.*

Proof. From the application of comparison result [45, 46], the proof of stochastic asymptotic stability follows immediately Moreover, the disease free equilibrium state is exponentially mean square stable. □

We now consider the following corollary to Theorem 4.1.

Corollary 4.1. *Let $r \in I(1, M)$ and $i \in I(1, n_r)$. Assume that $\sigma^r_i = \gamma^r_i = 0$, for all $r \in I(1, M)$ and $i \in I(1, n_r)$.*

$$\mathfrak{U}^{ru}_{ia} = \begin{cases} \dfrac{\dfrac{1}{(\delta^r_i)}}{\left[\frac{1}{2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu^{ru}_{ia}+\frac{3}{2}\mu^{rr}_{ii}}\right]} & \text{for } u = r, i = a \\ \dfrac{\dfrac{1}{(\delta^r_a)}}{\dfrac{1}{[\mu^{rr}_{ii}+\mu^{rr}_{ia}]}}, & \text{for } u = r, a \neq i \\ \dfrac{\dfrac{1}{(\delta^u_a)}}{\dfrac{1}{[\mu^{rr}_{ii}+\mu^{ru}_{ia}]}}, & \text{for } u \neq r, \end{cases} \tag{4.31}$$

$$\mathfrak{V}^{ru}_{ia} = \begin{cases} \dfrac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu^{ru}_{ia}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{rrv}_{iib}(S^{rr*}_{ii}+\mu^{rr}_{ii})+\frac{1}{2}d^{rr}_{ii}}{\eta^r_i+\varrho^r_i+\delta^r_i+d^r_i}, & \text{for } a = i, u = r \\ \dfrac{\frac{1}{2}\mu^{rr}_{ii}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{rrv}_{aib}(S^{rr*}_{ia}+\mu^{rr}_{ia})+\frac{1}{2}d^{rr}_{ia}}{\eta^r_a+\varrho^r_a+\delta^r_a+d^r_a}, & \text{for } a \neq i, u = r \\ \dfrac{\frac{1}{2}\mu^{rr}_{ii}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta^{urv}_{aib}(S^{ru*}_{ii}+\mu^{ru}_{ia})+\frac{1}{2}d^{ru}_{ia}}{\eta^u_a+\varrho^u_a+\delta^u_a+d^u_a}, & \text{for } u \neq r. \end{cases} \tag{4.32}$$

and

$$\mathfrak{W}^{ru}_{ia} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu^{ru}_{ia}+\frac{(\alpha^r_i)^2}{\mu^{rr}_{ii}}+\frac{1}{2}\mu^{rr}_{ii}\right]}{(\alpha^r_i+\delta^r_i)}, & \text{for } u = r, a = i, \\ \dfrac{\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\mu^{rr}_{ia}+\frac{(\alpha^r_a)^2}{\mu^{rr}_{ia}}}{(\alpha^r_a+\delta^r_a)}, & \text{for } u = r, a \neq i, \\ \dfrac{\left[\frac{1}{2}\mu^{rr}_{ii}+\frac{1}{2}\mu^{ru}_{ia}+\frac{(\alpha^u_a)^2}{\mu^{ru}_{ia}}\right]}{\alpha^u_a+\delta^u_a}, & \text{for } u \neq r \end{cases} \tag{4.33}$$

The equilibrium state E^{rr}_{ii} is globally, uniformly asymptotically stable provided that $\mathfrak{U}^{ru}_{ia}, \mathfrak{W}^{ru}_{ia} \leq 1$ and $\mathfrak{V}^{ru}_{ia} < 1$, for all $u \in I^r(1, M)$ and $a \in I^r_i(1, n_u)$.

Proof. Follows immediately from the hypotheses of Lemma 4.2, (letting $\sigma^r_i = \gamma^r_i = 0$), the conclusion of Theorem 4.1 and some algebraic manipulations. □

Remark 4.2. The presented results about the two-level large scale SIRS disease dynamic model depend on the underlying system parameters. In particular, the sufficient conditions are algebraically simple, computationally attractive and explicit in terms of the rate parameters. As a result of this, several scenarios can be discussed and exhibit practical course of action to control the disease. For simplicity, we present an illustration as follows: the conditions of $\sigma^r_i = \gamma^r_i = 0, \forall r, i$ in Corollary 4.1 signify that the arbitrary site s^r_i is a sink [17, 18] for all other sites in the inter and intra-regional accessible domain. This scenario is displayed in Figure 3. The condition

$\mathfrak{U}_{ia}^{ru} \leq 1$ exhibits that the average infectious period is smaller than the joint average life span of individuals in the intra and inter-regional accessible domain of site s_i^r. Furthermore, the condition $\mathfrak{V}_{ia}^{ru} < 1$ signifies that the magnitude of disease inhibitory processes for example, the magnitude of the recovery process is greater than the disease transmission process. A future detailed study of the disease dynamics in the two scale network dynamic structure for many real life scenarios using the presented two level large-scale SIRS disease dynamic model will appear elsewhere.

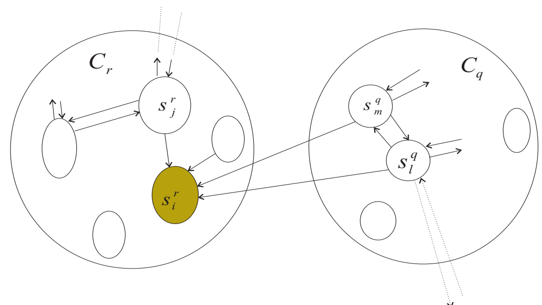

FIGURE 3. Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra reginal accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system [17, 18]. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites and regions.

5. EXAMPLE

By using the two scale mobility model [1], the mobility dynamic structure determined by the respective intra and interregional mobility data recorded in Tables 1& 2 [1, Section 6], and also the single-scale simulated data for an influenza pandemic study in [20], we develop a two-scale SIR influenza epidemic model. The compartmental framework for the SIR epidemic model is exhibited in Figure 2, where $\eta_i^r = \alpha_i^r = 0, \forall r \in I(1, M), i \in I(1, n_r)$. Furthermore, a diagram illustrating the inter-patch connections in the example for two scale dynamic epidemic model represented in this example is shown in Figure 4. In the absence of intra and interregional mobility return rates, based on the mobility structure and the probabilistic formulation of the mobility process, we simulate intra and interregional mobility return rates. We display the intra and inter-regional mobility return rates in Table 1 and Table 2 respectively.

The following assumptions are made concerning the influenza epidemic process represented in this example: (a_1) Every site in every region has one contact zone, and the contact zones are equivalent in size to the sites. In addition, the daily infection probability ($\beta = 0.6277$) of individuals in the one scale model in [20] is taken to be the probability of infection β_{aib}^{urv}, at all sites $s_a^u, s_i^r, s_b^v, r, u, v = 1, 2, 3; a, b, i = 1, 2, 3$ in the three regions $C_u, u = 1, 2, 3$. That is, $\beta_{aib}^{urv} = 0.6277$. (a_2) All infected individuals in the one scale model in [20], were assumed to be either hospitalized or die from disease related causes. Furthermore, the average hospitalization and disease related death

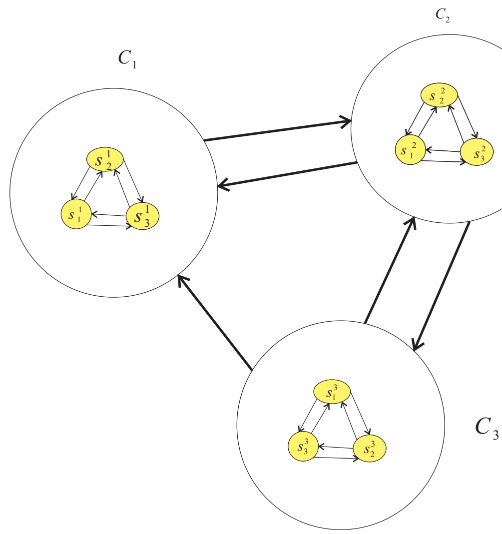

FIGURE 4. A two scale network of three spatial regions $C_r, r = 1, 2, 3$ of human habitation and three interconnected sites $s_i^r, i = 1, 2, 3$ in each region. The arrows represent direction of human mobility and summarize the homogeneities in the epidemic process at each site and region. C_1 & C_2, and C_2 & C_3 are symmetric in the human mobility process. C_1 is a sink for C_3 in human mobility. All sites in each region are completely symmetric in the human mobility process. The details of the two scale human mobility process represented in this example are given in [1].

$(\rho_{12}^{11}, \rho_{13}^{11}, \rho_{21}^{11}, \rho_{23}^{11})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{11}, \rho_{32}^{11})$	(0.013230408,0.001305838)
$(\rho_{12}^{22}, \rho_{13}^{22}, \rho_{21}^{22}, \rho_{23}^{22})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{22}, \rho_{32}^{22})$	(,0.013230408,0.001305838)
$(\rho_{12}^{33}, \rho_{13}^{33}, \rho_{21}^{33}, \rho_{23}^{33})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{33}, \rho_{32}^{33})$	(0.013230408,0.001305838)

TABLE 1. The intra-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 4 are simulated based on the mobility structure and the probabilistic formulation for the mobility process. cf. [1].

rate over all age groups were $\varrho = 0.05067$ and $d = 0.01838$ respectively. By assuming that all hospitalized persons in [20] recovered from the disease and were inoculated against influenza. At all sites of the two scale dynamic epidemic model represented in this example, we take the recovery and disease mortality rate to be $\varrho_a^u = 0.05067$ and $d_a^u = 0.01838, u = 1, 2, 3; a, i = 1, 2, 3$ respectively. (a_3) The population in this example assumed to be remote and lacking the high technological facilities found in the developed world. In the absence of data concerning average birth rates, we use the yearly birth rate data from [21] for the people of the Dominican republic, $B = \frac{births}{1000} = \frac{22.39}{1000}$ as an estimate. Furthermore, we assume this birth rate is the same for all residents of sites in the population. That is, the constant birth rate is

$(\rho_{11}^{12}, \rho_{12}^{12}, \rho 12_{13}, \rho 12_{21}, \rho 12_{22})$	(0.1995,0.035,0.0985,0.007892,0.02748)
$(\rho 12_{23}, \rho 12_{31}, \rho 12_{32}, \rho 12_{33})$	(0.075824,0.04256,0.009616,0.028628)
$(\rho_{11}^{21}, \rho_{12}^{21}, \rho 21_{13}, \rho 21_{21}, \rho 21_{22})$	(0.002096896,0.00175424,0.003460864,0.00043856,0.0001664)
$(\rho 21_{23}, \rho 21_{31}, \rho 21_{32}, \rho 21_{33})$	(0.00071504, 0.001944052,0.00119788,0.0001713912)
$(\rho_{11}^{23}, \rho_{12}^{23}, \rho 23_{13}, \rho 23_{21}, \rho 23_{22})$	(0.018512, 0.03290368,0.0272192,0.04883712,0.00151648)
$(\rho 23_{23}, \rho 23_{31}, \rho 23_{32}, \rho 23_{33})$	(0.0219232, 0.00383316,0.0025404,0.000414644)
$(\rho_{11}^{31}, \rho_{12}^{31}, \rho 31_{13}, \rho 31_{21}, \rho 31_{22})$	(0.001285712,0.00085328,0.001725008,0.0004380944,0.000379536)
$(\rho 31_{23}, \rho 31_{31}, \rho 31_{32}, \rho 31_{33})$	(0.0005991696 ,0.000000371428,0.00000026332,0.000000281252)
$(\rho_{11}^{32}, \rho_{12}^{32}, \rho 32_{13}, \rho 32_{21}, \rho 31_{22})$	(0.0003230096,0.00036224,0.0004619664,0.00043146104,0.0003741576)
$(\rho 32_{23}, \rho 32_{31}, \rho 32_{32}, \rho 32_{33})$	(0.00059126136, 0.000498339428,0.00042838332,0.000070993252)

Table 2. The inter-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 4 are simulated based on the mobility structure and the probabilistic formulation for the mobility process. cf. [1].

$B_a^u = \frac{births}{1000} = \frac{22.39}{1000}$ per year, for $u = 1,2,3$; $a, i = 1,2,3$. (a_4) In addition, using the average life span of the people of Dominican Republic [22], the natural death rate of the residents at all sites and regions are the same and is calculated as the reciprocal of the average life span of individuals in the population, that is, $\delta_a^u = \frac{1}{77.15 \times 365}$, $u = 1,2,3$; $a, i = 1,2,3$ per day.

Using the standard simple Euler method deterministic approximation scheme [47], we generate the trajectories for the residents of sites s_1^1, s_1^2 and s_1^3in regions C_1, C_2 and C_3 respectively, for the different population diseases classification (S, I, R), and current locations at some sites in the intra and inter-regional accessible domain of the sites. The solutions are displayed in Figure 5, Figure 6 and Figure 7 respectively. We note that the following initial conditions were used: for $r, u \in I(1,3), i, a \in I(1,3)$,

$$S_{ia}^{ru}(0) = \begin{cases} 9, for & r = u, i = a \\ 8, for & r = u, i \neq a \\ 7, for & r \neq u, \end{cases}$$

$$I_{ia}^{ru}(0) = \begin{cases} 6, for & r = u, i = a \\ 4, for & r = u, i \neq a \\ 3, for & r \neq u \end{cases}$$

and $R_{ia}^{ru}(0) = 2, \forall r, u, i, a \in I(1,3)$. Furthermore, the trajectories were generated over the time interval $t \in [0,1]$.

6. **CONCLUSION**

The recent high technological changes and scientific developments have led to many\variant structure type inter-patch connections interactions in the global human population. This has further afforded efficient mass flow of human beings, animals, goods and equipments between patches thereby causing the appearance of new disease strains and infectious agents at non-endemic zones. The presented two-scale network disease dynamic model characterizes the dynamics of an SIRS epidemic in a population with various scale levels created by the heterogeneities in the population. Furthermore, the SIRS epidemic has a proportional transfer to the susceptible class immediately after the infectious period. This work provides a mathematical and

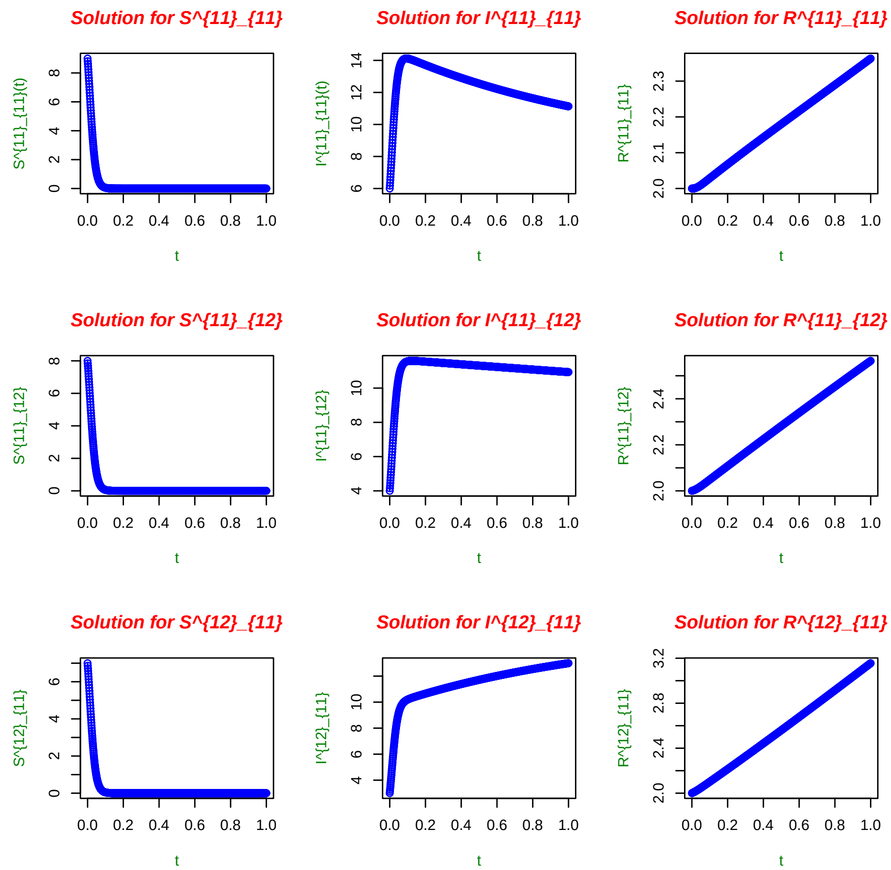

FIGURE 5. Trajectories of the disease classification (S, I, R) for residents of site s_1^1 in region C_1 at their current location in the two-scale spatial patch dynamic structure. Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^1 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_2^1 in home region C_1. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^1 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_1^2 in region C_2. These figures reflect the behavior of the disease affecting the residents of site s_1^1 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states.

probabilistic algorithmic tool to develop different levels of nested type disease transmission rates in the framework of the network-centric deterministic type dynamic equations.

The model validation results are developed and a positively invariant set for the dynamic model is defined. The detailed global asymptotic stability results of the disease free equilibrium are also exhibited in this paper by Lypunov direct method via characterizing the construction of the two scale dynamic structure motivated Lypunov

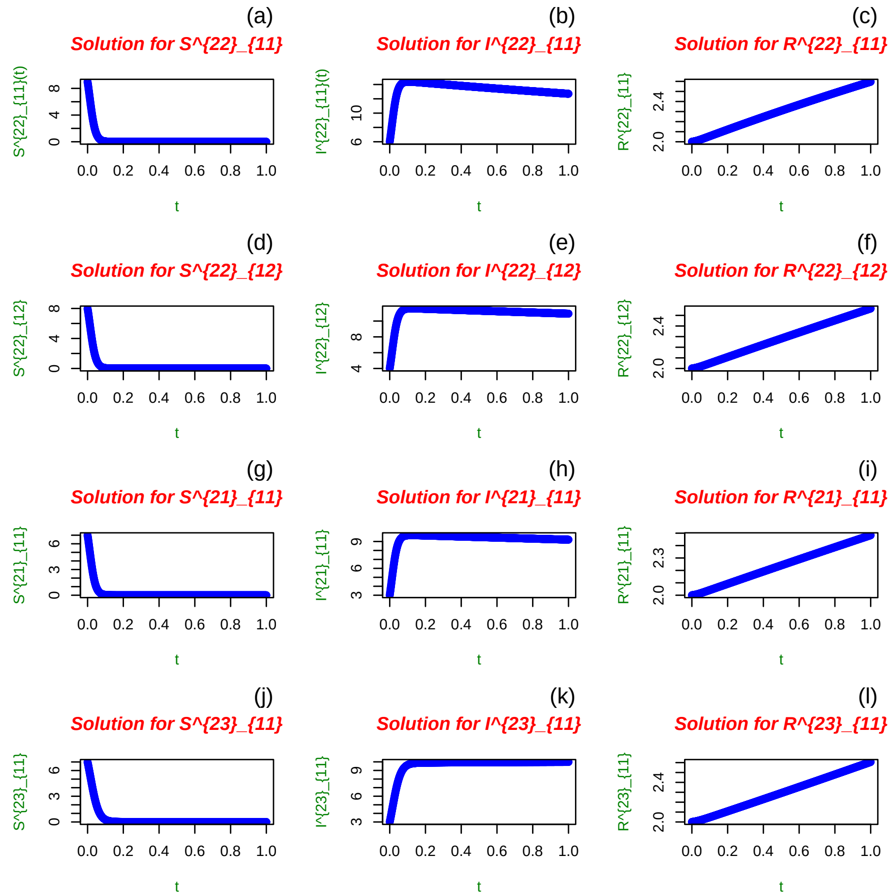

FIGURE 6. Trajectories of the disease classification (S, I, R) for residents of site s_1^2 in region C_2 at their current location in the two-scale spatial patch dynamic structure. Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^2 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_2^2 in home region C_2. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^2 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^3 in region C_3. These last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^2 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states.

function. Moreover, the system parameter dependent threshold values controlling the global asymptotic stability of the disease free equilibrium are developed. In fact the sufficient conditions are algebraically simple, computationally attractive and easy to interpret. Furthermore, a deduction to the global asymptotic stability results for a simple real life scenario is illustrated. Further detail study of the SIRS disease dynamic model the two scale network dynamic mobility structure real life scenarios

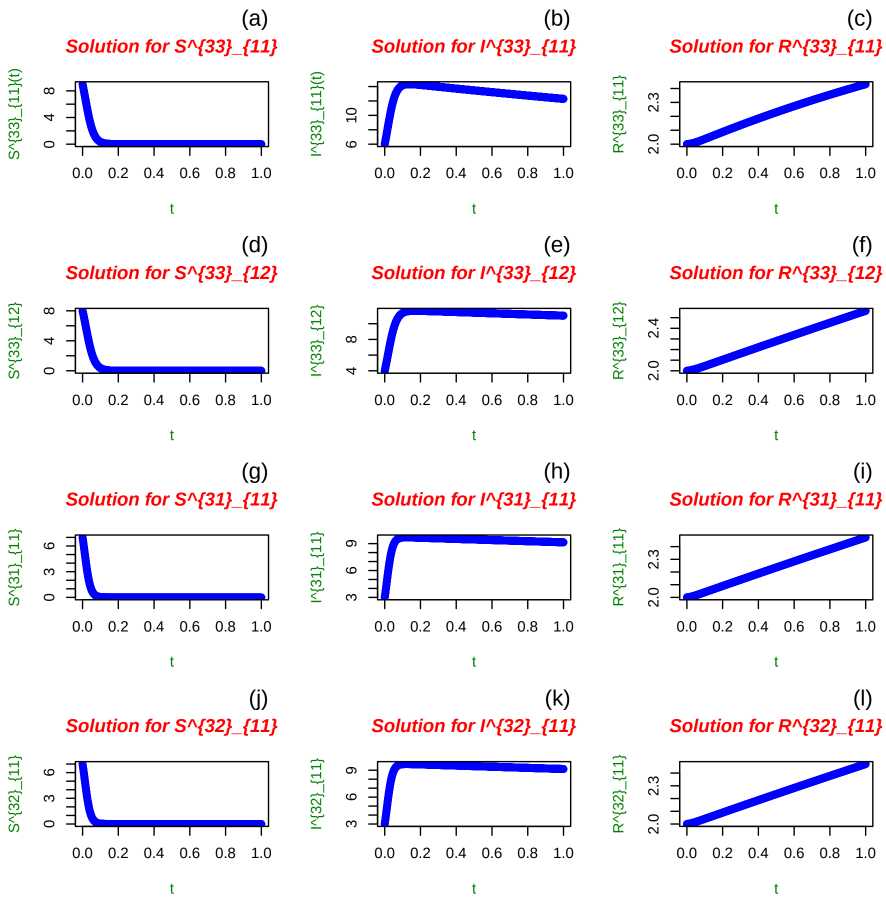

FIGURE 7. Trajectories of the disease classification (S, I, R) for residents of site s_1^3 in region C_3 at their current location in the two-scale spatial patch dynamic structure. Figures (a),(b) & (c) represent the trajectories of the different disease classes of residents of site s_1^3 at home. Figures (d),(e) & (f) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_2^3 in home region C_3. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^3 at the intra-regional level. Figures (g),(h) & (i) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^1 in region C_1. Figures (j),(k) & (l) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^2 in region C_2. The last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^3 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious(I) populations saturate to their equilibrium states.

will appear elsewhere. Simulation results for an SIR influenza epidemic represented by the two-scale network dynamic epidemic model for a specific scenario having a dynamic structure parallel to the earlier study [1] is also presented. A further detailed study of the endemic behavior of the epidemic process by examining the stability of the ideal endemic equilibrium of the dynamic epidemic model will also appear else where. In addition, a detailed study of the stochastic and hereditary features of the infectious agent such as the time-lag to infectiousness of exposed individuals in the population is currently underway and it will also appear elsewhere.

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REFERENCES

[1] D. Wanduku and G. S. Ladde (2010). A two-scale network dynamic model for human mobility process, Math. Biosci. Volume 229 (1) pages 1–15,

[2] Patterson, K. D. (1986). Pandemic Influenza 1700-1900, Totowa, NJ: Rowman & Littlefield

[3] van Hartesveldt, F. R. (Ed.) (1992a). Introduction, in The 1918-1919 Pandemic of Influenza: The Urban Impact in the Western World, Lewiston, NY: The Edwin Mellen Press, pp. 1–12.

[4] Stacey Knobler, Adel Mahmoud, Stanley Lemon, and Leslie Pray (2006), The Impact of Globalization on Infectious Disease Emergence and Control: Exploring the Consequences and Opportunities, Workshop Summary - Forum on Microbial Threats, Washington D.C.: The National Acedemies Press.

[5] L. Rvachev and I. Longini, A mathematical model for the global spread of influenza, Math Biosci 75 (1985), pp. 3–22.

[6] I. Longini, A mathematical model for predicting the geographic spread of new infectious agents, Math Biosci 90 (1988), pp. 367–383.

[7] W. Wang and G. Mulone, Threshold of disease transmission in a patch environment, J Math Anal Appl 285 (2003), pp. 321–335.

[8] W. Wang and X.-Q. Zhao, An epidemic model in a patchy environment, Math Biosci 190 (2004), pp. 97–112.

[9] W. Wang and X.-Q. Zhao, An age-structured epidemic model in a patchy environment, SIAM J Appl Math 65 (2005), pp. 1597–1614.

[10] C. Cosner, J. C. Beier, R.S.Cantrell, D. Impoinvil, L. Kapitanski, M. D. Potts, A.Troyo, and S. Ruan (2009), The effects of human movement on the persistence of vector borne diseases, Journal of Theoretical Biology Vol. 258, 550–560.

[11] J. Arino and P. Van den Driessche, A multi-city epidemic model, Math Popul Stud 10 (2003), pp. 175–193.

[12] J. Arino, J. R. Davis, D. Hartley, R. Jordan, J. M. Miller and P. van den Driessche, A multi-species epidemic model with spatial dynamics, Mathematical Medicine and Biology 22 (2005), pp. 129–142

[13] Watts DJ, Muhamad R, Medina DC, and Dodds PS. 2005. Multiscale, resurgent epidemics in a hierarchical metapopulation model. Proceedings of the National Academy of Sciences, USA 102: 11157–11162.

[14] Sattenspiel, L., and Dietz, K., 1995. A structured epidemic model incorporating geographic mobility among regions. Mathematical Biosciences 128, 7191

[15] L. Sattenspiel and D. A. Herring, Structured epidemic models and the spread of influenza in the central Canada subarctic, Human Biol 70 (1998), pp. 91–115

[16] L. Sattenspiel and D. A. Herring, Simulating the effect of quarantine on the spread of the 1918–1919 flu in central Canada, Bull Math Biol 65 (2003), pp. 1–26.

[17] G. S. Ladde, Cellular Systems-II. Stability of Compartmental Systems, Math. Biosci. 30 (1976), 1–21

[18] G. S. Ladde, Cellular Systems-I. Stability of Chemical Systems, Math Biosci. 29 (3/4) (1976), 309–330

[19] G. S. Ladde, Competitive Processes and Comparison Differential Systems, Trans. of American Mathematical Society, 221 (1976), 391–402

[20] Haber MJ, Shay DK, Davis XM, Patel R, Jin X, Weintraub E., Effectiveness of interventions to reduce contact rates during a simulated influenza pandemic. Emerg Infect Dis. (2007), vol. 13 (4).

[21] CIA World Factbook, http://www.indexmundi.com/g/g.aspx?c=dr&v=25

[22] CIA World Factbook, https://www.cia.gov/library/publications/the-world-factbook/rankorder/2102rank.html

[23] 2009 H1N1 Flu: International Situation Update http://www.cdc.gov/h1n1flu/updates/international/

[24] B. Buonomo, and S. Rionero: On the Lyapunov stability for SIRS epidemic models with general nonlinear incidence rate, Applied Mathematics and Computation, Volume 217, Issue 8, 15 December 2010, Pages 4010–4016

[25] Jianhua Pang, and Jing-an Cui, "An SIRS Epidemiological Model with Nonlinear Incidence Rate Incorporating Media Coverage," icic, vol. 3, pp.116–119, 2009 Second International Conference on Information and Computing Science, 2009

[26] Y. Jin, W. Wang and S. Xiao, An SIRS model with a nonlinear incidence rate, Chaos, Solitons Fractals 34 (2007), pp. 1482–1497.

[27] A. Korobeinikov, Lyapunov functions and global stability for SIR and SIRS epidemiological models with non-linear transmission, Bull. Math. Biol. 30 (2006), pp. 615–626.

[28] A. Korobeinikov, Global properties of infectious disease models with nonlinear incidence, Bull. Math. Biol. 69 (2007), pp. 1871–1886

[29] Zhixing Hu,Ping Bi, Wanbiao Ma, and Shigui Ruan: Bifurcations of an SIRS Epidemic Model with nonlinear incidence rate, Discrete And Continuous Dynamical Systems Series B, Volume 15, Number 1, January 2011, pp. 93–112

[30] Bifurcation Analysis of an SIRS Epidemic Model with Generalized Incidence M. E. Alexander and S. M. Moghadas SIAM Journal on Applied Mathematics Vol. 65, No. 5 (May - Jul., 2005), pp. 1794–1816

[31] Fuzhong Nian and Xingyuan Wang,Efficient immunization strategies on complex networks, Journal of Theoretical Biology, Volume 264, Issue 1, 7 May 2010, Pages 77–83

[32] W. Tan and X. Zhu, A stochastic model for the HIV epidemic in homosexual populations involving age and race, Math. Comput. Modelling 24 (1996), pp. 67–105.

[33] W. Tan and X. Zhu, A stochastic model of the HIV epidemic for heterosexual transmission involving married couples and prostitutes: I. The probabilities of HIV transmission and pair formation, Math. Comput. Modelling 24 (1996), pp. 47–107.

[34] Nirav Dalala, David Greenhalgh, Xuerong Mao, A stochastic model of AIDS and condom use, Journal of Mathematical Analysis and Applications Volume 325, Issue 1, 1 January 2007, Pages 36–53

[35] Nirav Dalal, David Greenhalgh, and Xuerong Mao, A stochastic model for internal HIV dynamics Original Research Article Journal of Mathematical Analysis and Applications, Volume 341, Issue 2, 15 May 2008, Pages 1084–1101

[36] Z. Jin, Z. Ma and M. Han, Global stability of an SIRS epidemic model with delays, Acta Math. Sci. 26 (2006), pp. 291–306.

[37] Ruan et al., 2006 S. Ruan, W. Wang and S.A. Levin, The effect of global travel on the spread of SARS, Mathematical Biosciences and Engineering 3 (2006), pp. 205–218.

[38] Rodriguez and Torres-Sorando, 2001 D.J. Rodriguez and L. Torres-Sorando, Models for infectious diseases in spatially heterogeneous environments, Bulletin of Mathematical Biology 63 (2001), pp. 547–571.

[39] J. Yu, D. Jiang and N. Shi, Global stability of two-group SIR model with random perturbation, J. Math. Anal. Appl. 360 (2009), pp. 235–244.

[40] E. Tornatore, S.M. Buccellato and P. Vetro, Stability of a stochastic SIR system, Phys. A 354 (2005), pp. 111–126.

[41] Zhen jin, Ma zhien, Han Maoan, Global Stability of an SIRS epidemic model with delays, Acta Mathematica Scientia, 26B, 291–306, 2006.

[42] Junjie Chen: An SIRS epidemic model ,Applied Mathematics - A Journal of Chinese Universities(2004), Volume 19, Number 1, 101–108

[43] G. S. Ladde, V. Lakshmikantham, Random Differential Inequalities, Academic press, New York, 1980

[44] W. Walter, Ordinary Differential Equations, Springer-Verlag, 1998, vol.182

[45] R. M. Murray, Z. Li and S. S. Sastry, A Mathematical Introduction to Robotic Manipulation. CRC Press, 1993.

[46] X. Liao, L.Q. Wang, P.Yu, Stability of Dynamical Systems, Elsevier, 2007

[47] Uri M. Ascher, Linda, R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-algebraic Equations, siam 1998