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FUNDAMENTAL PROPERTIES OF A TWO-SCALE NETWORK STOCHASTIC HUMAN EPIDEMIC DYNAMIC MODEL

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ABSTRACT. The non-uniform global spread of emergent infectious diseases of humans is closely interrelated with the large-scale structure of the human population, and the human mobility process in the population structure. The mobile population becomes the vector for the disease. We present an SIRS stochastic dynamic epidemic process in a two scale structured population. The variability caused by the fluctuating environment is assumed to manifest mainly in the transmission process. We investigate the stochastic asymptotic stability of the disease free equilibrium of the scale structured mobile population, under environmental fluctuations and its impact on the emergence, propagation and resurgence of the disease. The presented results are demonstrated by numerical simulation results.

Key Words: Disease-free steady state, Stochastic asymptotic stability, Threshold value, Positively invariant set, Lyapunov function

AMS (MOS) Subject Classification: 39A10

1. INTRODUCTION

The recent advent of high technology in the areas such as communication and transportation has increased the rate and effects of globalization in many aspects of the human species. Of particular importance is the rate of globalization of human infectious diseases [4]. For instance, the 2009 H1N1 flu pandemic [26] is a result of the many inter-patch connections facilitated human transportation. Several mathematical models describing the dynamics of infectious diseases of humans have been studied. Models describing the dynamics of insect vector born diseases [10, 39], influenza [5], HIV [35, 36, 38] and AIDS [37] are studied.

There has also been many studies [5, 6, 8, 9, 10, 16, 17, 12, 13, 29, 40, 41, 15] describing the dynamics of human mobility and disease in meta-populations. Generally, these models can be called multi-group models as they describe the dynamics of diseases in a network of the patches of a meta-population. These models can be further categorized into two general classes based on the modeling approach, namely: Langrangian [40, 41, 15, 10, 16, 17] and Eulerian [12, 13, 29, 8, 9, 5, 6] models. In

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addition, individuals in the population bases on their residence and also their current location. In Langrangian models, individuals do not change their residence, but are allowed to visit other patches in the meta-population. The Eulerian models on the other hand label individuals in the population based only on the current location. Moreover, this model can be considered to be migration models because only the present location of individuals is important.

Many authors have investigated the dynamics of diseases described with SIRS models. A significant portion of SIRS models study the dynamics of the disease under variant incident rates [27, 28, 29, 30, 31, 32, 33]. Using Lyapunov functions, the local nonlinear and global stability of the equilibria is established [27]. By constructing a Lyapunov function based on the structure of the biological system [30, 18, 19], the existence, uniqueness and global stability of the endemic equilibrium are investigated. Furthermore, the bifurcation and stability analysis of the disease free and endemic equilibria, are investigated in [29, 32, 33]. SIRS epidemic models have also been described and studied using complex network of human contacts [34]. In [45], a special SIRS epidemic model is formulated with a proportional direct transfer from the infectious state to the susceptible state immediately after the infectious period.

Stochastic models offer a better representation of the reality. Several stochastic models describing single and multi-group disease dynamics have been investigated [42, 43, 37, 38]. Assuming random perturbation about the endemic equilibrium of a two-group SIR model, the stochastic asymptotic stability of the endemic equilibrium via constructing a Lypunov function according to the structure the system is established in [42]. Also, the stability of the competitive equilibrium [48], disease free equilibrium for SIRS [44] and SIR [43] single-group epidemic models are studied. Furthermore, by showing the existence of nonnegative solution for a stochastic model, the stochastic asymptotic stability behavior of the equilibria is proved in [37, 38, 48, 49].

In more complex meta-population structures, the understanding of the dynamics of infectious diseases is still in the infancy level. This is due to the high degree of heterogeneities and complexity of spatial human population structures. Recently, Wanduku and Ladde [1] characterized various patterns of static behavior of multi-scale structured meta-population human mobility process described by the following Langrangian type dynamic model.

$$\frac{dN_{ii}^{rr}}{dt} = \sum_{k=1}^{n_r} \rho_{ik}^{rr} N_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} N_{il}^{rq} - (\gamma_i^r + \sigma_i^r) N_{ii}^{rr}, \tag{1.1}$$

$$\frac{dN_{ij}^{rr}}{dt} = \sigma_{ij}^{rr} N_{ii}^{rr} - \rho_{ij}^{rr} N_{ij}^{rr}, i \neq j, \tag{1.2}$$

$$\frac{dN_{il}^{rq}}{dt} = \gamma_{il}^{rq} N_{ii}^{rr} - \rho_{il}^{rq} N_{il}^{rq}, r \neq q, \tag{1.3}$$

$$i \in I(1, n_r), \quad l \in I_i^r(1, n_q); \quad r, q \in I^r(1, M),$$

where for all $r, u \in I(1, M), i \in I(1, n_r), a \in I(1, n_u)$, N_{ia}^{ru} is the number of residents of site s_i^r in region C_r visiting site s_a^u in region C_u. Furthermore, the scale of this human mobility dynamic model is two, where the scale represents the intra and inter-regional levels of human interaction. All the parameters in (1.1)–(1.3) are nonnegative. Moreover, σ_{ij}^{rr} and γ_{il}^{rq} are the intra and inter-regional visiting rates of residents of site s_i^r in region C_r to sites s_j^r and s_l^q in regions C_r and C_q respectively. In addition, ρ_{ij}^{rr} and ρ_{il}^{rq} are the intra and inter-regional return rates of residents of site s_i^r in region C_r, from sites s_j^r and s_l^q in regions C_r and C_q respectively. The probabilistic formulation of these mobility rates is exhibited in [1].

In this paper we incorporate the multi-scale structured meta-population human mobility process (1.1)–(1.3) into an SIRS human epidemic model under the influence of random environmental fluctuations. The resulting two-scale network structured SIRS human epidemic stochastic dynamic model is an extension, expansion and generalization of the structured deterministic epidemic model [15], under the influence of mobility process.The presented stochastic two-scale network human dynamic epidemic process is described by a large-scale system of Ito-Doob stochastic differential equations. In addition to well defined underlying system parameter domains for disease eradication in the large-scale two level dynamic structure, the results are algebraically simple, computationally attractive and explicit system parameter dependent threshold values. Furthermore, the presented simulation results exhibit the fact that the human mobility structure of the two-scale network dynamic epidemic model is isomorphic to the human mobility structure of the simulated example in [1].

The work is organized as follows. In Section 2 we describe the general stochastic SIRS epidemic process under the influence of mobility process [46]. In Section 3, the model validation is exhibited. The existence and asymptotic stability of the disease free equilibrium is shown in Section 4. We present simulation results in Section 5. Finally a few conclusions are drawn in Section 6.

2. LARGE SCALE TWO LEVEL SIRS EPIDEMIC PROCESS

In this section, we define the structure of the SIRS epidemic dynamic process in the two-scale network population dynamic structure. The human mobility dynamic structure of the intra and inter-regional levels of the SIRS epidemic dynamic model of this study are exhibited in [1, Fig. 1] and [1, Fig. 2] respectively. Furthermore, the characterization of the human mobility hierarchic process in the two-scale population dynamic structure is also exhibited in [1]. The general SIRS disease structure with dual conversions to the susceptible class from the infectious and immune populations

exhibited in this study is inspired by the work [45]. We make the following definitions related to the SIRS disease process.

Definition 2.1 (Endemic population decomposition and aggregation)**.** For each $r \in I(1, M)$, let $i \in I_i^r(1, n_r)$. The total population N_{i0}^{rr} of residents of site s_i^r at time t is distributed among the sites in their intra and inter regional domain $C(s_i^r)$, and it is partitioned into three general disease compartments namely, susceptible (S), infectious (I) and removals (R) (those who were previously sick and have acquired immunity from the disease). That is, A_{il}^{rq} is the number of residents of site s_i^r whose disease status is of type A, $A \in \{S, I, R\}$, and are visiting to site s_l^q, $l \in I_i^q(1, n_q)$ in region C_q, where $q \in I^r(1, M)$. Furthermore, when $r = q$, A_{ik}^{rr} is the number of residents of site s_i^r with disease status $A \in \{S, I, R\}$, and are visiting to site $s_k^r, k \in I_i^r(1, n_r)$ in their home region C_r. Moreover, when $k = i$, A_{ii}^{rr} is the number of residents of site s_i^r who have disease status of type A, $A \in \{S, I, R\}$ and remain as permanent residents at their home site. Hence N_i^{rr} is given by

$$N_{i0}^{rr} = S_{i0}^{rr} + I_{i0}^{rr} + R_{i0}^{rr}, \tag{2.1}$$

where

$$S_{i0}^{rr} = \sum_{q=1}^{M} \sum_{k=1}^{n_q} S_{ik}^{rq}, \quad I_{i0}^{rr} = \sum_{q=1}^{M} \sum_{k=1}^{n_q} I_{ik}^{rq}, \quad \text{and} \quad R_{i0}^{rr} = \sum_{q=1}^{M} \sum_{k=1}^{n_q} R_{ik}^{rq}. \tag{2.2}$$

Remark 2.1. We note that the effective population $eff(N_{i0}^{rr})$ present at the site s_i^r at anytime is different from the census population or the total number of residents N_{i0}^{rr} (2.1) with permanent residence site s_i^r. At anytime t, the effective community size of site s_i^r is made up of the permanent residents of site s_i^r and all visitors of to site s_i^r. This is as given below

$$eff(N_i^{rr}) = \sum_{q=1}^{M} \sum_{k=1}^{n_q} S_{ki}^{qr} + \sum_{q=1}^{M} \sum_{k=1}^{n_q} I_{ki}^{qr} + \sum_{q=1}^{M} \sum_{k=1}^{n_q} R_{ki}^{qr}. \tag{2.3}$$

$eff(N_i^{rr})$ represents the population that is at risk for infection at site s_i^r and it is the population size resulted by the mobility process in the two-scale network structure.

Definition 2.2 (Disease Transmission Process)**.** The disease transmission process in any site s_i^r in region C_r in a mobile population necessitates: (1) a susceptible person to travel from site s_k^u in region C_u to site s_i^r, ($u = r$ and $k = i$ if there is no traveling), (2) an infectious person traveling from site s_l^q in region $C_q, q \neq r$ to site s_i^r, (3) the susceptible and infectious persons meeting at a contact zone z (which may be the home, market place or recreational facility etc) in site s_i^r with a probability p of a person being at a zone z at anytime t, and (4) β is the probability of the infectious agent being transmitted from the infectious person to the susceptible person knowing that the contact between the susceptible and the infectious individual took place.

Let n_{ri} be the number of contact zones denoted by $z^r_{i_b}$, $b \in \{1, 2, \ldots, n_{r_i}\} \equiv I(1, n_{r_i})$ at each site s^r_i. Furthermore, let $p^r_{i_b}$ be the probability that a member of the effective population would be in a zone $z^r_{i_b}$ at a time t; in addition, we assume that the events of visiting contact zones are independent, and the probability $p^r_{i_b}$ of being in a given zone $z^r_{i_b}$ is independent of the permanent residence of the individual. In each zone $z^r_{i_b}$, there is random mixing and transmission of the infectious agent from an infectious person to a susceptible person via a direct contact between the two individuals. Moreover, let β^{ruv*}_{ikj} be the probability that the transmission takes place given that the contact occurs in any zone $z^r_{i_b}$, $\forall\, b \in I(1, n_{r_i})$ in site s^r_i between a susceptible S^{ur}_{ki} from site s^u_k in region C_u and an infectious individual I^{vr}_{mi} from site s^v_m in region C_v. Then the infectious rate (average number of contacts per individual per unit time required to transmit the disease), $\beta^{ruv*}_{i_bkm}$, in zone $z^r_{i_b}$ between S^{ur}_{ki} and I^{vr}_{mi} is given by

$$\beta^{ruv*}_{i_bkm} = (p^r_{i_b})^2 \beta^{ruv*}_{ikm}, \tag{2.4}$$

whenever $v, u \in I(1, M)$, and $v \neq u$. The infection process in zone $z^r_{i_b}$ is illustrated by the following transition.

$$S^{ur}_{ki} + I^{vr}_{mi} \xrightarrow{\beta^{ruv*}_{i_bkm}} I^{ur}_{ki} + I^{vr}_{mi}. \tag{2.5}$$

Hence, the net conversion rate to the infectious class from the susceptible class during the disease transmission process at the site s^r_i in region C_r of the meta-population with M regions is given by

$$\sum_{v=1}^{M}\sum_{u=1}^{M}\sum_{m=1}^{n_v}\sum_{k=1}^{n_u}\sum_{b=1}^{n_{r_i}} \beta^{ruv*}_{i_bkm} I^{vr}_{mi} S^{ur}_{ki} \tag{2.6}$$

We set

$$\beta^{ruv}_{ikm} = \sum_{b=1}^{n_{r_i}} \beta^{ruv*}_{i_bkm} \tag{2.7}$$

We further assume that the disease status of an individual in the population does not affect travel rates and the mobility pattern.

A diagram illustrating the disease transmission and mobility processes in the two scale dynamic structure described in Definition 2.2 is exhibited in Figure 1.

Definition 2.3 (Acquisition and Loss of Immunity Process)**.** Environmental conditions changes impact the immunity systems of individuals in the large scale two level population dynamic structure. This leads to dependence of the acquisition and loss of immunity rates of residents of all sites in all regions in the two-scale structured population, on the current locations of the residents in the population dynamic structure. In each site s^r_i, let $\frac{1}{\varrho^r_i}$ be the average active infectious period of infected individual

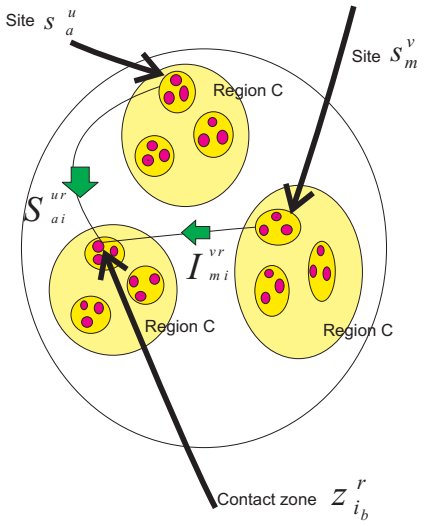

FIGURE 1. Shows the movement of susceptible (S_{ai}^{ur}) and infective (I_{mi}^{vr}) from arbitrary home site s_a^u in region C_u and from site s_m^v in region C_v, to visit an arbitrary contact zone $z_{i_b}^r$ in site s_i^r, which is in region C_r. Disease transmission takes place in zone $z_{i_b}^r$.

(I) who recovered from the disease and acquired immunity (R),immediately after the infectious period. Also, let $\frac{1}{\eta_i^r}$ be the average infectious period of infected person in site s_i^r, who is recovered from the disease and become susceptible (S), immediately, after the infectious period. Furthermore, let $\frac{1}{\alpha_i^r}$ be the average immunity period of removal person (R) in site s_i^r, who has lost his/her their immunity and become susceptible (S) again immediately after the immunity period. The recovery process of an infected person in site s_i^r as well as the loss of immunity of a removal person is illustrated in the following disease transition processes:

$$I_{ki}^{ur} \xrightarrow{\varrho_i^r} R_{ki}^{ur}, \quad I_{ki}^{ur} \xrightarrow{\eta_i^r} S_{ki}^{ur}, \quad R_{ki}^{ur} \xrightarrow{\alpha_i^r} S_{ki}^{ur}, \tag{2.8}$$

for $u \in I(1, M)$ and $k \in I(1, n_u)$.

Definition 2.4 (Population Demography)**.** The current SIRS infectious disease involves time scales that are comparable with the life-time of individuals in the population. Furthermore, all births occur at home site and deaths occur at current locations of residents in the two-scale population structure. Let B_i^r be a constant birthrate of the human population at site s_i^r and at time t. We assume that every new born is a susceptible and becomes a resident of the site of birth. Let δ_i^r be the per capita natural mortality rate, and let d_i^r be the per capita disease related mortality rate of all members of the effective population at site s_i^r.

A compartmental framework illustrating the different process and stages in the SIRS epidemic described above is exhibited in Figure 2.

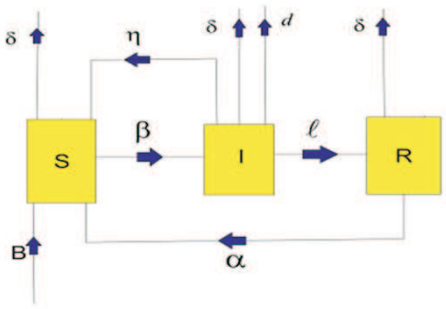

FIGURE 2. Compartmental framework summarizing the transition stages in the SIRS epidemic process. All the parameters presented in this figure are define in Section 2 for particular sites and regions.

From Definitions 2.1–2.4, the complete SIRS epidemic model under the influence of a large scale two-level population mobility process[1] is described by:

(2.9)
$$\frac{dS_{il}^{rq}}{dt} = \begin{cases} [B_i^r + \sum_{k=1}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} \\ \quad -(\gamma_i^r + \sigma_i^r + \delta_i^r) S_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}], \quad \text{for } q = r,\ l = i \\ [\sigma_{ij}^{rr} S_{ii}^{rr} + \eta_j^r I_{ij}^{rr} + \alpha_j^r R_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) S_{ij}^{rr} \\ \quad - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}], \quad \text{for } q = r, l = j,\ j \neq i, \\ [\gamma_{il}^{rq} S_{ii}^{rr} + \eta_l^q I_{il}^{rq} + \alpha_l^q R_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q) S_{il}^{rq} \\ \quad - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}], \quad \text{for } q \neq r, \end{cases}$$

(2.10)
$$\frac{dI_{il}^{rq}}{dt} = \begin{cases} [\sum_{k=1}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \eta_i^r I_{ii}^{rr} - \varrho_i^r I_{ii}^{rr} \\ \quad -(\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) I_{ii}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur}], \quad \text{for } q = r, l = i \\ [\sigma_{ij}^{rr} I_{ii}^{rr} - \eta_j^r I_{ij}^{rr} - \varrho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) I_{ij}^{rr} \\ \quad + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur}], \quad \text{for } q = r,\ l = j, i \neq j, \\ [\gamma_{il}^{rq} I_{ii}^{rr} - \eta_l^q I_{il}^{rq} - \varrho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q + d_l^q) I_{il}^{rq} \\ \quad + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq}], \quad \text{for } q \neq r, \end{cases}$$

(2.11)
$$\frac{dR_{il}^{rq}}{dt} = \begin{cases} [\sum_{k=1}^{n_r} \rho_{ik}^{rr} R_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} R_{il}^{rq} + \varrho_i^r I_{ii}^{rr} - (\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r) R_{ii}^{rr}], \\ \quad \text{for } q = r,\ l = i \\ [\sigma_{ij}^{rr} R_{ii}^{rr} + \varrho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \alpha_j^r + \delta_j^r) R_{ij}^{rr}], \quad \text{for } q = r,\ l = j,\ i \neq j, \\ [\gamma_{il}^{rq} R_{ii}^{rr} + \varrho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \alpha_l^q + \delta_l^q) R_{il}^{rq}], \quad \text{for } q \neq r, \end{cases}$$

where $i \in I(1, n_r)$, $l \in I_i^r(1, n_q)$; $r \in I(1, M)$, $q \in I^r(1, M)$. Furthermore, the parameters B_i^r, η_a^u, α_a^u, δ_a^u and d_a^u are nonnegative, and ϱ_a^u is positive for $r, u \in I(1, M)$,

$i \in I(1, n_r)$, and $a \in I(1, n_u)$. Also, at time $t = t_0$, and for each $r \in I(1, M)$, and $i \in I(1, n_r)$, $(S^{rr}_{ii}(t_0), S^{rr}_{ij}(t_0), S^{rq}_{il}(t_0)) = (S^{rr}_{ii0}, S^{rr}_{ij0}, S^{rq}_{il0})$, $(I^{rr}_{ii}(t_0), I^{rr}_{ij}(t_0), I^{rq}_{il}(t_0)) = (I^{rr}_{ii0}, I^{rr}_{ij0}, I^{rq}_{il0})$, $(R^{rr}_{ii}(t_0), R^{rr}_{ij}(t_0), R^{rq}_{il}(t_0)) = (R^{rr}_{ii0}, R^{rr}_{ij0}, R^{rq}_{il0})$, whenever $j \in I^r_i(1, n_r)$ and $l \in I^r_i(1, n_q)$. Furthermore, we denote $n = \sum_{u=1}^{M} n_u$. We now incorporate the effects of the random environmental perturbations into the modeling epidemic dynamic process described in (2.9)–(2.11).

The random fluctuations lead to variabilities in the disease transmission, human mobility, birth and death processes of the system. In this work, we assume that the effects of the fluctuating environment manifest mainly as variations in the infectious rate β. Generally, we represent the variability in the infectious rate by a white noise process as:

$$\beta \to \beta + v\xi(t), \quad dw(t) = \xi(t)dt, \quad \text{and} \quad \text{var}(\beta(t)) = v^2, \tag{2.12}$$

where $\xi(t)$ is the standard white noise process, and $w(t)$ is corresponding normalized Wiener process or a homogenous Brownian motion process with the following properties: $w(0) = 0$, $E(w(t)) = 0$ and $var(w(t)) = t$.

Given $t \geq t_0$, we let $(\Omega, \digamma, P)$ be a complete probability space, and $\digamma_t$ is a filtration (that is sub σ-algebra $\digamma_t$ satisfies the following: given $t_1 \leq t_2 \Rightarrow \digamma_{t_1} \subset \digamma_{t_2}$; $E \in \digamma_t$ and $P(E) = 0 \Rightarrow E \in \digamma_0$), for each $r \in I(1, M)$, and $i \in I(1, n_r)$, the variability in the infectious process at sites s^r_i, s^r_j and s^q_l between a susceptible from site s^u_k and an infective from an arbitrary site s^v_m, can be represented as follows:

$$\begin{aligned}
\beta^{ruv}_{ikm} &\to \beta^{ruv}_{ikm} + v^{ruv}_{ikm}\xi^{ruv}_{ikm}(t), \quad dw^{ruv}_{ikm}(t) = \xi^{ruv}_{ikm}(t)dt \\
\beta^{ruv}_{jkm} &\to \beta^{ruv}_{jkm} + v^{ruv}_{jkm}\xi^{ruv}_{jkm}(t), dw^{ruv}_{jkm}(t) = \xi^{ruv}_{jkm}(t)dt \\
\beta^{quv}_{lkm} &\to \beta^{quv}_{lkm} + v^{quv}_{lkm}\xi^{quv}_{lkm}(t), dw^{quv}_{lkm}(t) = \xi^{quv}_{lkm}(t)dt
\end{aligned} \tag{2.13}$$

and

$$var(\beta^{ruv}_{ikm}(t)) = (v^{ruv}_{ikm})^2, \quad var(\beta^{ruv}_{jkm}(t)) = (v^{ruv}_{jkm})^2, \quad var(\beta^{quv}_{lkm}(t)) = (v^{quv}_{lkm})^2, \tag{2.14}$$

where $q, u, v \in I^r(1, M)$, $k \in I^u_i(1, n_u)$, $m \in I^v_i(1, n_v)$, and $l \in I^r_i(1, n_q)$.

We substitute (2.13) into (2.9)–(2.11), and obtain the following two level large scale stochastic epidemic model under the influence of human mobility process [1]

(2.15)
$$dS_{il}^{rq} = \begin{cases} \big[B_i^r + \sum_{k=1}^{n_r} \rho_{ik}^{rr} S_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} S_{ia}^{rq} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} \\ \quad -(\gamma_i^r + \sigma_i^r + \delta_i^r) S_{ii}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} \big] dt \\ \quad -\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t) \big], \quad \text{for } q=r,\ l=i, \\ \big[\sigma_{ij}^{rr} S_{ii}^{rr} + \eta_j^r I_{ij}^{rr} + \alpha_j^r R_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) S_{ij}^{rr} - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} \big] dt \\ \quad -\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t) \big], \quad \text{for } q=r,\ l=j, j\neq i, \\ \big[\gamma_{il}^{rq} S_{ii}^{rr} + \eta_l^q I_{il}^{rq} + \alpha_l^q R_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q) S_{il}^{rq} \\ \quad - \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} \big] dt - \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t) \big], \quad \text{for } q\neq r, \end{cases}$$

(2.16)
$$dI_{il}^{rq} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} I_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} I_{ia}^{rq} - \eta_i^r I_{ii}^{rr} - \varrho_i^r I_{ii}^{rr} \\ \quad -(\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) I_{ii}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} \big] dt \\ \quad +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{iia}^{rru} S_{ii}^{rr} I_{ai}^{ur} dw_{iia}^{rru}(t) \big], \quad \text{for } q=r,\ l=i \\ \big[\sigma_{ij}^{rr} I_{ii}^{rr} - \eta_j^r I_{ij}^{rr} - \varrho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \delta_j^r + d_j^r) I_{ij}^{rr} + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} \big] dt \\ \quad +\big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{jia}^{rru} S_{ij}^{rr} I_{aj}^{ur} dw_{jia}^{rru}(t) \big], \quad \text{for } q=r,\ l=j,\ j\neq i, \\ \big[\gamma_{il}^{rq} I_{ii}^{rr} - \eta_l^q I_{il}^{rq} - \varrho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \delta_l^q + d_l^q) I_{il}^{rq} \\ \quad + \sum_{u=1}^{M} \sum_{a=1}^{n_u} \beta_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} \big] dt + \big[\sum_{u=1}^{M} \sum_{a=1}^{n_u} v_{lia}^{qru} S_{il}^{rq} I_{al}^{uq} dw_{lia}^{qru}(t) \big], \quad \text{for } q\neq r, \end{cases}$$

(2.17)
$$dR_{il}^{rq} = \begin{cases} \big[\sum_{k=1}^{n_r} \rho_{ik}^{rr} R_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{l=1}^{n_q} \rho_{il}^{rq} R_{il}^{rq} + \varrho_i^r I_{ii}^{rr} - (\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r) R_{ii}^{rr} \big] dt, \\ \quad \text{for } q=r,\ l=i \\ \big[\sigma_{ij}^{rr} R_{ii}^{rr} + \varrho_j^r I_{ij}^{rr} - (\rho_{ij}^{rr} + \alpha_j^r + \delta_j^r) R_{ij}^{rr} \big] dt, \quad \text{for } q=r,\ l=j,\ j\neq i, \\ \big[\gamma_{il}^{rq} R_{ii}^{rr} + \varrho_l^q I_{il}^{rq} - (\rho_{il}^{rq} + \alpha_l^q + \delta_l^q) R_{il}^{rq} \big] dt, \quad \text{for } q\neq r, \end{cases}$$

where $i \in I(1,n_r)$, $l \in I_i^r(1,n_q)$; $r \in I(1,M)$, $q \in I^r(1,M)$; all parameters are as defined before. At time $t=t_0$, for each $r \in I(1,M)$ and $i \in I(1,n_r)$, $(S_{ii}^{rr}(t_0), S_{ij}^{rr}(t_0), S_{il}^{rq}(t_0)) = (S_{ii0}^{rr}, S_{ij0}^{rr}, S_{il0}^{rq})$, $(I_{ii}^{rr}(t_0), I_{ij}^{rr}(t_0), I_{il}^{rq}(t_0)) = (I_{ii0}^{rr}, I_{ij0}^{rr}, I_{il0}^{rq})$, $(R_{ii}^{rr}(t_0), R_{ij}^{rr}(t_0), R_{il}^{rq}(t_0)) = (R_{ii0}^{rr}, R_{ij0}^{rr}, R_{il0}^{rq})$, whenever $j \in I_i^r(1,n_r)$ and $l \in I_i^r(1,n_q)$, where the random variables $(S_{ii}^{rr}(0), S_{ij}^{rr}(0), S_{il}^{rq}(0))$, $(I_{ii}^{rr}(0), I_{ij}^{rr}(0), I_{il}^{rq}(0))$ and $(R_{ii}^{rr}(0), R_{ij}^{rr}(0), R_{il}^{rq}(0))$ are $\mathcal{F}_0$-measurable, and are independent of $w(t)$ whenever $t \geq t_0$.

We express the state of system (2.15)–(2.17) in vector form and use it, subsequently. We denote

$$\begin{aligned} x_{ia}^{ru} &= (S_{ia}^{ru}, I_{ia}^{ru}, R_{ia}^{ru})^T \in \mathbb{R}^3 \\ x_{i0}^{ru} &= (x_{i1}^{ruT}, x_{i2}^{ruT}, \ldots, x_{i,nu}^{ruT})^T \in \mathbb{R}^{3n_u}, \\ x_{00}^{ru} &= (x_{10}^{ruT}, x_{20}^{ruT}, \ldots, x_{n_r0}^{ruT})^T \in \mathbb{R}^{3n_r n_u}, \end{aligned}$$

$$
\begin{aligned}
x_{00}^{r0} &= (x_{00}^{r1T}, x_{00}^{r2T}, \dots, x_{00}^{rMT})^T \in \mathbb{R}^{3n_r \sum_{u=1}^{M} n_u}, \\
x_{00}^{00} &= (x_{00}^{10}, x_{00}^{20}, \dots, x_{00}^{M0})^T \in \mathbb{R}^{3\left(\sum_{r=1}^{M} n_r\right)\left(\sum_{u=1}^{M} n_u\right)}, \qquad (2.18)
\end{aligned}
$$

where $r, u \in I(1, M)$, $i \in I(1, n_r)$, $a \in I_i^r(1, n_u)$. We set $n = \sum_{u=1}^{M} n_u$.

Definition 2.5. 1. p-norm in $\mathbb{R}^{3n^2}$: Let $z_{00}^{00} \in \mathbb{R}^{3n^2}$ be an arbitrary vector defined in (2.18), where $z_{ia}^{ru} = (z_{ia1}^{ru0}, z_{ia2}^{ru0}, z_{ia3}^{ru0})^T$ whenever $r, u \in I(1, M)$, $i \in I(1, n_r)$, $a \in I_i^r(1, n_u)$. The $p-norm$ on $\mathbb{R}^{3n^2}$ is defined as follows

$$
\|z_{00}^{00}\|_p = \left(\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} \sum_{j=1}^{3} |z_{iaj}^{ru0}|^p \right)^{\frac{1}{p}} \qquad (2.19)
$$

whenever $1 \le p < \infty$, and

$$
\bar{z} \equiv \|z_{00}^{00}\|_p = \max_{1 \le r,u \le M, 1 \le i \le n_r, 1 \le a \le n_u, 1 \le j \le 3} |z_{iaj}^{ru0}|, \qquad (2.20)
$$

whenever $p = \infty$. Let

$$
\underline{k} \equiv k_{00\,\min}^{00} = \min_{1 \le r,u \le M, 1 \le i \le n_r, 1 \le a \le n_u} |k_{ia}^{ru}|. \qquad (2.21)
$$

2. Closed Ball in $\mathbb{R}^{3n^2}$: Let $z_{00}^{*00} \in \mathbb{R}^{3n^2}$ be fixed. The closed ball in $\mathbb{R}^{3n^2}$ with center at z_{00}^{*00} and radius $r > 0$ denoted $\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z_{00}^{*00}; r)$ is the set

$$
\bar{\mathfrak{B}}_{\mathbb{R}^{3n^2}}(z_{00}^{*00}; r) = \{ z_{00}^{00} \in \mathbb{R}^{3n^2} : \|z_{00}^{00} - z_{00}^{*00}\|_p \le r \} \qquad (2.22)
$$

3. MODEL VALIDATION RESULTS

We now show that the initial value problem associated with the system (2.15)–(2.17) has a unique solution. We observe that the rate functions of the system are nonlinear and locally Lipschitz continuous with respect to x_{00}^{00} but do not satisfy the linear growth condition. As a result of this the classical existence and uniqueness results [46] are not applicable. Therefore, we use the Lyapunov energy function method (cf. [37, 38, 46, 47]) to prove the existence and uniqueness of solution process of the system. We first state and prove two lemmas that are useful for the proof of the existence and uniqueness result. From (2.15)–(2.17), define the vector $y_{00}^{00} \in \mathbb{R}^{n^2}$ as follows: For $i \in I(1, n_r)$, $l \in I_i^r(1, n_q)$, $r \in I(1, M)$ and $q \in I^r(1, M)$,

$$
\begin{aligned}
y_{ia}^{ru} &= S_{ia}^{ru} + I_{ia}^{ru} + R_{ia}^{ru} \in \mathbb{R}_+ = [0, \infty) \\
y_{i0}^{ru} &= (y_{i1}^{ru}, y_{i2}^{ru}, \dots, y_{i,nu}^{ru})^T \in \mathbb{R}_+^{n_u}, \\
y_{00}^{ru} &= (y_{10}^{ruT}, y_{20}^{ruT}, \dots, y_{n_r 0}^{ruT})^T \in \mathbb{R}_+^{n_r n_u}, \\
y_{00}^{r0} &= (y_{00}^{r1T}, y_{00}^{r2T}, \dots, y_{00}^{rMT})^T \in \mathbb{R}_+^{n_r \sum_{u=1}^{M} n_u}, \\
y_{00}^{00} &= (y_{00}^{10T}, y_{00}^{20T}, \dots, y_{00}^{M0T})^T \in \mathbb{R}_+^{\left(\sum_{r=1}^{M} n_r\right)\left(\sum_{u=1}^{M} n_u\right)}, \qquad (3.1)
\end{aligned}
$$

and obtain

(3.2)

$$dy_{il}^{rq} = \begin{cases} \big[B_i^r + \sum_{k\neq i}^{n_r} \rho_{ik}^{rr} y_{ik}^{rr} + \sum_{q\neq r}^{M} \sum_{a=1}^{n_q} \rho_{ia}^{rq} y_{ia}^{rq} - (\gamma_i^r + \sigma_i^r + \delta_i^r) y_{ii}^{rr} - d_i^r I_{ii}^{rr}\big] dt, \\ \quad \text{for } q = r,\ l = i \\ \big[\sigma_{ij}^{rr} y_{ii}^{rr} - (\rho_{ij}^{rr} + \delta_j^r) y_{ij}^{rr} - d_j^r I_{ij}^{rr}\big] dt, \quad \text{for } q = r,\ a = j \quad \text{and } i \neq j, \\ \big[\gamma_{il}^{rq} y_{ii}^{rr} - (\rho_{il}^{rq} + \delta_l^q) y_{il}^{rq} - d_l^q I_{il}^{rq}\big] dt, \quad \text{for } q \neq r,\ y_{il}^{rq}(t_0) \geq 0, \end{cases}$$

In the following, we show that the solution process of the initial value problem (3.2) is nonnegative. That is for all $t \geq 0$, $y_{ia}^{ru}(t) \geq 0$ is nonnegative, whenever $y_{ia}^{ru}(t_0) \geq 0$.

Lemma 3.1. *Let $r, u \in I(1, M)$, $i \in I^r(1, n_r)$ and $a \in I_i^r(1, n_u)$. For all $t \geq t_0$, from (3.1), if $y_{ia}^{ru}(t_0) \geq 0$, then $y_{ia}^{ru}(t) \geq 0$.*

Proof. It follows from (3.1) and (2.15)–(2.17) that the system (3.2) is of the form $u' = A(t, u)w(t, u), u(t_0) \geq 0$, in [20, equation (8)], and satisfies the quasimonotonicity condition. Furthermore, from Remark 4 in [20], we assert that this system (3.2) has nonnegative solutions whenever $y_{il}^{rq}(0) \geq 0$, $\forall\ i \in I(1, n_r), l \in I_i^r(1, n_q), r \in I(1, M)$, and $q \in I^r(1, M)$. □

Remark 3.1. From the decomposition described in (2.1), we observe that $y_{ia}^{ru}(t) = N_{ia}^{ru} = S_{ia}^{ru}(t) + I_{ia}^{ru}(t) + R_{ia}^{ru}(t)$. Furthermore, that $N_{i0}^{rr} = \sum_{u=1}^{M} \sum_{a=1}^{n_u} y_{ia}^{ru}$. Therefore, Lemma 3.1 established that for any nonnegative initial endemic population, the number of residents of site s_i^r present at home, y_{ii}^{rr}, or visiting any given site s_{ia}^{ru} in any other region C_u, y_{ia}^{ru}, is nonnegative. This implies that the total population of residents of site s_i^r present at home and also visiting sites in regions in their intra and intra-regional accessible domains, $N_{i0}^{rr}(t)$, is nonnegative. Moreover, Lemma 3.1 exibits that the effective population at any site in any region given by (2.3) is nonnegative at all time $t \geq t_0$. Furthermore, $R_+^{n^2} = \{y \in R^{n^2} : y \geq 0\}$ is a self-invariant set with respect to (3.2).

In the following lemma, we use Lemma 3.1 to find an upper bound for the solution process of (2.15)–(2.17)

Lemma 3.2. *Let $\mu = \min_{1\leq u\leq M, 1\leq a\leq n_u}(\delta_a^u)$. If*

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} y_{ia}^{ru}(t_0) \leq \frac{1}{\mu} \sum_{r=1}^{M} \sum_{i=1}^{n_r} B_i^r, \tag{3.3}$$

then

$$\sum_{r=1}^{M} \sum_{u=1}^{M} \sum_{i=1}^{n_r} \sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu} \sum_{r=1}^{M} \sum_{i=1}^{n_r} B_i^r, \quad \textit{for } t \geq 0,\ \textit{a.s.} \tag{3.4}$$

Proof. From 3.1, define

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} dy_{ia}^{ru} = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left[dy_{ii}^{rr} + \sum_{a\neq i}^{n_r} dy_{ia}^{rr} + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r} dy_{ia}^{ru}\right] \tag{3.5}$$

From (2.15)–(2.17) and (3.5), one can see that

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} dy_{ia}^{ru} = \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r - \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(\delta_a^u y_{ia}^{ru} + d_a^u I_{ia}^{ru})\right] dt \tag{3.6}$$

From Lemma 3.1, and (3.6), we have

$$d\left\{\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}\right\} \leq \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r - \mu\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} y_{ia}^{ru}\right] dt \tag{3.7}$$

for a nonnegative differential of t. We note that (3.7) is a first order deterministic differential inequality [46], and its solution is given by

$$\sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \left[\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} y_{ia}^{ru}(t_0)\right] e^{-\mu t} \tag{3.8}$$

Therefore, (3.4) is satisfied provided (3.3) is valid. □

Remark 3.2. From Lemma 3.2, we conclude that a closed ball in R^{3n^2} under the sum norm with radius $r = \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r$ is self-invariant with regard to a two-scale network dynamic of human epidemic process that is under the influence of human mobility process [1].

Prior to presenting the model validation result, we need to establish an auxiliary result. this result provides a fundamental tool in the context of the energy Lyapunov function approach.

Lemma 3.3. *Let us assume that the hypotheses of Lemma* 3.2 *be satisfied. Let* V *be a function defined by* $V : \mathbb{R}_+^{3n^2} \times \mathbb{R}_+ \to \bar{\mathbb{R}}_+$ *as follows*

$$V(x_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{ia}^{ru}), \tag{3.9}$$

where

$$V_{ia}^{ru}(x_{ia}^{ru}) = \left[(S_{ia}^{ru} - 1 - \log S_{ia}^{ru}) + (I_{ia}^{ru} - 1 - \log I_{ia}^{ru}) + (R_{ia}^{ru} - 1 - \log R_{ia}^{ru})\right]. \tag{3.10}$$

Furthermore, let us denote

$$\begin{aligned} M_{001}^{00} &= \max_{1\leq r,q\leq M, q\neq r, 1\leq l\leq n_q} 1 + \frac{S_{il}^{rq}}{I_{il}^{rq}}, \\ M_{002}^{00} &= \max_{1\leq r,q\leq M, q\neq r, 1\leq l\leq n_q} 1 + \frac{(S_{il}^{rq})^2}{(I_{il}^{rq})^2}, \\ N_{00}^{00} &= \max_{1\leq r,q\leq M, q\neq r, 1\leq l\leq n_q} 1 + S_{il}^{rq}, \end{aligned}$$

$$\begin{aligned}
\beta_{000}^{000} &= \max_{1\le r,q,u\le M, q\ne r, 1\le l, a\le n_{q,u}} \beta_{ail}^{urq} \\
v_{000}^{000} &= \max_{1\le r,q,u\le M, q\ne r, 1\le l, a\le n_{q,u}} v_{ail}^{urq} \\
(\rho_{00}^{00}, \alpha_0^0, \delta_0^0, d_0^0, \sigma_{i0}^{00}, \varrho_0^0) &= \max_{1\le r,u\le M, 1\le a\le n_u} (\rho_{ia}^{ru}, \alpha_a^u, \delta_a^u, d_a^u, \sigma_{ia}^{ru}, \varrho_a^u), \qquad (3.11)
\end{aligned}$$

Then there exists $\tilde{K} > 0$ *such that*

$$dV(x_{00}^{00}) \le \tilde{K}dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1 - \frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right) v_{aib}^{urv} I_{ba}^{vu} dw_{aib}^{urv} \qquad (3.12)$$

Proof. For $r, u \in I(1, M)$, $i \in I^r(1, n_r)$ and $a \in I_i^r(1, n_u)$, under the assumptions of Lemma 3.2, and the definitions of S_{ia}^u, I_{ia}^u and R_{ia}^u, the function defined in (3.9) belongs to $V \in C^{2,1}(\mathbb{R}_+^{3n^2} \times \mathbb{R}_+, \bar{\mathbb{R}}_+)$. Moreover, we rewrite (3.9) as

$$\begin{aligned}
V(x_{00}^{00}) &= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{00}^{00}), \\
&= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{V_{ii}^{rr}(x_{00}^{00}) + \sum_{a\ne i}^{n_r} V_{ia}^{rr}(x_{00}^{00}) + \sum_{u\ne r}^{M}\sum_{a=1}^{n_u} V_{ia}^{ru}(x_{00}^{00})\right\}, \qquad (3.13)
\end{aligned}$$

where

$$V_{ia}^{ru}(x_{00}^{00}) = (S_{ia}^{ru} - 1 - \log S_{ia}^{ru}) + (I_{ia}^{ru} - 1 - \log I_{ia}^{ru}) + (R_{ia}^{ru} - 1 - \log R_{ia}^{ru}). \qquad (3.14)$$

From (3.13) and (3.14), it follows that

$$dV(x_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\left\{dV_{ii}^{rr}(x_{00}^{00}) + \sum_{a\ne i}^{n_r} dV_{ia}^{rr}(x_{00}^{00}) + \sum_{u\ne r}^{M}\sum_{a=1}^{n_u} dV_{ia}^{ru}(x_{00}^{00})\right\}, \qquad (3.15)$$

where

$$\begin{aligned}
dV_{ia}^{ru}(x_{00}^{00}) &= \left[\left(1 - \frac{1}{S_{ia}^{ru}}\right) dS_{ia}^{ru} + \frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right] \\
&\quad + \left[\left(1 - \frac{1}{I_{ia}^{ru}}\right) dI_{ia}^{ru} + \frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right] \\
&\quad + \left[\left(1 - \frac{1}{R_{ia}^{ru}}\right) dR_{ia}^{ru} + \frac{1}{2(R_{ia}^{ru})^2}(dR_{ia}^{ru})^2\right]. \qquad (3.16)
\end{aligned}$$

In the following, by considering positive differential of t ($0 < \Delta t \approx dt$), using the nature of the rate coefficients of (2.15)–(2.17) and definitions (3.11), we carefully estimate the three terms in the righthand side of (3.16). This is achieved by the usage of nested argument process. □

Site level: the estimates on terms in the righthand side of (3.16) for the case of $u = r$, and $a = i$

$$\left[\left(1 - \frac{1}{S_{ii}^{rr}}\right) dS_{ii}^{rr} + \frac{1}{2(S_{ii}^{rr})^2}(dS_{ii}^{rr})^2\right]$$

$$\leq \Bigg\{ B_i^r + \sum_{b=1}^{n_r} \rho_{ib}^{rr} S_{ib}^{rr} + \sum_{v\neq r}^{M} \sum_{b=1}^{n_v} \rho_{ib}^{rv} S_{ib}^{rv} + \eta_i^r I_{ii}^{rr} + \alpha_i^r R_{ii}^{rr} + (\gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r)$$
$$+ \sum_{v=1}^{M} \sum_{b=1}^{n_v} \beta_{iib}^{rrv} I_{bi}^{vr} + \frac{1}{2} \sum_{v=1}^{M} \sum_{b=1}^{n_v} (v_{iib}^{rrv})^2 (I_{bi}^{vr})^2 \Bigg\} dt$$
$$+ \left[(1 - S_{ii}^{rr}) \sum_{v=1}^{M} \sum_{b=1}^{n_v} v_{iib}^{rrv} I_{bi}^{vr} dw_{iib}^{rrv} \right], \tag{3.17}$$

$$\left[\left(1 - \frac{1}{I_{ii}^{rr}} \right) dI_{ii}^{rr} + \frac{1}{2(I_{ii}^{rr})^2} (dI_{ii}^{rr})^2 \right] \leq \Bigg\{ \sum_{b=1}^{n_r} \rho_{ib}^{rr} I_{ib}^{rr} + \sum_{v\neq r}^{M} \sum_{b=1}^{n_v} \rho_{ib}^{rv} I_{ib}^{rv}$$
$$+ (\varrho_i^r + \eta_i^r + \gamma_i^r + \sigma_i^r + \delta_i^r + d_i^r) + \sum_{v=1}^{M} \sum_{b=1}^{n_v} \beta_{iib}^{rrv} \frac{S_{ii}^{rr} I_{bi}^{vr}}{I_{ii}^{rr}}$$
$$+ \frac{1}{2(I_{ii}^{rr})^2} (S_{ii}^{rr})^2 \sum_{v=1}^{M} \sum_{b=1}^{n_v} (v_{iib}^{rrv})^2 (I_{bi}^{vr})^2 \Bigg\} dt$$
$$+ \left(S_{ii}^{rr} - \frac{S_{ii}^{rr}}{I_{ii}^{rr}} \right) \left[\sum_{v=1}^{M} \sum_{b=1}^{n_v} v_{iib}^{rrv} I_{bi}^{vr} dw_{iib}^{rrv} \right], \tag{3.18}$$

and

$$\left(1 - \frac{1}{R_{ii}^{rr}} \right) dR_{ii}^{rr} \leq \left[\sum_{b=1}^{n_r} \rho_{ib}^{rr} R_{ib}^{rr} + \sum_{v\neq r}^{M} \sum_{b=1}^{n_v} \rho_{ib}^{rv} R_{ib}^{rv} + \varrho_i^r I_{ii}^{rr} + (\gamma_i^r + \alpha_i^r + \sigma_i^r + \delta_i^r + d_i^r) \right] dt. \tag{3.19}$$

Regional Level: The estimated on terms in the righthand side of (3.16) for the case of $u = r$ and $a \neq i$:

$$\sum_{a\neq i}^{n_r} \left[\left(1 - \frac{1}{S_{ia}^{rr}} \right) dS_{ia}^{rr} + \frac{1}{2(S_{ia}^{rr})^2} (dS_{ia}^{rr})^2 \right]$$
$$\leq \sum_{a\neq i}^{n_r} \Bigg\{ \Bigg[\sigma_{ia}^{rr} S_{ii}^{rr} + \eta_a^r I_{ia}^{rr} + \alpha_a^r R_{ia}^{rr} + \sum_{v=1}^{M} \sum_{b=1}^{n_v} \beta_{aib}^{rrv} I_{ba}^{vr} + (\sigma_{ia}^{rr} + \delta_a^r + d_a^r)$$
$$+ \frac{1}{2} \left(\sum_{v=1}^{M} \sum_{b=1}^{n_v} (v_{aib}^{rrv})^2 (I_{bi}^{vr})^2 \right) \Bigg] dt + (1 - S_{ia}^{rr}) \sum_{v=1}^{M} \sum_{b=1}^{n_v} v_{aib}^{rrv} I_{bi}^{vr} dw_{aib}^{rrv} \Bigg\}, \tag{3.20}$$

$$\sum_{a\neq i}^{n_r} \left[\left(1 - \frac{1}{I_{ia}^{rr}} \right) dI_{ia}^{rr} + \frac{1}{2(I_{ia}^{rr})^2} (dI_{ia}^{rr})^2 \right]$$
$$\leq \sum_{a\neq i}^{n_r} \Bigg\{ \Bigg[\sigma_{ia}^{rr} I_{ii}^{rr} + \sum_{v=1}^{M} \sum_{b=1}^{n_v} \beta_{aib}^{rrv} S_{ia}^{rr} I_{ba}^{vr} + (\varrho_a^r + \eta_a^r + \rho_{ia}^{rr} + \delta_a^r + d_a^r)$$

$$+\frac{(S_{ia}^{rr})^2}{2(I_{ia}^{rr})^2}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(I_{ba}^{vr})^2\Bigg]\,dt+\left(S_{ia}^{rr}-\frac{S_{ia}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]\Bigg\},$$

and

$$\sum_{a\neq i}^{n_r}\left(1-\frac{1}{R_{ia}^{rr}}\right)dR_{ia}^{rr}\le\sum_{a\neq i}^{n_r}\left[\sigma_{ia}^{rr}R_{ii}^{rr}+\varrho_a^r I_{ia}^{rr}+(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r+d_a^r)\right]dt. \tag{3.22}$$

Interregional Level: the estimate on terms in the righthand side of (3.16) for the case of $u\neq r$, $a\in I(1,n_u)$:

(3.23)

$$\begin{aligned}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{S_{ia}^{ru}}\right)dS_{ia}^{ru}+\frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right]&\le\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{[\gamma_{ia}^{ru}S_{ii}^{rr}+\eta_a^u I_{ia}^{rr}+\alpha_a^u R_{ia}^{ru}\\&+(\rho_{ia}^{ru}+\delta_a^u+d_a^u)+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}I_{ba}^{vr}+\frac{1}{2}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(I_{ba}^{vr})^2\right]\Bigg]dt\\&+(1-S_{ia}^{ru})\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{urv}I_{ba}^{vr}dw_{aib}^{urv}\Bigg\},\end{aligned}$$

(3.24)

$$\begin{aligned}&\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{I_{ia}^{ru}}\right)dI_{ia}^{ru}+\frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right]\\&\le\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\Bigg[\gamma_{ia}^{ru}I_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}S_{ia}^{ru}I_{ba}^{vu}\\&\quad+(\eta_a^r+\varrho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)+\frac{(S_{ia}^{ru})^2}{2(I_{ia}^{ru})^2}\left(\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(I_{ba}^{vr})^2\right)\Bigg]dt\\&\quad+\left(S_{ia}^{ru}-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{urv}I_{ba}^{vr}dw_{aib}^{urv}\Bigg\},\end{aligned}$$

and

$$\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left(1-\frac{1}{R_{ia}^{ru}}\right)dR_{ia}^{ru}\le\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\gamma_{ia}^{ru}R_{ii}^{rr}+\varrho_a^u I_{ia}^{ru}+(\rho_{ia}^{ru}+\alpha_a^u\delta_a^u+d_a^u)\right]dt. \tag{3.25}$$

From (3.16) and (3.17)–(3.19), the first term in the righthand side of (3.13) can be estimated as follows:

(3.26)

$$\begin{aligned}\sum_{r=1}^{M}\sum_{i=1}^{n_r}dV_{ii}^{rr}(x_{00}^{00})&=\sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\left[\left(1-\frac{1}{S_{ii}^{rr}}\right)dS_{ii}^{rr}+\frac{1}{2(S_{ii}^{rr})^2}(dS_{ii}^{rr})^2\right]\\&+\left[\left(1-\frac{1}{I_{ii}^{rr}}\right)dI_{ii}^{rr}+\frac{1}{2(I_{ii}^{rr})^2}(dI_{ii}^{rr})^2\right]\end{aligned}$$

$$
\begin{aligned}
&+\left[\left(1-\frac{1}{R_{ii}^{rr}}\right)dR_{ii}^{rr}+\frac{1}{2(R_{ii}^{rr})^2}(dR_{ii}^{rr})^2\right]\Bigg\}\\
&\le\sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[B_i^r+\sum_{b=1}^{n_r}\rho_{ib}^{rr}(S_{ib}^{rr}+I_{ib}^{rr}+R_{ib}^{rr})+\sum_{b\neq r}^{M}\sum_{b=1}^{n_v}\rho_{ib}^{rv}(S_{ib}^{rv}+I_{ib}^{rv}+R_{ib}^{rv})\\
&+(\varrho_i^r+\eta_i^r+\alpha_i^r)(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+3(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r+d_i^r)\\
&+\left(1+\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{iib}^{rrv}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})\\
&+\frac{1}{2}\left(1+\frac{(S_{ii}^{rr})^2}{(I_{ii}^{rr})^2}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{iib}^{rrv})^2(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})^2\Bigg]dt\\
&+\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right]\Bigg\},\\
&\le\Bigg\{\Bigg[\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r+\rho_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ib}^{rv}+I_{ib}^{rv}+R_{ib}^{rv})\\
&+(\varrho_0^0+\eta_0^0+\alpha_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+3\sum_{r=1}^{M}\sum_{i=1}^{n_r}(\gamma_0^0+\sigma_0^0+\alpha_0^0+\delta_0^0+d_0^0)\\
&+M_{001}^{00}\beta_{000}^{000}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})\\
&+M_{002}^{00}(v_{000}^{000})^2\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{bi}^{vr}+I_{bi}^{vr}+R_{bi}^{vr})^2\Bigg]dt\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right]\Bigg\}
\end{aligned}
$$

From Lemma 3.2, (3.26) becomes

$$
\sum_{r=1}^{M}\sum_{i=1}^{n_r}dV_{ii}^{rr}\le\tilde{K}_1dt+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right],\tag{3.27}
$$

where

$$
\begin{aligned}
\tilde{K}_1=\Bigg\{\Bigg[&\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r+\rho_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r(\varrho_0^0+\eta_0^0+\alpha_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r\\
&+3\sum_{r=1}^{M}\sum_{i=1}^{n_r}(\gamma_0^0+\sigma_0^0+\alpha_0^0+\delta_0^0+d_0^0)\\
&+M_{001}^{00}\beta_{000}^{000}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r
\end{aligned}\tag{3.28}
$$

$$+\ M_{002}^{00}(v_{000}^{000})^2\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r\right)^2\Bigg]\Bigg\}>0.$$

Similarly from (3.16) and (3.20)–(3.22) the second term in the righthand side of (3.13) is estimated as

(3.29)

$$\begin{aligned}
&\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}dV_{ia}^{rr}\\
&\quad=\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{S_{ia}^{rr}}\right)dS_{ia}^{rr}+\frac{1}{2(S_{ia}^{rr})^2}(dS_{ia}^{rr})^2\right]\\
&\qquad+\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{I_{ia}^{rr}}\right)dI_{ia}^{rr}+\frac{1}{2(I_{ia}^{rr})^2}(dI_{ia}^{rr})^2\right]\\
&\qquad+\sum_{a\neq i}^{n_r}\left[\left(1-\frac{1}{R_{ia}^{rr}}\right)dR_{ia}^{rr}+\frac{1}{2(R_{ia}^{rr})^2}(dR_{ia}^{rr})^2\right]\\
&\quad\le\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\big\{\sigma_{00}^{00}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+(\eta_0^0+\alpha_0^0+\varrho_0^0)(S_{ia}^{rr}+I_{ia}^{rr}+R_{ia}^{rr})\\
&\qquad+\beta_{000}^{000}N_{00}^{00}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})\\
&\qquad+3(\rho_{00}^{00}+\varrho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0)+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})^2\right]\Bigg\}dt\\
&\qquad+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]\\
&\quad=\Bigg\{\sigma_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr})+(\eta_0^0+\alpha_0^0+\varrho_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}(S_{ia}^{rr}+I_{ia}^{rr}+R_{ia}^{rr})\\
&\qquad+\beta_{000}^{000}N_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})\\
&\qquad+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}3(\rho_{00}^{00}+\varrho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0)\\
&\qquad+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vr}+I_{ba}^{vr}+R_{ba}^{vr})^2\right]\Bigg\}dt\\
&\qquad+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right].
\end{aligned}$$

Again from and Lemma 3.2, the above random differential inequality reduces to

$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} dV_{ia}^{rr} \leq \tilde{K}_2 dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v} v_{aib}^{rrv} I_{ba}^{vr} dw_{aib}^{rrv}\right]. \tag{3.30}$$

where

(3.31)

$$\begin{aligned}\tilde{K}_2 = \Bigg\{ & \sigma_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + (\eta_0^0+\alpha_0^0+\varrho_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r + \beta_{000}^{000}N_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \\ & + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r} 3(\rho_{00}^{00}+\varrho_0^0+\eta_0^0+\alpha_0^0+\delta_0^0+d_0^0) + \frac{M_{002}^{00}(v_{000}^{000})^2}{2}\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r\right)^2\Bigg\}\end{aligned}$$

Finally from (3.13), (3.16) and (3.23)–(3.25), the third term in (3.13) is estimated as below we get

(3.32)

$$\begin{aligned}&\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u} dV_{ia}^{ru} \\ &= \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{S_{ia}^{ru}}\right)dS_{ia}^{ru} + \frac{1}{2(S_{ia}^{ru})^2}(dS_{ia}^{ru})^2\right] \\ &\quad + \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{I_{ia}^{ru}}\right)dI_{ia}^{ru} + \frac{1}{2(I_{ia}^{ru})^2}(dI_{ia}^{ru})^2\right] \\ &\quad + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\left[\left(1-\frac{1}{R_{ia}^{ru}}\right)dR_{ia}^{ru} + \frac{1}{2(R_{ia}^{ru})^2}(dR_{ia}^{ru})^2\right] \\ &\leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\{\gamma_{ia}^{ru}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr}) + (\eta_a^u+\alpha_a^u+\varrho_a^u)(S_{ia}^{ru}+I_{ia}^{ru}+R_{ia}^{ru}) \\ &\quad + 3(\eta_a^u+\varrho_a^u+\alpha_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u) \\ &\quad + (1+S_{ia}^{ru})\sum_{v=1}^{M}\sum_{b=1}^{n_v}\beta_{aib}^{urv}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu}) \\ &\quad + \frac{1}{2}\left(1+\frac{(S_{ia}^{ru})^2}{(I_{ia}^{ru})^2}\right)\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})^2\Bigg\}dt \\ &\quad + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv} \\ &\leq \Bigg\{\gamma_{00}^{00}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}(S_{ii}^{rr}+I_{ii}^{rr}+R_{ii}^{rr}) + (\eta_0^0+\alpha_0^0+\varrho_0^0)\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}(S_{ia}^{ru}+I_{ia}^{ru}+R_{ia}^{ru})\end{aligned}$$

$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}3(\eta_0^0+\varrho_0^0+\alpha_0^0+\rho_{00}^{00}+\delta_0^0+d_0^0)$$
$$+N_{00}^{00}\beta_{000}^{000}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})$$
$$+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(S_{ba}^{vu}+I_{ba}^{vu}+R_{ba}^{vu})^2\Bigg\}dt$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}.$$

By using Lemma 3.2, differential inequality (3.33) becomes

(3.33)
$$\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}dV_{ia}^{ru}\leq\tilde{K}_3dt+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv},$$

where

(3.34)
$$\tilde{K}_3=\Bigg\{\gamma_{00}^{00}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r+(\eta_0^0+\alpha_0^0+\varrho_0^0)\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}3(\eta_0^0+\varrho_0^0+\alpha_0^0+\rho_{00}^{00}+\delta_0^0+d_0^0)$$
$$+N_{00}^{00}\beta_{000}^{000}\frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r+\frac{M_{002}^{00}(v_{000}^{000})^2}{2}\frac{1}{\mu^2}\left(\sum_{r=1}^{M}\sum_{i=1}^{n_r}B_i^r\right)^2\Bigg\}$$

Hence, from (3.27), (3.30) and (3.33), we arrive at the following stochastic differential inequality

(3.35)
$$dV(x_{00}^{00}(t))\leq(\tilde{K}_1+\tilde{K}_2+\tilde{K}_3)\,dt+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\left(1-\frac{S_{ii}^{rr}}{I_{ii}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{iib}^{rrv}I_{bi}^{vr}dw_{iib}^{rrv}\right]$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}\left(1-\frac{S_{ia}^{rr}}{I_{ia}^{rr}}\right)\left[\sum_{v=1}^{M}\sum_{b=1}^{n_v}v_{aib}^{rrv}I_{ba}^{vr}dw_{aib}^{rrv}\right]$$
$$+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1-\frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right)v_{aib}^{urv}I_{ba}^{vu}dw_{aib}^{urv}.$$

Therefore choosing $\tilde{K}=\tilde{K}_1+\tilde{K}_2+\tilde{K}_3>0$, and combining the last three summations, concludes the proof of the theorem.

We now show the existence of a unique solution of the system (2.15)–(2.17) in the following theorem.

Theorem 3.1. *Given any initial condition $x_{00}^{00}(t_0) \in \mathbb{R}_+^{3n^2}$ under the assumptions of Lemma 4.1, there is a unique solution process of the system (2.15)–(2.17) in $\mathbb{R}_+^{3n^2}$, for $t \geq t_0$, almost surely.*

Proof. Given that the rate functions of the system are locally Lipschitz continuous in x_{00}^{00}, it follows that for any initial value $x_{00}^{00}(t_0) \in \mathbb{R}_+^{3n^2}$, there is a unique local solution of the system (2.15)–(2.17) $x_{00}^{00}(t)$, for $t \in (t_0, t_e)$, where at $t = t_e$ is the first exit time of x_{00}^{00}. Therefore to show the solution process of the system exists for all $t \geq t_0$, it suffices to show that $t_e = \infty$. □

Let $k_{00}^{00} \in \mathbb{R}_+^{n^2}$. From (2.20) and (2.21), we have

$$\|k_{00}^{00}\|_\infty = \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} |k_{ia}^{ru}|, \quad k_{00min}^{00} = \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} |k_{ia}^{ru}|. \tag{3.36}$$

We denote

$$k \equiv k_{00min}^{00}. \tag{3.37}$$

We choose $k_{00}^{*00} \in \mathbb{R}_+^{n^2}$ with each component k_{ia}^{*ru}, sufficiently large such that $S_{ia}^{ru}(t_0)$, $I_{ia}^{ru}(t_0)$, $R_{ia}^{ru}(t_0) \in [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \equiv \bar{\mathfrak{B}}_{\mathbb{R}}(\frac{\frac{1}{k_{ia}^{*ru}} + k_{ia}^{*ru}}{2}; \frac{\frac{1}{k_{ia}^{*ru}} - k_{ia}^{*ru}}{2})$, for $i \in I(1, n_r), a \in I(1, n_u)$, and $r, u \in I(1, M)$. In other words, from (2.18), $x_{00}^{00}(t_0) \in \prod_{r=1}^{M}\prod_{u=1}^{M}\prod_{i=1}^{n_i}\prod_{a=1}^{n_u}[\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \times [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}] \times [\frac{1}{k_{ia}^{*ru}}, k_{ia}^{*ru}]$. From (3.37) let $k_0 \equiv k_{00min}^{*00}$.

Let $k_{00}^{00} \in \mathbb{R}_+^{n^2}$ be an arbitrary vector whose components k_{ia}^{ru} satisfy $k_{ia}^{ru} \geq k_{ia}^{*ru}, \forall i \in I(1, n_i), a \in I(1, n_u)$, and $r, u \in I(1, M))$. And let the local solution $x_{00}^{00}(t) \in \prod_{r=1}^{M}\prod_{u=1}^{M}\prod_{i=1}^{n_i}\prod_{a=1}^{n_u}[\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}] \times [\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}] \times [\frac{1}{k_{ia}^{ru}}, k_{ia}^{ru}]$, for $t \in (0, t_e)$ where t_e is the first hitting time of the solution process. For $t \leq t_e$, it follows that $S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t) \in [\frac{1}{||k_{00}^{00}||_\infty}, k_{00min}^{00}]$, for all $i \in I(1, n_r), a \in I(1, n_u), r, u \in I(1, M)$. Using (3.37), define a stoping time for the process as follows

$$\begin{aligned} \tau_k = \inf\Bigg\{ & t \in (0, t_e) : \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} (S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t)) \\ & \leq \frac{1}{||k_{00}^{00}||_\infty}, \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} (S_{ia}^{ru}(t), I_{ia}^{ru}(t), R_{ia}^{ru}(t)) \geq k \Bigg\}, \quad \text{and} \\ \tau_k(t) = \min\{t, \tau_k\}, & \quad \text{for } t \geq t_0 \end{aligned} \tag{3.38}$$

where k is defined in (3.37). Furthermore, we set $inf \varnothing = \infty$.

It follows from (3.38) that τ_k increases as $k \to \infty$. We let $\tau_\infty = \lim_{k\to\infty} \tau_k$. From (3.38) it implies that

$$\tau_\infty \leq t_e \quad a.s. \tag{3.39}$$

Therefore to show $t_e = \infty$, we only show that $\tau_\infty = \infty$ a.s.

On the contrary suppose $\tau_\infty < \infty$, then $\exists\, T > 0$, such that for a given $0 < \epsilon < 1$, $P(\tau_\infty \leq T) > \epsilon$. This means that $\{\tau_k\}$ is a finite sequence. Moreover, from the definition of a finite sequence there exists a vector $k_{00}^{100} \in \mathbb{R}^{n^2}$, with $k_{00min}^{100} \equiv k_1 \geq k_0$, (where $k_1 \equiv k_{00min}^{100}$ is defined by (3.37) and (3.36)),

$$P(\tau_k \leq T) \geq \epsilon, \tag{3.40}$$

whenever $k \geq k_1$. From (3.14), (3.13) can be rewritten as

$$\begin{aligned} V(x_{00}^{00}) = \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u} &[(S_{ia}^{ru} - 1 - \log S_{ia}^{ru}) + (I_{ia}^{ru} - 1 - \log I_{ia}^{ru}) \\ &+ (R_{ia}^{ru} - 1 - \log R_{ia}^{ru})]\,. \end{aligned} \tag{3.41}$$

From Lemma 3.2& 3.3, the stopped solution process (2.15)–(2.17) satisfies the following stochastic inequality for some $\tilde{K} > 0$.

$$dV(x_{00}^{00}(\tau_k(t))) \leq \tilde{K}dt + \sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v}\left(1 - \frac{S_{ia}^{ru}}{I_{ia}^{ru}}\right) v_{aib}^{urv} I_{ba}^{vu} dw_{aib}^{urv} \tag{3.42}$$

Furthermore, for $t_1 \leq T$, integrating both sides of (3.42) on $[0, t_1 \wedge \tau_k]$, and taking the expected values of both sides, it implies that

$$\begin{aligned} E(V(x_{00}^{00}(t))) &\leq V(x_{00}^{00}(0)) + \tilde{K}(t_1 \wedge \tau_k) \\ &\leq V(x_{00}^{00}(t)) + \tilde{K}T \end{aligned} \tag{3.43}$$

Given that $k \geq k_1$, we set $E_k = \{\tau_k \leq T\}$. Then from (3.40), we see that $P(E_k) \geq \epsilon$. If $\omega \in E_k$, then ω is an event at the stopping time where at least one of $S_{ia}^{ru}(\tau_k, \omega)$, $I_{ia}^{ru}(\tau_k, \omega)$, or $R_{ia}^{ru}(\tau_k, \omega)$ whenever $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$ is $\frac{1}{||k_{00}^{00}||_\infty}$ or $k \equiv k_{min}$. This implies from (3.41) that

$$V(x_{00}^{00}(\tau_k, \omega)) \geq [k_{min} - 1 - \log k_{\min}] \wedge \left[\frac{1}{\|k_{00}^{00}\|_\infty} - 1 - \log \|k_{00}^{00}\|_\infty\right], \forall \omega \in E_k. \tag{3.44}$$

It follows from (3.43) and (3.44) that

$$\begin{aligned} V(x_{i0}^{00}(0)) + \tilde{K}T &\geq E(I_{E_k(\omega)} V(x_{00}^{00}(\tau_k, \omega))) \\ &\geq \epsilon\left\{[k_{\min} - 1 - \log k_{min}] \wedge \left[\frac{1}{\|k_{00}^{00}\|_\infty} - 1 - \log \|k_{00}^{00}\|_\infty\right]\right\}, \end{aligned} \tag{3.45}$$

where $I_{E_k(\omega)}$ is the indicator function of E_k.

Hence as $k = k_{\min} \to \infty$, (3.45) implies that $V(x_{00}^{00}(t_0)) + \tilde{K}T \to \infty$ which leads to a contradiction to the existence of a local solution. Therefore, we must have $\tau_\infty = \infty$, and the rest of the proof follows.

Remark 3.3. For any $r \in I(1,M)$ and $i \in I(1,n_r)$, Lemmas 3.1, 3.2, 3.3 and Theorem 3.1 show that there exists a positive self-invariant set for system (2.15)–(2.17) given by

(3.46)
$$A = \left\{ (S_{ia}^{ru}, I_{ia}^{ru}, R_{ia}^{ru}) : y_{ia}^{ru}(t) \geq 0 \quad \text{and} \quad \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} y_{ia}^{ru}(t) \leq \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \right\}$$

whenever $u \in I^r(1,M)$ and $a \in I_i^r(1,n_u)$. We shall denote

$$\bar{B} \equiv \frac{1}{\mu}\sum_{r=1}^{M}\sum_{i=1}^{n_r} B_i^r \tag{3.47}$$

4. EXISTENCE AND ASYMPTOTIC BEHAVIOR OF DISEASE FREE EQUILIBRIUM

In this section, we study the existence and the asymptotic behavior of the disease free equilibrium state of the system (2.15)–(2.17). The disease free equilibrium is obtained by solving the system of algebraic equations obtained by setting the drift and the diffusion parts of the system of stochastic differential equations to zero. In addition, conditions that $I = R = 0$ in the event when there is no disease in the population. We summarize the results as follows.

For any $r, u \in I(1,M)$, $i \in I(1,n_r)$ and $a \in I(1,n_u)$, let

$$D_i^r = \gamma_i^r + \sigma_i^r + \delta_i^r - \sum_{a=1}^{n_r}\frac{\rho_{ia}^{rr}\sigma_{ia}^{rr}}{\rho_{ia}^{rr}+\delta_a^r} - \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\frac{\rho_{ia}^{rr}\gamma_{ia}^{ru}}{\rho_{ia}^{ru}+\delta_a^u} > 0. \tag{4.1}$$

Furthermore, let $(S_{ia}^{ru*}, I_{ia}^{ru*}, R_{ia}^{ru*})$, be the equilibrium state of the system (2.15)–(2.17). One can see that the disease free equilibrium state is given by $E_{ia}^{ru} = (S_{ia}^{ru*}, 0, 0)$, where

$$S_{ia}^{ru*} = \begin{cases} \frac{B_i^r}{D_i^r}, & \text{for } u = r,\ a = i, \\ \frac{B_i^r}{D_i^r}\frac{\sigma_{ij}^{rr}}{\rho_{ij}^{rr}+\delta_j^r}, & \text{for } u = r,\ a \neq i, \\ \frac{B_i^r}{D_i^r}\frac{\gamma_{ia}^{ru}}{\rho_{ia}^{ru}+\delta_a^u}, & \text{for } u \neq r. \end{cases} \tag{4.2}$$

The asymptotic stability property of E_{ia}^{ru} will be established by verifying the conditions of the stochastic version of the Lyapunov second method given in [21, Theorem 2.4], [46], and [21, Theorem 4.4], [46] respectively. In order to study the qualitative properties of (2.15)–(2.17) with respect to the equilibrium state $(S_{ia}^{ru*}, 0, 0)$, first, we to use the change of variable. For this purpose, we use the following transformation:

$$\begin{cases} U_{ia}^{ru} &= S_{ia}^{ru} - S_{ia}^{ru*} \\ V_{ia}^{ru} &= I_{ia}^{ru} \\ W_{ia}^{ru} &= R_{ia}^{ru}. \end{cases} \tag{4.3}$$

By employing this transformation, system (2.15)–(2.17) is transformed into the following forms

(4.4)
$$dU_{il}^{rq} = \begin{cases} \big[\sum_{q\neq r}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}U_{ia}^{rq} + \eta_i^r V_{ii}^{rr} + \alpha_i^r W_{ii}^{rr} \\ \quad -(\gamma_i^r+\sigma_i^r+\delta_i^r)U_{ii}^{rr} - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}\big]dt \\ \quad -\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\big],\ \text{for } q=r,\ l=i \\ \big[\sigma_{ij}^{rr}U_{ii}^{rr} + \eta_j^r V_{ij}^{rr} + \alpha_j^r W_{ij}^{rr} - (\rho_{ij}^{rr}+\delta_j^r)U_{ij}^{rr} - \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}\big]dt \\ \quad -\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\big],\ \text{for } q=r,\ l=j,\ j\neq i, \\ \big[\gamma_{il}^{rq}U_{ii}^{rr} + \eta_l^q V_{il}^{rq} + \alpha_l^q W_{il}^{rq} - (\rho_{il}^{rq}+\delta_l^q)U_{il}^{rq} \\ \quad -\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}S_{il}^{rq}I_{al}^{uq}\big]dt - \big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\big], \\ \quad \text{for } q\neq r, \end{cases}$$

(4.5)
$$dV_{il}^{rq} = \begin{cases} \big[\sum_{q=1}^{M}\sum_{a=1}^{n_q}\rho_{ia}^{rq}V_{ia}^{rq} - (\eta_i^r+\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)W_{ii}^{rr} \\ \quad +\sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}\big]dt \\ \quad +\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}dw_{iia}^{rru}(t)\big],\ \text{for } q=r,\ l=i \\ \big[\sigma_{ij}^{rr}V_{ii}^{rr} - (\eta_j^r+\varrho_j^r+\rho_{ij}^{rr}+\delta_j^r+d_j^r)V_{ij}^{rr} + \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}\big]dt \\ \quad +\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{jia}^{rru}(S_{ij}^{rr*}+U_{ij}^{rr})V_{aj}^{ur}dw_{jia}^{rru}(t)\big],\ \text{for } q=r,\ l=j, j\neq i, \\ \big[\gamma_{il}^{rq}V_{ii}^{rr} - (\eta_l^q+\varrho_l^q+\rho_{il}^{rq}+\delta_l^q+d_l^q)V_{il}^{rq} \\ \quad \sum_{u=1}^{M}\sum_{a=1}^{n_u}\beta_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}\big]dt \\ \quad +\big[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\upsilon_{lia}^{qru}(S_{il}^{rq*}+U_{il}^{rq})V_{al}^{uq}dw_{lia}^{qru}(t)\big],\ \text{for } q\neq r, \end{cases}$$

and

(4.6)
$$dW_{il}^{rq} = \begin{cases} \big[\sum_{q\neq r}^{M}\sum_{l=1}^{n_q}\rho_{il}^{rq}W_{il}^{rq} + \varrho_i^r V_{ii}^{rr} - (\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)W_{ii}^{rr}\big]dt,\ \text{for } q=r,\ l=i \\ \big[\sigma_{ij}^{rr}W_{ii}^{rr} + \varrho_j^r V_{ij}^{rr} - (\rho_{ij}^{rr}+\alpha_j^r+\delta_j^r)W_{ij}^{rr}\big]dt,\ \text{for } q=r,\ l=j,\ j\neq i \\ \big[\gamma_{il}^{rq}W_{ii}^{rr} + \varrho_l^q V_{il}^{rq} - (\rho_{il}^{rq}+\alpha_l^q+\delta_l^q)W_{il}^{rq}\big]dt,\ \text{for } q\neq r \end{cases}$$

We state and prove the following lemmas that would be useful in the proofs of the stability results.

Lemma 4.1. *Let $V:\mathbb{R}^{3n^2}\times\mathbb{R}_+\to\mathbb{R}_+$ be a function defined by*

$$V(\tilde{x}_{00}^{00}) = \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}V(\tilde{x}_{ia}^{ru}), \tag{4.7}$$

where,

$$V(\tilde{x}_{ia}^{ru}) = (S_{ia}^{ru} - S_{ia}^{ru*} + I_{ia}^{ru})^2 + c_{ia}^{ru}(I_{ia}^{ru})^2 + (R_{ia}^{ru})^2 \tag{4.8}$$
$$\tilde{x}_{00}^{00} = (U_{ia}^{ru}, V_{ia}^{ru}, W_{ia}^{ru})^T \quad \text{and} \quad c_{ia}^{ru}\geq 0.$$

Then $V \in \mathcal{C}^{2,1}(\mathbb{R}^{3n^2} \times \mathbb{R}_+, \mathbb{R}_+)$, and it satisfies

$$b(\|\tilde{x}_{00}^{00}\|) \leq V(\tilde{x}_{00}^{00}(t)) \leq a(\|\tilde{x}_{00}^{00}\|) \tag{4.9}$$

where

(4.10)

$$b(\|\tilde{x}_{00}^{00}\|) = \min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} \left\{\frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\right\} \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} \left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right]$$

$$a(\|\tilde{x}_{00}^{00}\|) = \max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u} \{c_{ia}^{ru}+2\} \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} \left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right].$$

Proof. From (4.6), (4.7) can be written as

$$\begin{aligned}
V(x_{ia}^{ru}) &= (U_{ia}^{ru} + V_{ia}^{ru})^2 + c_{ia}^{ru}(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&= (U_{ia}^{ru})^2 + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + 2\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right)\left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right) + (W_{ia}^{ru})^2 \\
&= \left(-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}+1\right)(U_{ia}^{ru})^2 + \left(-\left(1+\frac{c_{ia}^{ru}}{2}\right)+c_{ia}^{ru}+1\right)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&\quad + \left[\left(\frac{1}{\sqrt{1+\frac{c_{ia}^{ru}}{2}}}U_{ia}^{ru}\right) + \left(\sqrt{1+\frac{c_{ia}^{ru}}{2}}V_{ia}^{ru}\right)\right]^2
\end{aligned}$$

Therefore, by nothing the fact that $\min\{1-\frac{1}{1+\frac{c_{ia}^{ru}}{2}}, \frac{c_{ia}^{ru}}{2}, 1\}$, we have

$$V(x_{ia}^{ru}) \geq \frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right] \tag{4.11}$$

Hence from (4.11) we have

$$\begin{aligned}
V(\tilde{x}_{00}^{00}) &\geq \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u} \frac{c_{ia}^{ru}}{2+c_{ia}^{ru}}\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2)\right] \\
&\geq b(\|\tilde{x}_{00}^{00}\|).
\end{aligned} \tag{4.12}$$

On the other hand, it follows from (4.7) that

$$\begin{aligned}
V(x_{ia}^{ru}) &= (U_{ia}^{ru})^2 + 2U_{ia}^{ru}V_{ia}^{ru} + (c_{ia}^{ru}+1)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&\leq 2(U_{ia}^{ru})^2 + (c_{ia}^{ru}+2)(V_{ia}^{ru})^2 + (W_{ia}^{ru})^2 \\
&\leq (c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2 + (V_{ia}^{ru})^2 + (W_{ia}^{ru})^2\right]
\end{aligned} \tag{4.13}$$

Thus, from (4.11) and (4.13) we have

$$V(x_{00}^{00}(t)) \leq \sum_{r=1}^{M}\sum_{u=1}^{M}\sum_{i=1}^{n_r}\sum_{a=1}^{n_u}(c_{ia}^{ru}+2)\left[(U_{ia}^{ru})^2+(V_{ia}^{ru})^2+(W_{ia}^{ru})^2\right] \leq a(\|\tilde{x}_{00}^{00}\|) \tag{4.14}$$

Therefore from (4.7), (4.12) and (4.14), we establish the desired inequality. □

Remark 4.1. Lemma 4.1 shows that the Lyapunov function V defined in (4.7) is positive definite ((4.12)), decrescent and radially unbounded ((4.14)) function [21, 46].

We now state the following lemma

Lemma 4.2. *Assume that the hypothesis of Lemma* 4.1 *are satisfied. For each* $r,u,v \in I(1,M)$, $i \in I(1,n_r)$, $a \in I(1,n_u)$ *and* $b \in I(1,n_v)$, *let*

$$d_{ai}^{ur} = \sum_{v=1}^{M}\sum_{b=1}^{n_r} c_{ba}^{vu}\left[\beta_{abi}^{uvr}\left(\frac{S_{ba}^{vu*}}{\mu_{ba}^{vu}}+\frac{\bar{B}^2}{\mu_{ba}^{vu}}\right)+(v_{abi}^{uvr})^2(S_{ba}^{vu*}+\bar{B})^2\right]. \tag{4.15}$$

for some positive numbers c_{ia}^{ru}. *Furthermore, let*

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\left[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{3}{2}\mu_{ii}^{rr}\right]}{(\gamma_i^r+\sigma_i^r+\delta_i^r)} & \text{for } u=r,\ i=a \\ \dfrac{\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{rr}\right]}{(\rho_{ia}^{rr}+\delta_a^r)}, & \text{for } u=r,\ a\neq i \\ \dfrac{\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\frac{3}{2}\mu_{ia}^{ru}\right]}{(\rho_{ia}^{ru}+\delta_a^u)}, & \text{for } u\neq r, \end{cases} \tag{4.16}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \frac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu_{ia}^{ru}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{iib}^{rrv}(S_{ii}^{rr*}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\eta_i^r+\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r}, & \text{for } a=i,\ u=r \\ \frac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{rrv}(S_{ia}^{rr*}+\mu_{ia}^{rr})+\frac{1}{2}d_{ai}^{rr}}{\eta_a^r+\varrho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r}, & \text{for } a\neq i,\ u=r \\ \frac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{urv}(S_{ii}^{ru*}+\mu_{ia}^{ru})+\frac{1}{2}d_{ai}^{ur}}{\eta_a^u+\varrho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u}, & \text{for } u\neq r. \end{cases} \tag{4.17}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\frac{1}{2}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}\right]}{(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)}, & \text{for } u=r,\ a=i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}+\frac{(\alpha_a^r)^2}{\mu_{ia}^{rr}}\right]}{(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)}, & \text{for } u=r,\ a\neq i, \\ \dfrac{\left[\frac{1}{2}\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}+\frac{(\alpha_a^u)^2}{\mu_{ia}^{ru}}\right]}{(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)}, & \text{for } u\neq r \end{cases} \tag{4.18}$$

for some suitably defined positive number μ_{ia}^{ru}, *depending on* δ_a^u, *for all* $r,u \in I^r(1,M)$, $i \in I(1,n)$ *and* $a \in I_i^r(1,n_r)$. *Assume that* $\mathfrak{U}_{ia}^{ru} \leq 1$, $\mathfrak{V}_{ia}^{ru} < 1$ *and* $\mathfrak{W}_{ia}^{ru} \leq 1$. *There exist positive numbers* ϕ_{ia}^{ru}, ψ_{ia}^{ru} *and* φ_{ia}^{ru} *such that the differential operator LV*

associated with Ito-Doob type stochastic system (2.15)–(2.17) *satisfies the following inequality*

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \;\leq\; & \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg[-[\phi_{ii}^{rr}(U_{ii}^{rr})^2+\psi_{ii}^{rr}(V_{ii}^{rr})^2+\varphi_{ii}^{rr}(W_{ii}^{rr})^2] \\ & -\sum_{a\neq i}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2+\psi_{ia}^{rr}(V_{ia}^{rr})^2+\varphi_{ia}^{rr}(W_{ia}^{rr})^2] \\ & -\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{rr})^2+\psi_{ia}^{ru}(V_{ia}^{ru})^2+\varphi_{ia}^{ru}(W_{ia}^{ru})^2]\Bigg]. \end{aligned} \tag{4.19}$$

Moreover,

$$LV(\tilde{x}_{00}^{00}) \leq -cV(\tilde{x}_{00}^{00}) \tag{4.20}$$

where a positive constant c is defined by

$$c=\frac{\min_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}(\phi_{ia}^{ru},\psi_{ia}^{ru},\varphi_{ia}^{ru})}{\max_{1\leq r,u\leq M,1\leq i\leq n_r,1\leq a\leq n_u}\{C_{ia}^{ru}+2\}} \tag{4.21}$$

Proof. The computation of differential operator [46, 21] applied to the Lyapunov function V in (4.7) with respect to the large-scale system of Ito-Doob type stochastic differential equation (2.15)–(2.17) is as follows:

(4.22)

$$\begin{aligned} LV(\tilde{x}_{ii}^{rr}) = {} & 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(1+C_{ii}^{rr})\rho_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr}+\rho_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\ & +\rho_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr}]+2\alpha_i^rU_{ii}^{rr}W_{ii}^{rr}+2(\alpha_i^r+\varrho_i^r)V_{ii}^{rr}W_{ii}^{rr} \\ & -2[\varrho_i^r+d_i^r+2(\gamma_i^r+\sigma_i^r+\delta_i^r)]V_{ii}^{rr}U_{ii}^{rr}-2(\gamma_i^r+\sigma_i^r+\delta_i^r)(U_{ii}^{rr})^2 \\ & -2[c_{ii}^{rr}\eta_i^r+2(c_{ii}^{rr}+1)(\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)](V_{ii}^{rr})^2 \\ & -2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)(W_{ii}^{rr})^2 \\ & +2\sum_{u=1}^{M}\sum_{a=1}^{n_u}c_{ii}^{rr}\beta_{iia}^{rru}(S_{ii}^{rr*}+U_{ii}^{rr})V_{ai}^{ur}V_{ii}^{rr} \\ & +c_{ii}^{rr}\sum_{u=1}^{M}\sum_{a=1}^{n_u}(v_{iia}^{rru})^2(S_{ii}^{rr*}+U_{ii}^{rr})^2(V_{ai}^{ur})^2, \\ & \text{for } u=r,\ a=i \end{aligned} \tag{4.22}$$

(4.23)

$$\begin{aligned} \sum_{a\neq i}^{n_r}LV(\tilde{x}_{ia}^{rr}) = {} & \sum_{a\neq r}^{n_r}\{2(1+c_{ia}^{rr})\sigma_{ia}^{rr}V_{ia}^{rr}V_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}V_{ia}^{rr}U_{ii}^{rr}+2\sigma_{ia}^{rr}U_{ia}^{rr}V_{ii}^{rr} \\ & +2\sigma_{ia}^{rr}W_{ia}^{rr}W_{ii}^{rr} \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&- 2[c_{ia}^{rr}\eta_a^r + 2(c_{ia}^{rr}+1)(\varrho_a^r + \rho_{ia}^{rr} + \delta_a^r)](V_{ia}^{rr})^2 - 2(\rho_{ia}^{rr}+\delta_a^r)(U_{ia}^{rr})^2 \\
&- 2(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)(W_{ia}^{rr})^2 + 2\alpha_a^r W_{ia}^{rr}U_{ia}^{rr} + 2(\alpha_a^r+\varrho_a^r)V_{ia}^{rr}W_{ia}^{rr} \\
&- 2[\varrho_a^r + d_a^r + 2(\rho_{ia}^{rr}+\delta_a^r)]V_{ia}^{rr}U_{ia}^{rr}\} \\
&+ 2\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr} \\
&+ \sum_{a\neq r}^{n_r} c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{rrv})^2(S_{ia}^{rr*}+U_{ia}^{rr})^2(V_{ba}^{vr})^2\ ,\ \text{for } u=r,\ a\neq i
\end{aligned}
$$

(4.24)

$$
\begin{aligned}
\sum_{u=1}^{M}\sum_{a=1}^{n_r} LV(\tilde{x}_{ia}^{ru}) = \sum_{u=1}^{M}\sum_{a=1}^{n_u}\Bigg\{ & 2(1+c_{ia}^{ru})\gamma_{ia}^{ru}V_{ia}^{ru}V_{ii}^{rr} + 2\gamma_{ia}^{ru}U_{ia}^{ru}U_{ii}^{rr} \\
&+ 2\gamma_{ia}^{ru}V_{ia}^{ru}U_{ii}^{rr} + 2\gamma_{ia}^{ru}U_{ia}^{ru}V_{ii}^{rr} \\
&+ 2\gamma_{ia}^{ru}W_{ia}^{ru}W_{ii}^{rr} - 2[c_{ia}^{ru}\eta_a^u + 2(c_{ia}^{ru}+1)(\varrho_a^u + \rho_{ia}^{ru} + \delta_a^u + d_a^u)](V_{ia}^{ru})^2 \\
&- 2(\rho_{ia}^{ru}+\delta_a^u)(U_{ia}^{ru})^2 \\
&- 2(\rho_{ia}^{ru}+\alpha_a^u+\delta_a^u)(W_{ia}^{rr})^2 + 2\alpha_a^u W_{ia}^{ru}U_{ia}^{ru} + 2(\alpha_a^u+\varrho_a^u)V_{ia}^{ru}W_{ia}^{ru} \\
&- 2[\varrho_a^u + d_a^u + 2(\rho_{ia}^{ru}+\delta_a^u)]V_{ia}^{ru}U_{ia}^{ru}\Bigg\} \\
&+ 2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} \\
&+ \sum_{u=1}^{M}\sum_{a=1}^{n_r} c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_v}(v_{aib}^{urv})^2(S_{ia}^{ru*}+U_{ia}^{ru})^2(V_{ba}^{vu})^2,\ \text{for } u\neq r
\end{aligned}
$$

By using (3.46) and the algebraic inequality

$$
2ab \le \frac{a^2}{g(c)} + b^2 g(c) \tag{4.25}
$$

where $a, b, c \in \mathbb{R}$, and the function g is such that $g(c) \ge 0$. The sixth term in (4.22), (4.23) and (4.24) is estimated as follows:

$$
\begin{aligned}
&2\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}+U_{ii}^{rr})V_{bi}^{vr}V_{ii}^{rr} \\
&\le \sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}(S_{ii}^{rr*}g_i^r(\delta_i^r) + g_i^r(\delta_i^r))(V_{ii}^{rr})^2 \\
&\quad + \sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ii}^{rr}\beta_{iib}^{rrv}\left(\frac{S_{ii}^{rr*}}{g_i^r(\delta_i^r)}) + \frac{\bar{B}^2}{g_i^r(\delta_i^r)}\right)(V_{bi}^{vr})^2
\end{aligned} \tag{4.26}
$$

$$\sum_{a\neq r}^{n_r} 2\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}+U_{ia}^{rr})V_{ba}^{vr}V_{ia}^{rr}$$
$$\leq \sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}(S_{ia}^{rr*}g_i^r(\delta_a^r)+g_i^r(\delta_a^r))(V_{ia}^{rr})^2$$
$$+\sum_{a\neq r}^{n_r}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{rr}\beta_{aib}^{rrv}\left(\frac{S_{ia}^{rr*}}{g_i^r(\delta_a^r)}\right)+\frac{\bar{B}^2}{g_i^r(\delta_a^r)}\Big)(V_{bi}^{vr})^2$$

and

$$2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}+U_{ia}^{ru})V_{ba}^{vu}V_{ia}^{ru} \tag{4.27}$$
$$\leq \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}(S_{ia}^{ru*}g_i^r(\delta_a^u)+g_i^r(\delta_a^u))(V_{ia}^{ru})^2$$
$$+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\sum_{v=1}^{M}\sum_{b=1}^{n_v} c_{ia}^{ru}\beta_{aib}^{urv}\left(\frac{S_{ia}^{ru*}}{g_i^r(\delta_a^u)}\right)+\frac{\bar{B}^2}{g_i^r(\delta_a^u)}\Big)(V_{ba}^{vu})^2$$

From (4.22), (4.23) and repeated usage of inequality (4.25) and (4.27) coupled with algebraic manipulations and simplifications, we have the following inequality

(4.28)

$$LV(\tilde{x}_{00}^{00}) \leq \sum_{r=1}^{M}\sum_{i=1}^{n_r}\Bigg\{\Bigg[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+3\mu_{ii}^{rr}+2\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+2\sum_{a\neq r}^{M}\sum_{a=1}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}$$
$$-2(\gamma_i^r+\sigma_i^r+\delta_i^r)\Big](U_{ii}^{rr})^2$$
$$+\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}[(2+c_{ii}^{rr})\mu_{ia}^{ru}]+\mu_{ii}^{rr}+\frac{(\varrho_i^r)^2}{\mu_{ii}^{rr}}+\frac{(\varrho_i^r+d_i^r)^2}{\mu_{ii}^{rr}}+4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r)^2}{\mu_{ii}^{rr}}$$
$$-2[c_{ii}^{rr}\eta_i^r+(c_{ii}^{rr}+1)(\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)]+\sum_{a\neq r}^{n_r}\frac{(2+c_{ia}^{rr})(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}$$
$$+\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\frac{(2+c_{ia}^{ru})(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}+c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{iib}^{rrv}(S_{ii}^{rr*}\mu_{ii}^{rr}+\mu_{ii}^{rr})\Bigg](V_{ii}^{rr})^2$$
$$+\Bigg[\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+2\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}+\mu_{ii}^{rr}+\sum_{a\neq i}^{n_r}\frac{(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}}+\sum_{a\neq r}^{M}\sum_{a=1i}^{n_r}\frac{(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}$$
$$-2(\gamma_i^r+\sigma_i^r+\alpha_i^r+\delta_i^r)\Big](W_{ii}^{rr})^2$$
$$+\sum_{a\neq i}^{n_r}\Bigg\{\Bigg[2\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+2\mu_{ii}^{rr}+3\mu_{ia}^{rr}-2(\rho_{ia}^{rr}+\delta_a^r)\Bigg](U_{ia}^{rr})^2+\Bigg[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}$$
$$(2+c_{ia}^{rr})\mu_{ii}^{rr}-2[c_{ia}^{rr}\eta_a^r+(1+c_{ia}^{rr})(\eta_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)]+\frac{(\varrho_a^r+d_a^r)^2}{\mu_{ia}^{rr}}$$

$$
\begin{aligned}
&+4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}}+\mu_{ia}^{rr}+c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{rrv}(S_{ia}^{rr}\mu_{ia}^{rr}+\mu_{ia}^{rr})\Bigg](V_{ia}^{rr})^2\\
&+\left[\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}}+\mu_{ii}^{rr}+\mu_{ia}^{rr}+\frac{2(\alpha_a^r)^2}{\mu_{ia}^{rr}}-2(\rho_{ia}^{rr}+\alpha_a^r+\delta_a^r)\right](W_{ia}^{rr})^2\Bigg\}\\
&+\sum_{u\neq r}^{M}\sum_{a=1}^{n_u}\Bigg\{\left[2\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+2\mu_{ii}^{rr}+3\mu_{ia}^{ru}-2(\rho_{ia}^{ru}+\delta_a^u)\right](U_{ia}^{ru})^2\\
&+\left[(2+c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+(2+c_{ia}^{ru})\mu_{ii}^{rr}-2[c_{ia}^{ru}\eta_a^u+(1+c_{ia}^{ru})(\eta_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)]\right.\\
&\left.+\frac{(\varrho_a^u+d_a^u)^2}{\mu_{ia}^{ru}}+4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}}+\mu_{ia}^{ru}+c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\beta_{aib}^{urv}(S_{ia}^{ru*}\mu_{ia}^{ru}+\mu_{ia}^{ru})\right](V_{ia}^{ru})^2\\
&+\left[\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}}+\mu_{ii}^{rr}+\mu_{ia}^{ru}+\frac{2(\alpha_a^u)^2}{\mu_{ia}^{ru}}-2(\rho_{ia}^{ru}\alpha_a^u+\delta_a^u)\right](W_{ia}^{ru})^2\Bigg\}\Bigg\}\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}c_{ii}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{iib}^{rrv}(\frac{S_{ii}^{rr*}}{\mu_{ii}^{rr}}+\frac{\bar{B}^2}{\mu_{ii}^{rr}})+(v_{iib}^{rrv})^2(S_{ii}^{rr*}+\bar{B})^2\right](V_{bi}^{vr})^2\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{a\neq i}^{n_r}c_{ia}^{rr}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{aib}^{rrv}(\frac{S_{ia}^{rr*}}{\mu_{ia}^{rr}}+\frac{\bar{B}^2}{\mu_{ia}^{rr}})+(v_{aib}^{rrv})^2(S_{ia}^{rr*}+\bar{B})^2\right](V_{ba}^{vr})^2\\
&+\sum_{r=1}^{M}\sum_{i=1}^{n_r}\sum_{u\neq r}^{M}\sum_{a=1}^{n_r}c_{ia}^{ru}\sum_{v=1}^{M}\sum_{b=1}^{n_r}\left[\beta_{aib}^{urv}(\frac{S_{ia}^{ru*}}{\mu_{ia}^{ru}}+\frac{\bar{B}^2}{\mu_{ia}^{ru}})+(v_{aib}^{urv})^2(S_{ia}^{ru*}+\bar{B})^2\right](V_{ba}^{vu})^2,
\end{aligned}
$$

where $\mu_{ia}^{ru}=g_i^r(\delta_a^u)$, g_i^r is appropriately defined by (4.25).

For each $r,u\in I(1,M)$, $i\in I(1,n_r)$, and $a\in I(1,n_u)$, using algebraic manipulations and (4.16), (4.17) and (4.18), the coefficients of $(U_{ia}^{ru})^2$, $(V_{ia}^{ru})^2$ and $(W_{ia}^{ru})^2$ in (4.28) defined by ϕ_{ia}^{ru}, ψ_{ia}^{ru} and φ_{ia}^{ru} respectively:

$$
\phi_{ia}^{ru}=\begin{cases}2(\gamma_i^r+\sigma_i^r+\delta_i^r)(1-\mathfrak{U}_{ia}^{ru}), & \text{for } u=r,\ a=i\\ 2(\rho_{ia}^{rr}+\delta_a^r+\delta_a^r)(1-\mathfrak{U}_{ia}^{ru}), & \text{for } u=r,\ a\neq i\\ 2(\rho_{ia}^{ru}+\delta_a^u+\delta_a^u)(1-\mathfrak{U}_{ia}^{ru}), & \text{for } u\neq r,\end{cases}\tag{4.29}
$$

$$
\psi_{ia}^{ru}=\begin{cases}[2c_{ii}^{rr}(1-\mathfrak{V}_{ii}^{rr})(\eta_i^r+\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r)-\mathfrak{E}_{ii}^{rr}]\\ \quad+2(\varrho_i^r+\gamma_i^r+\sigma_i^r+\delta_i^r+d_i^r),\ \text{for } u=r,\ a=i\\ [2c_{ia}^{rr}(1-\mathfrak{V}_{ia}^{rr})(\eta_a^r+\varrho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r)-\mathfrak{E}_{ia}^{rr}]\\ \quad+2(\varrho_a^r+\rho_{ia}^{rr}+\delta_a^r+d_a^r),\ \text{for } u=r,\ a\neq i\\ [2c_{ia}^{ru}(1-\mathfrak{V}_{ia}^{ru})(\eta_a^u+\varrho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u)-\mathfrak{E}_{ia}^{ru}]\\ \quad+2(\varrho_a^u+\rho_{ia}^{ru}+\delta_a^u+d_a^u),\ \text{for } u\neq r\end{cases}\tag{4.30}
$$

and

$$\varphi_{ia}^{ru} = \begin{cases} 2(\gamma_i^r + \sigma_i^r + \alpha_i^r + \delta_i^r)(1 - \mathfrak{W}_{ia}^{ru}), & \text{for } u = r,\ a = i, \\ 2(\rho_{ia}^{rr} + \delta_a^r)(1 - \mathfrak{W}_{ia}^{ru}), & \text{for } u = r,\ a \neq i, \\ 2(\rho_{ia}^{ru} + \delta_a^u)(1 - \mathfrak{W}_{ia}^{ru}), & \text{for } u \neq r \end{cases} \tag{4.31}$$

where

$$\mathfrak{E}_{ia}^{ru} = \begin{cases} \Big[2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru} + \mu_{ii}^{rr} + \frac{(\varrho_i^r)^2}{\mu_{ii}^{rr}} + \frac{(\varrho_i^r + d_i^r)^2}{\mu_{ii}^{rr}} + 4\frac{(\gamma_i^r+\sigma_i^r+\delta_i^r}{\mu_{ii}^{rr}} + \sum_{a\neq r}^{n_r}\frac{(2+c_{ia}^{rr})(\sigma_{ia}^{rr})^2}{\mu_{ii}^{rr}} \\ \quad + \sum_{u\neq r}^{M}\sum_{a=1}^{n_r}\left(\frac{(2+c_{ia}^{ru})(\gamma_{ia}^{ru})^2}{\mu_{ii}^{rr}}\right) + \sum_{b\neq i}^{n_r} c_{bi}^{rr} d_{ii}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{bi}^{vr} d_{ii}^{rr}\Big], \\ \quad \text{for } u = r,\ a = i, \\ (2 + c_{ii}^{rr})\frac{(\rho_{ia}^{rr})^2}{\mu_{ia}^{rr}} + \frac{(\varrho_a^r)^2}{\mu_{ia}^{rr}} + 2\mu_{ii}^{rr} + \frac{(\varrho_a^r + d_a^r)^2}{\mu_{ia}^{rr}} \\ \quad +4\frac{(\rho_{ia}^{rr}+\delta_a^r)^2}{\mu_{ia}^{rr}} + \mu_{ia}^{rr} + \sum_{b\neq i}^{n_r} c_{ba}^{rr} d_{ia}^{rr} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{ba}^{vr} d_{ia}^{rr},\ \text{for } u = r,\ a \neq i, \\ (2 + c_{ii}^{rr})\frac{(\rho_{ia}^{ru})^2}{\mu_{ia}^{ru}} + \frac{(\varrho_a^u)^2}{\mu_{ia}^{ru}} + 2\mu_{ii}^{rr} + \frac{(\varrho_a^u + d_a^u)^2}{\mu_{ia}^{ru}} \\ \quad +4\frac{(\rho_{ia}^{ru}+\delta_a^u)^2}{\mu_{ia}^{ru}} + \mu_{ia}^{ru} + \sum_{b\neq i}^{n_r} c_{ba}^{ru} d_{ia}^{ru} + \sum_{v\neq r}^{M}\sum_{b=1}^{n_r} c_{ba}^{vu} d_{ia}^{ru}, u \neq r \end{cases} \tag{4.32}$$

Under the assumptions on $\mathfrak{U}_{ia}^{ru}$, $\mathfrak{V}_{ia}^{ru}$ and $\mathfrak{W}_{ia}^{ru}$, it is clear that $\phi_{ia}^{ru}, \psi_{ia}^{ru}$ and φ_{ia}^{ru} are positive for suitable choice of c_{ia}^{ru} defined in (4.8). We substitute (4.15), (4.29), (4.30) and (4.32) into (4.28). Thus inequality (4.28) can be rewritten as

$$\begin{aligned} LV(\tilde{x}_{00}^{00}) \leq \sum_{r=1}^{M}\sum_{i=1}^{n_r} - \big\{&[\phi_{ii}^{rr}(U_{ii}^{rr})^2 + \psi_{ii}^{rr})(V_{ii}^{rr})^2\ \varphi_{ii}^{rr}(W_{ii}^{rr})^2] \\ &+ \sum_{a\neq r}^{n_r}[\phi_{ia}^{rr}(U_{ia}^{rr})^2 + \psi_{ia}^{rr}(V_{ia}^{rr})^2 + \varphi_{ia}^{rr}(W_{ia}^{rr})^2] \\ &+ \sum_{u\neq r}^{M}\sum_{a=1}^{n_u}[\phi_{ia}^{ru}(U_{ia}^{ru})^2 + \psi_{ia}^{ru}(V_{ia}^{ru})^2 + \varphi_{ia}^{ru}(W_{ia}^{ru})^2]\big\} \end{aligned} \tag{4.33}$$

This proves the inequality (4.19). Now, the validity of (4.20) follows from (4.19), that is,

$$LV(\tilde{x}_{00}^{00}) \leq -cV(\tilde{x}_{00}^{00}),$$

where $c = \frac{\min_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}(\phi_{ia}^{ru}, \psi_{ia}^{ru}, \varphi_{ia}^{ru})}{\max_{1\leq r,u\leq M, 1\leq i\leq n_r, 1\leq a\leq n_u}\{C_{ia}^{ru}+2\}}$. This establishes the result. □

We now formally state the stochastic stability theorems for the disease free equilibria.

Theorem 4.1. *Given $r, u \in I(1, M)$, $i \in I(1, n_r)$ and $a \in I(1, n_u)$. Let us assume that the hypotheses of Lemma 4.2 are satisfied. Then the disease free solutions E_{ia}^{ru}, are asymptotically stable in the large. Moreover, the solutions E_{ia}^{ru} are exponentially mean square stable.*

Proof. From the application of comparison result [21, 46], the proof of stochastic asymptotic stability follows immediately Moreover, the disease free equilibrium state is exponentially mean square stable. We now consider the following corollary to Theorem 4.1. □

Corollary 4.1. *Let $r \in I(1,M)$ and $i \in I(1,n_r)$. Assume that $\sigma_i^r = \gamma_i^r = 0$, for all $r \in I(1,M)$ and $i \in I(1,n_r)$.*

$$\mathfrak{U}_{ia}^{ru} = \begin{cases} \dfrac{\frac{1}{(\delta_i^r)}}{\left[\frac{1}{2\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{3}{2}\mu_{ii}^{rr}}\right]} & \text{for } u=r,\ i=a \\ \dfrac{\frac{1}{(\delta_a^r)}}{\left[\mu_{ii}^{rr}+\mu_{ia}^{rr}\right]}, & \text{for } u=r,\ a\neq i \\ \dfrac{\frac{1}{(\delta_a^u)}}{\left[\mu_{ii}^{rr}+\mu_{ia}^{ru}\right]}, & \text{for } u\neq r, \end{cases} \tag{4.34}$$

$$\mathfrak{V}_{ia}^{ru} = \begin{cases} \dfrac{\sum_{u=1}^{M}\sum_{a=1}^{n_u}\frac{1}{2}\mu_{ia}^{ru}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{iib}^{rrv}(S_{ii}^{rr*}+\mu_{ii}^{rr})+\frac{1}{2}d_{ii}^{rr}}{\eta_i^r+\varrho_i^r+\delta_i^r+d_i^r}, & \text{for } a=i,\ u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{rrv}(S_{ia}^{rr*}+\mu_{ia}^{rr})+\frac{1}{2}d_{ia}^{rr}}{\eta_a^r+\varrho_a^r+\delta_a^r+d_a^r}, & \text{for } a\neq i,\ u=r \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\sum_{v=1}^{M}\sum_{b=1}^{n_v}\frac{1}{2}\beta_{aib}^{urv}(S_{ii}^{ru*}+\mu_{ia}^{ru})+\frac{1}{2}d_{ia}^{ru}}{\eta_a^u+\varrho_a^u+\delta_a^u+d_a^u}, & \text{for } u\neq r. \end{cases} \tag{4.35}$$

and

$$\mathfrak{W}_{ia}^{ru} = \begin{cases} \dfrac{\left[\frac{1}{2}\sum_{u=1}^{M}\sum_{a=1}^{n_u}\mu_{ia}^{ru}+\frac{(\alpha_i^r)^2}{\mu_{ii}^{rr}}+\frac{1}{2}\mu_{ii}^{rr}\right]}{(\alpha_i^r+\delta_i^r)}, & \text{for } u=r,\ a=i, \\ \dfrac{\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{rr}+\frac{(\alpha_a^r)^2}{\mu_{ia}^{rr}}}{(\alpha_a^r+\delta_a^r)}, & \text{for } u=r,\ a\neq i, \\ \dfrac{\left[\frac{1}{2}\mu_{ii}^{rr}+\frac{1}{2}\mu_{ia}^{ru}+\frac{(\alpha_a^u)^2}{\mu_{ia}^{ru}}\right]}{\alpha_a^u+\delta_a^u}, & \text{for } u\neq r \end{cases} \tag{4.36}$$

The equilibrium state E_{ii}^{rr} is stochastically asymptotically stable provided that $\mathfrak{U}_{ia}^{ru}$, $\mathfrak{W}_{ia}^{ru} \leq 1$ and $\mathfrak{V}_{ia}^{ru} < 1$, for all $u \in I^r(1,M)$ and $a \in I_i^r(1,n_u)$.

Proof. Follows immediately from the hypotheses of Lemma 4.2, (letting $\sigma_i^r = \gamma_i^r = 0$), the conclusion of Theorem 4.1 and some algebraic manipulations. □

Remark 4.2. The presented results about the two-level large scale SIRS disease dynamic model depend on the underlying system parameters. In particular, the sufficient conditions are algebraically simple, computationally attractive and explicit in terms of the rate parameters. As a result of this, several scenarios can be discussed and exhibit practical course of action to control the disease. For simplicity, we present an illustration as follows: the conditions of $\sigma_i^r = \gamma_i^r = 0$, $\forall\, r, i$ in Corollary 4.1 signify that the arbitrary site s_i^r is a sink[18, 19] for all other sites in the inter and intra-regional accessible domain. This scenario is displayed in Figure 3. The condition $\mathfrak{U}_{ia}^{ru} \leq 1$ exhibits that the average infectious period is smaller than the joint average life span of individuals in the intra and inter-regional accessible domain of site s_i^r.

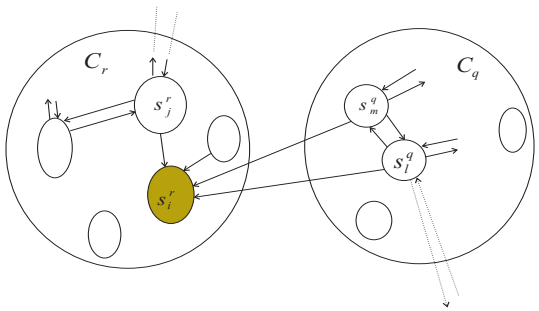

FIGURE 3. Shows that residents of site s_i^r are present only at their home site s_i^r. Hence they isolate every site from their inter and intra reginal accessible domain $C(s_i^r)$. Site s_i^r is a 'sink' in the context of the compartmental system [18, 19]. The arrows represent a transport network between any two sites and regions. Furthermore, the dotted lines and arrows indicate connection with other sites and regions.

Furthermore, the condition $\mathfrak{V}_{ia}^{ru} < 1$ signifies that the magnitude of disease inhibitory processes for example, the magnitude of the recovery process is greater than the disease transmission process. A future detailed study of the disease dynamics in the two scale network dynamic structure for many real life scenarios using the presented two level large-scale SIRS disease dynamic model will appear elsewhere.

5. EXAMPLE

By using the two scale mobility model [1], the mobility dynamic structure determined by the respective intra and interregional mobility data recorded in [1, Tables 1& 2, Section 6], and also the influenza pandemic simulation model in [22], we develop a two-scale SIR influenza epidemic dynamic model. The compartmental framework for the SIR epidemic model is exhibited in Figure 2, where $\eta_i^r = \alpha_i^r = 0$, $\forall\, r \in I(1, M)$, $i \in I(1, n_r)$. Furthermore, a diagram illustrating the inter-patch connections in the example for two scale dynamic epidemic model represented in this example is shown in Figure 4. In the absence of intra and interregional mobility return rates, based on the mobility structure and the probabilistic formulation of the mobility process, we simulate intra and interregional mobility return rates. We display the intra and inter-regional mobility return rates in Table 1 and Table 2 respectively.

The following assumptions are made concerning the influenza epidemic process represented in this example: (a_1)The population structure and influenza transmission process at every site s_i^r, $r = 1, 2, 3$, $i = 1, 2, 3$ in region $C_r, r = 1, 2, 3$ is similar to the population structure and the influenza transmission process represented in the simulation model of [22]. That is, we assume that every person in site s_i^r belongs to one age dependent stratum (ages$\geq$ 0). In addition, each individual belongs to three mixing or contact groups $z_j, j = 1, 2, 3$, for example, household, marketplace,

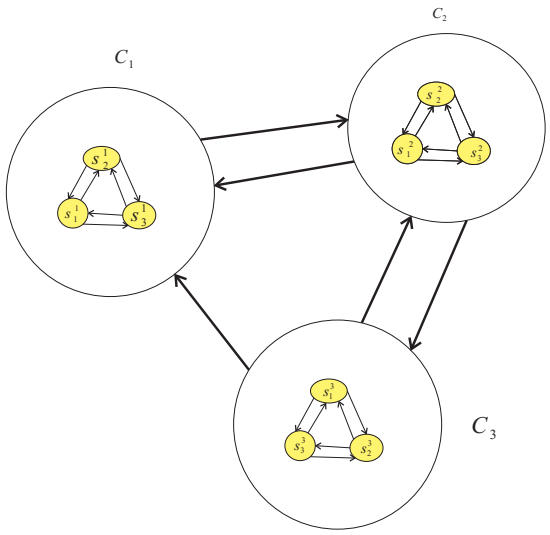

FIGURE 4. A two scale network of three spatial regions C_r, $r = 1, 2, 3$ of human habitation and three interconnected sites s_i^r, $i = 1, 2, 3$ in each region. The arrows represent direction of human mobility and summarize the homogeneities in the epidemic process at each site and region. C_1 & C_2, and C_2 & C_3 are symmetric in the human mobility process. C_1 is a sink for C_3 in human mobility. All sites in each region are completely symmetric in the human mobility process. The details of the two scale human mobility process represented in this example are given in [1].

$(\rho_{12}^{11}, \rho_{13}^{11}, \rho_{21}^{11}, \rho_{23}^{11})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{11}, \rho_{32}^{11})$	(0.013230408,0.001305838)
$(\rho_{12}^{22}, \rho_{13}^{22}, \rho_{21}^{22}, \rho_{23}^{22})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{22}, \rho_{32}^{22})$	(,0.013230408,0.001305838)
$(\rho_{12}^{33}, \rho_{13}^{33}, \rho_{21}^{33}, \rho_{23}^{33})$	(0.000092504,0.000177496,0.164327,0.0001173)
$(\rho_{31}^{33}, \rho_{32}^{33})$	(0.013230408,0.001305838)

TABLE 1. The intra-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 4 are simulated based on the mobility structure and the probabilistic formulation for the mobility process cf. [1].

and the community. In each day, a susceptible person, A, has contacts with other individuals in his or her contact zones. The probability of acquiring infection depends on (a) the number of different persons A has contacts within the contact group, (b) the time duration, in minutes, of all contacts (c) the rate of infection transmission per-minute if the contacted person is infectious (see [22]). We assume that in a given day, a susceptible person makes three contacts in mixing group z_1, ten contacts in mixing group z_2, and three contacts in mixing group z_3. In addition, each contacted person is infectious. Furthermore, the time duration d and the per minute influenza transmission rate λ per contact in all contact zones are [zone z_1: $d \approx 92$ minutes,

$(\rho_{11}^{12}, \rho_{12}^{12}, \rho 12_{13}, \rho 12_{21}, \rho 12_{22})$	(0.1995,0.035,0.0985,0.007892,0.02748)
$(\rho 12_{23}, \rho 12_{31}, \rho 12_{32}, \rho 12_{33})$	(0.075824,0.04256,0.009616,0.028628)
$(\rho_{11}^{21}, \rho_{12}^{21}, \rho 21_{13}, \rho 21_{21}, \rho 21_{22})$	(0.002096896,0.00175424,0.003460864,0.00043856,0.0001664)
$(\rho 21_{23}, \rho 21_{31}, \rho 21_{32}, \rho 21_{33})$	(0.00071504, 0.001944052,0.00119788,0.0001713912)
$(\rho_{11}^{23}, \rho_{12}^{23}, \rho 23_{13}, \rho 23_{21}, \rho 23_{22})$	(0.018512, 0.03290368,0.0272192,0.04883712,0.00151648)
$(\rho 23_{23}, \rho 23_{31}, \rho 23_{32}, \rho 23_{33})$	(0.0219232, 0.00383316,0.0025404,0.000414644)
$(\rho_{11}^{31}, \rho_{12}^{31}, \rho 31_{13}, \rho 31_{21}, \rho 31_{22})$	(0.001285712, 0.00085328, 0.001725008, 0.0004380944, 0.000379536)
$(\rho 31_{23}, \rho 31_{31}, \rho 31_{32}, \rho 31_{33})$	(0.0005991696, 0.000000371428, 0.00000026332, 0.000000281252)
$(\rho_{11}^{32}, \rho_{12}^{32}, \rho 32_{13}, \rho 32_{21}, \rho 31_{22})$	(0.0003230096, 0.00036224, 0.0004619664, 0.00043146104, 0.0003741576)
$(\rho 32_{23}, \rho 32_{31}, \rho 32_{32}, \rho 32_{33})$	(0.00059126136, 0.000498339428, 0.00042838332, 0.000070993252)

TABLE 2. The inter-regional return rates of residents of sites in the two scale network of spatial patches illustrated in Figure 4 are simulated based on the mobility structure and the probabilistic formulation for the mobility process cf. [1].

$\lambda = 0.00062]$, [zone z_2: $d \approx 120$ minutes, $\lambda = 0.00061$] and [zone z_3: $d \approx 51$ minutes, $\lambda = 0.00061$]. Furthermore, we assume that the number and duration of contacts are the same on weekdays and weekend days. We utilize the probability model $1 - \exp(-\lambda d)$ for the influenza transmission occurring during a contact of d minutes and a transmission rate λ (see [22]) to find the infection probability β_{aib}^{urv} of the two-scale SIRS epidemic dynamic model. It is easy to see that the infection probability per day for a susceptible person at site s_i^r is $\beta_{aib}^{urv} = 0.6277$. (a_2) In the absence of data for the recovery and disease related death processes, we take the recovery and disease mortality rate to be $\varrho_a^u = 0.05067$ and $d_a^u = 0.01838$, $u = 1, 2, 3$; $a, i = 1, 2, 3$ respectively. (a_3) The population in this example assumed to be remote and lacking the high technological facilities found in the developed world. Furthermore, we assume that influenza is highly endemic in this population. As a result, we can assume that the time duration of the epidemic is comparable with the average life span of individuals in the population. In the absence of data concerning average birth rates, we use the yearly birth rate data from [23] for the people of the Dominican republic, $B = \frac{births}{1000} = \frac{22.39}{1000}$ as an estimate. Furthermore, we assume this birth rate is the same for all residents of sites in the population. That is, the constant birth rate is $B_a^u = \frac{births}{1000} = \frac{22.39}{1000}$ per year, for $u = 1, 2, 3$; a, $i = 1, 2, 3$. (a_4) In addition, using the average life span of the people of Dominican Republic [24], the natural death rate of the residents at all sites and regions are the same and is calculated as the reciprocal of the average life span of individuals in the population, that is, $\delta_a^u = \frac{1}{77.15 \times 365}$, $u = 1, 2, 3$; $a, i = 1, 2, 3$ per day. (a_5) The effects of the fluctuating environment on the dynamics of the influenza epidemic is assumed to be the same at all sites and regions. We take the standard deviation of the environmental fluctuations to be $v_{aib}^{urv} = 0.5$, r, u, $v = 1, 2, 3$; a, b, $i = 1, 2, 3$.

Using the standard Euler-Maruyama method stochastic approximation scheme [25], we generate the trajectories for the residents of sites s_1^1, s_1^2 and s_1^3in regions C_1, C_2 and C_3 respectively, for the different population diseases classification (S, I, R), and current locations at some sites in the intra and inter-regional accessible domain of the sites. The solutions are displayed in Figure 5, Figure 6 and Figure 7 respectively. We note that the following initial conditions were used: for $r, u \in I(1,3)$, $i, a \in I(1,3)$,

$$S_{ia}^{ru}(0) = \begin{cases} 9, & \text{for } r = u,\ i = a \\ 8, & \text{for } r = u,\ i \neq a \\ 7, & \text{for } r \neq u, \end{cases}$$

$$I_{ia}^{ru}(0) = \begin{cases} 6, & \text{for } r = u,\ i = a \\ 4, & \text{for } r = u,\ i \neq a \\ 3, & \text{for } r \neq u \end{cases}$$

and $R_{ia}^{ru}(0) = 2, \forall r, u, i, a \in I(1,3)$. Furthermore, the trajectories were generated over the time interval $t \in [0,1]$.

6. CONCLUSION

The recent high technological changes and scientific developments have led to many\variant structure types inter-patch connections interactions in the global human population. This has further afforded efficient mass flow of human beings, animals, goods and equipments between patches thereby causing the appearance of new disease strains and infectious agents at non-endemic zones. The presented two-scale network disease dynamic model characterizes the dynamics of an SIRS epidemic in a population with various scale levels created by the heterogeneities in the population. Moreover, the disease dynamics is subject to random environmental perturbations at the disease transmission stage of the disease. Furthermore, the SIRS epidemic has a proportional transfer to the susceptible class immediately after the infectious period. This work provides a mathematical and probabilistic algorithmic tool to develop different levels nested type disease transmission rates as well as the variability in the transmission process in the framework of the network-centric Ito-Doob type dynamic equations.

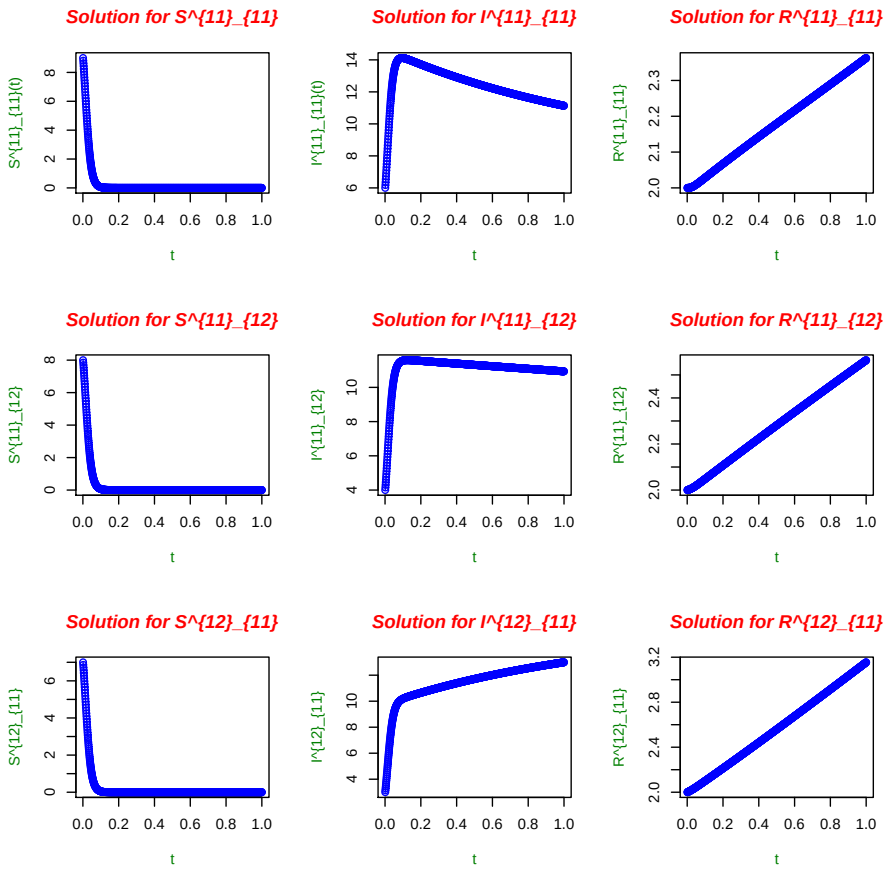

FIGURE 5. Trajectories of the disease classification (S, I, R) for residents of site s_1^1 in region C_1 at their current location in the two-scale spatial patch dynamic structure. Figures (a), (b) & (c) represent the trajectories of the different disease classes of residents of site s_1^1 at home. Figures (d), (e) & (f) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_2^1 in home region C_1. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^1 at the intra-regional level. Figures (g), (h) & (i) represent the trajectories of the different disease classes of residents of site s_1^1 visiting site s_1^2 in region C_2. These figures reflect the behavior of the disease affecting the residents of site s_1^1 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious (I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

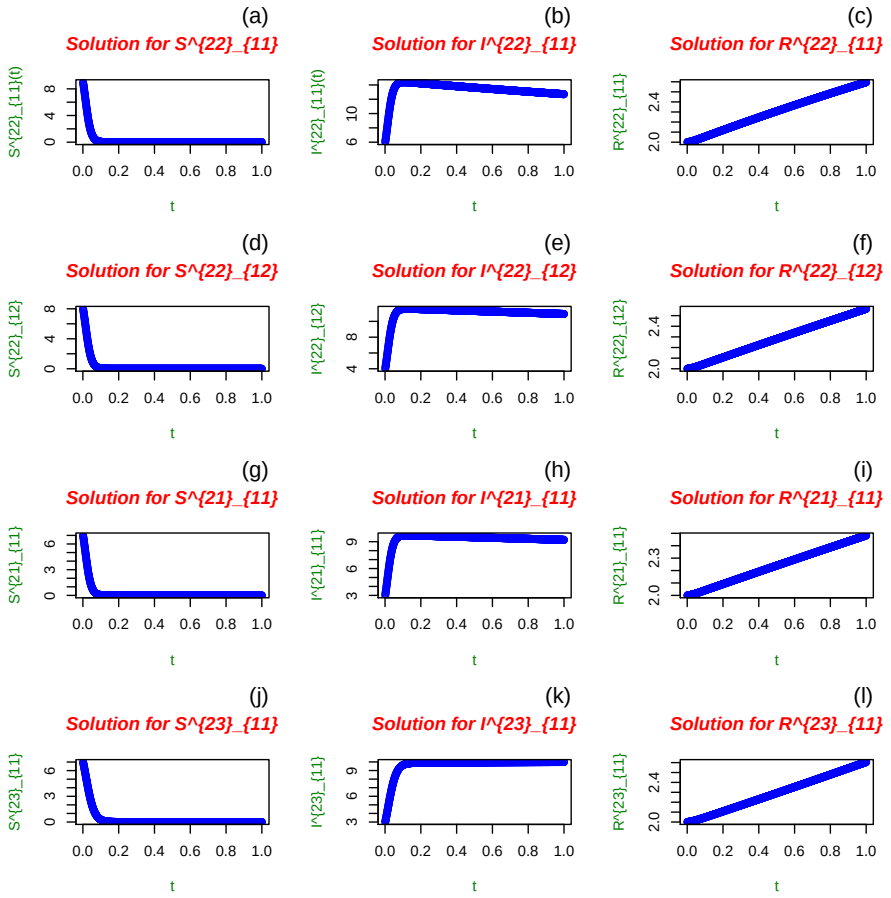

FIGURE 6. Trajectories of the disease classification (S, I, R) for residents of site s_1^2 in region C_2 at their current location in the two-scale spatial patch dynamic structure. Figures (a), (b) & (c) represent the trajectories of the different disease classes of residents of site s_1^2 at home. Figures (d), (e) & (f) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_2^2 in home region C_2. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^2 at the intra-regional level. Figures (g), (h) & (i) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^1 in region C_1. Figures (j), (k) & (l) represent the trajectories of the different disease classes of residents of site s_1^2 visiting site s_1^3 in region C_3. These last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^2 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious (I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

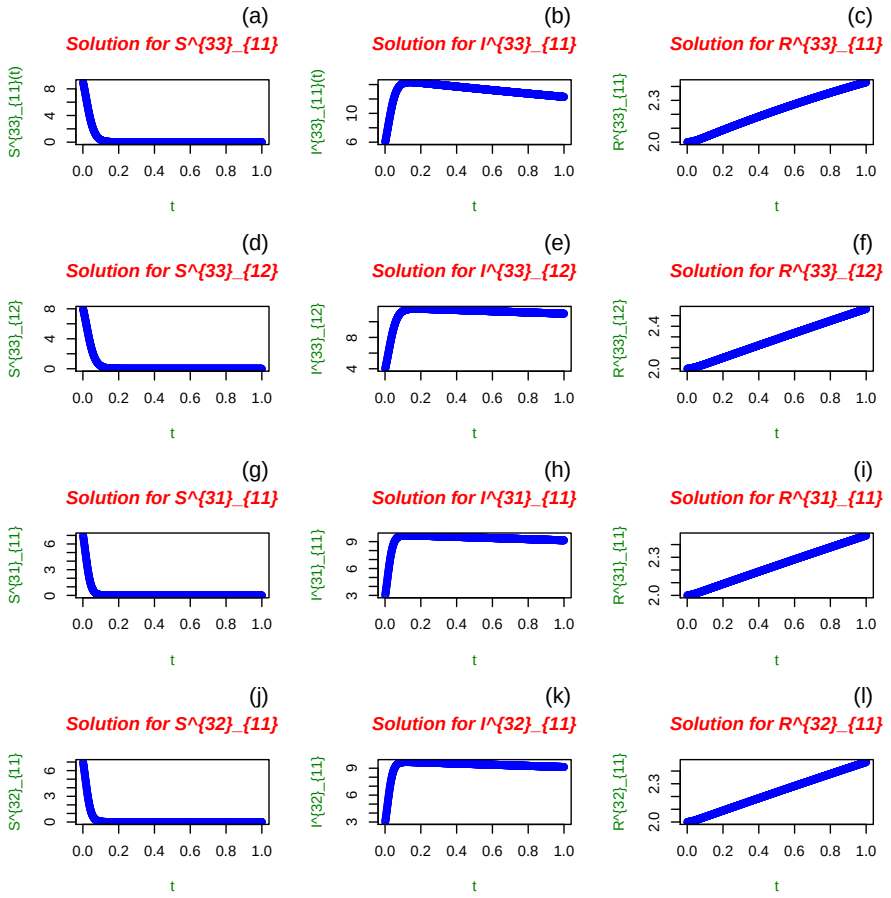

FIGURE 7. Trajectories of the disease classification (S, I, R) for residents of site s_1^3 in region C_3 at their current location in the two-scale spatial patch dynamic structure. Figures (a), (b) & (c) represent the trajectories of the different disease classes of residents of site s_1^3 at home. Figures (d), (e) & (f) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_2^3 in home region C_3. These two groups of figures are representative of the disease dynamics of influenza affecting the residents of site s_1^3 at the intra-regional level. Figures (g), (h) & (i) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^1 in region C_1. Figures (j), (k) & (l) represent the trajectories of the different disease classes of residents of site s_1^3 visiting site s_1^2 in region C_2. The last two groups of figures reflect the behavior of the disease affecting the residence of site s_1^3 at the inter-regional level. Furthermore, we observe that the trajectories of the susceptible (S) and infectious (I) populations saturate to their equilibrium states. We further note that the trajectory paths are random in character but because of the scale of the pictures presented in this figure, they appear to be smooth.

The model validation results are developed and a positively invariant set for the dynamic model is defined. Moreover, the globalization of the solution existence is obtained via the construction of the two-scale dynamic structure motivated Lypunov

function. The detailed stochastic asymptotic stability results of the disease free equilibrium are also exhibited in this paper. Moreover, the system parameter dependent threshold values controlling the stochastic asymptotic stability of the disease free equilibrium are also defined. Furthermore, a deduction to the stochastic asymptotic stability results for a simple real life scenario is illustrated. Further detail study of the SIRS disease dynamic model the two scale network dynamic mobility structure real life scenarios will appear elsewhere. Simulation results for an SIR influenza epidemic represented by the two-scale network dynamic epidemic model for a specific scenario having a dynamic structure parallel to the earlier study [1] is also presented.

We note that the disease dynamics is subject to random environmental perturbations from other related processes such as the mobility, recovery, birth and death processes. The stochastic variability due to the disease transmission incorporated in the epidemic dynamic model will be extended to the stochastic variability in the mobility, recovery and birth and death processes. A further detailed study of the oscillation of the epidemic process about the ideal endemic equilibrium of the dynamic epidemic model will also appear else where. In addition, a detailed study of the hereditary features of the infectious agent such as the time-lag to infectiousness of exposed individuals in the population is currently underway and it will also appear elsewhere.

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