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A NUMERICAL INVESTIGATION OF THE ROOTS OF THE SECOND KIND λ-BERNOULLI POLYNOMIALS

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ABSTRACT. In this paper we consider a new type of the Apostol Bernoulli numbers and polynomials. We call them the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$. We also observe the behavior of complex roots of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$, using numerical investigation. Finally, we give a table for the solutions of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$.

AMS (MOS) Subject Classification. 11B68, 11S40, 11S80

1. INTRODUCTION

Several mathematicians have studied the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, q-Bernoulli numbers and polynomials (see [1–16]). These numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In this paper, we introduce the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$. In order to study the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$, we must understand the structure of the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$. Therefore, using computer, a realistic study for the second kind λ-Bernoulli numbers $E_{n,q}$ and polynomials $B_{n,\lambda}(x)$ is very interesting. It is the aim of this paper to observe an interesting phenomenon of 'scattering' of the zeros of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ in complex plane. The outline of this paper is as follows. We introduce the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$. In section 2, we construct the λ-Bernoulli numbers and polynomials. Some interesting results are obtained. In Section 3, we describe the beautiful zeros of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ using a numerical investigation. Finally, we investigate the roots of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$. Also we carried out computer experiments for doing demonstrate a remarkably regular structure of the complex roots of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$. Throughout this paper, we always make use of the following

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notations: $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of natural numbers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers.

The classical Bernoulli polynomials $B_n(x)$ are usually defined by means of the following generating functions:

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \qquad (|z| < 2\pi)$$

The classical Bernoulli numbers $B_n := B_n(0)$.

We begin by recalling here Apostol's definitions as follows:

Definition 1 (Apostol [1]). The Apostol-Bernoulli polynomials $B_n(x; \lambda)$ are defined by means of the generating function:

$$\frac{t}{\lambda e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x; \lambda) \frac{t^n}{n!} \qquad (|t + \log \lambda| < 2\pi) \tag{1}$$

with of course,

$$B_n(x) = B_n(x; 1) \text{ and } B_n(\lambda) := B_n(0; \lambda),$$

where $B_n(\lambda)$ denotes the so-called λ-Bernoulli numbers.

2. THE SECOND KIND λ-BERNOULLI NUMBERS AND POLYNOMIALS

In this section, we introduce the second kind λ-Bernoulli numbers $B_{n,\lambda}$ and polynomials $B_{n,\lambda}(x)$ and investigate their properties. Based on Apostol's idea, it follows that we define the second kind λ-Bernoulli polynomials and numbers.

Definition 2. The second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ and numbers $B_{n,\lambda}$ are defined by means of the generating functions

$$\frac{2te^t}{\lambda e^{2t} - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}, \qquad (|2t + \log \lambda| < 2\pi), \tag{2}$$

and

$$\frac{2te^t}{\lambda e^{2t} - 1} = \sum_{n=0}^{\infty} B_{n,\lambda} \frac{t^n}{n!}, \tag{3}$$

respectively.

Setting $\lambda = 1$ in (2) and (3), we can obtain the corresponding definitions for the second kind Bernoulli polynomials $B_n(x)$ and numbers B_n respectively. More studies and results in this subject we may see reference [13].

By simple calculation, the second kind λ-Bernoulli polynomials tune into the following generating function:

$$\sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!} = \frac{2te^t}{\lambda e^{2t} - 1} e^{xt} = -2t \sum_{n=0}^{\infty} \lambda^n e^{(2n+1+x)t} \tag{4}$$

Setting $x = 0$ in above generating function, we can obtain the corresponding definitions for the second kind λ-Bernoulli numbers as following generating function:

$$\sum_{n=0}^{\infty} B_{n,\lambda}\frac{t^n}{n!} = -2t\sum_{n=0}^{\infty} \lambda^n e^{(2n+1)t} \tag{5}$$

By using computer, the second kind λ-Bernoulli numbers $B_{n,\lambda}$ can be determined explicitly. A few of them are

$$\begin{aligned}
B_{0,\lambda} &= 0,\\
B_{1,\lambda} &= \frac{2}{-1+\lambda},\\
B_{2,\lambda} &= \frac{4}{-1+\lambda} - \frac{8\lambda}{(-1+\lambda)^2},\\
B_{3,\lambda} &= \frac{6}{-1+\lambda} - \frac{48\lambda}{(-1+\lambda)^2} + \frac{48\lambda^2}{(-1+\lambda)^3}\\
B_{4,\lambda} &= \frac{8}{-1+\lambda} - \frac{208\lambda}{(-1+\lambda)^2} + \frac{576\lambda^2}{(-1+\lambda)^3} - \frac{384\lambda^3}{(-1+\lambda)^4}.
\end{aligned}$$

By the above definition, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2te^t}{\lambda e^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} B_{n,\lambda}\frac{t^n}{n!}\sum_{k=0}^{\infty} x^k\frac{t^k}{k!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}B_{k,\lambda}x^{n-k}\right)\frac{t^n}{n!}.
\end{aligned} \tag{6}$$

By using comparing coefficients of $\frac{t^n}{n!}$, we have the following theorem.

Theorem 3. *For any positive integer n, we have*

$$B_{n,\lambda}(x) = \sum_{k=0}^{n}\binom{n}{k}B_{k,\lambda}x^{n-k}. \tag{7}$$

The second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ can be determined explicitly. A few of them are

$$\begin{aligned}
B_{0,\lambda}(x) &= 0,\\
B_{1,\lambda}(x) &= \frac{2}{-1+\lambda},\\
B_{2,\lambda}(x) &= \frac{4}{-1+\lambda} - \frac{8\lambda}{(-1+\lambda)^2} + \frac{4x}{-1+\lambda},\\
B_{3,\lambda}(x) &= \frac{6}{-1+\lambda} - \frac{48\lambda}{(-1+\lambda)^2} + \frac{48\lambda^2}{(-1+\lambda)^3} + \frac{12x}{-1+\lambda} - \frac{24\lambda x}{(-1+\lambda)^2} + \frac{6x^2}{-1+\lambda}.
\end{aligned}$$

For $n = 1, \dots, 10$, we can draw a plot of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$, respectively. This shows the ten plots combined into one. We display the shape of $B_{n,\lambda}(x)$, $\lambda = -1/2$, $-6 \le x \le 6$ (Figure 1).

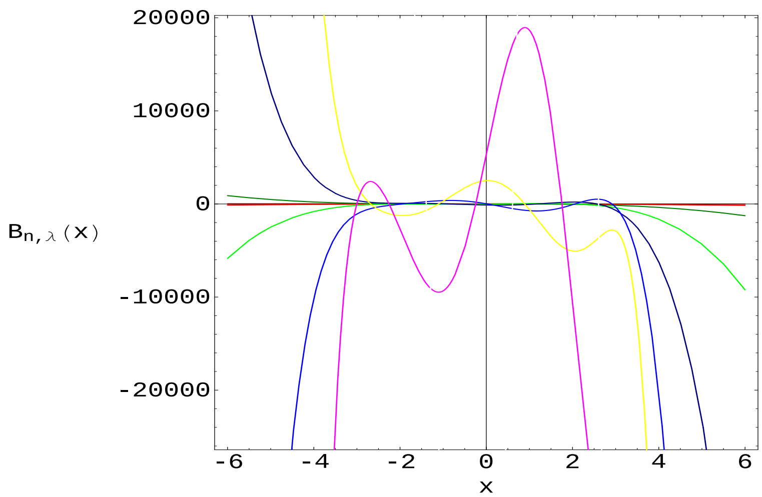

FIGURE 1. Curve of $B_{n,\lambda}(x)$

The following basic properties of the second kind λ- Bernoulli polynomials $B_{n,\lambda}(x)$ are derived from (2), (3), (4), and (5). We, therefore, choose to omit the details involved.

Proposition 4 (Difference equation)**.**

$$\lambda B_{n,\lambda}(x+2) - B_{n,\lambda}(x) = 2n(x+1)^{n-1}.$$

Proposition 5 (Differential relation)**.**

$$\frac{\partial}{\partial x} B_{n,\lambda}(x) = nB_{n-1,\lambda}(x).$$

Proposition 6 (Integral formula)**.**

$$\int_a^b B_{n-1,\lambda}(x)dx = \frac{1}{n}\big(B_{n,\lambda}(b) - B_{n,\lambda}(a)\big).$$

Proposition 7 (Complement formula)**.**

$$B_{n,\lambda}(x) = \frac{(-1)^n}{\lambda} B_{n,\lambda^{-1}}(-x).$$

Since

$$\begin{aligned}\sum_{n=0}^{\infty} B_{n,\lambda}(x+y)\frac{t^n}{n!} &= \frac{2te^t}{\lambda e^{2t}-1} e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} \sum_{k=0}^{\infty} y^n \frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n}\binom{n}{k} B_{k,\lambda}(x) y^{n-k}\Big)\frac{t^n}{n!},\end{aligned}$$

we have the following addition theorem.

Theorem 8. *The second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ satisfies the following relation:*

$$B_{n,\lambda}(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_{k,\lambda}(x)y^{n-k}.$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}&=\frac{2te^t}{\lambda e^{2t}-1}e^{xt}=\frac{2t}{\lambda^m e^{2mt}-1}e^{(1+x)t}\sum_{k=0}^{m-1}(\lambda e^{2t})^k\\&=\sum_{k=0}^{m-1}\lambda^k\frac{2t}{\lambda^m e^{2mt}-1}e^{(2k+1+x)t}\\&=\frac{1}{m}\sum_{k=0}^{m-1}\lambda^k\sum_{n=0}^{\infty}B_{n,\lambda^m}\Big(\frac{2k+1+x-m}{m}\Big)\frac{(mt)^n}{n!}\\&=\sum_{n=0}^{\infty}\left(m^{n-1}\sum_{k=0}^{m-1}\lambda^k B_{n,\lambda^m}\Big(\frac{2k+1+x-m}{m}\Big)\right)\frac{t^n}{n!}.\end{aligned}$$

Hence we have the below distribution theorem.

Theorem 9. *For $n\in\mathbb{N}$, we have*

$$B_{n,\lambda}(x)=m^{n-1}\sum_{k=0}^{m-1}\lambda^k B_{n,\lambda^m}\Big(\frac{2k+1+x-m}{m}\Big).$$

By using generating functions, we also obtain the following λ-odd sum.

Theorem 10. *For $n\in\mathbb{N}$, we have*

$$\sum_{j=0}^{m-1}\lambda^j(2j+1)^{n-1}=\frac{\lambda^m B_{n,\lambda}(2m)-B_{n,\lambda}}{2n}$$

3. ZEROS OF THE SECOND KIND λ-BERNOULLI POLYNOMIALS $B_{n,\lambda}(x)$

In this section, we investigate the zeros of the second λ-Bernoulli polynomials $B_{n,\lambda}(x)$. We investigate the beautiful zeros of the $B_{n,\lambda}(x)$ by using a computer. We plot the zeros of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ for $n=15,20,25,30,\lambda=-1/2$ and $x\in\mathbb{C}$ (Figure 2).

In Figure 2 (top-left), we choose $n=15$ and $\lambda=-1/2$. In Figure 2 (top-right), we choose $n=20$ and $\lambda=-1/2$. In Figure 2 (bottom-left), we choose $n=25$ and $\lambda=-1/2$. In Figure 2 (bottom-right), we choose $n=30$ and $\lambda=-1/2$.

We plot the zeros of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ for $n=30$, $\lambda=-5,-10,-20,-30$ and $x\in\mathbb{C}$ (Figure 3). In Figure 3 (top-left), we choose $n=30$ and $\lambda=-5$. In Figure 3 (top-right), we choose $n=30$ and $\lambda=-10$. In Figure 3

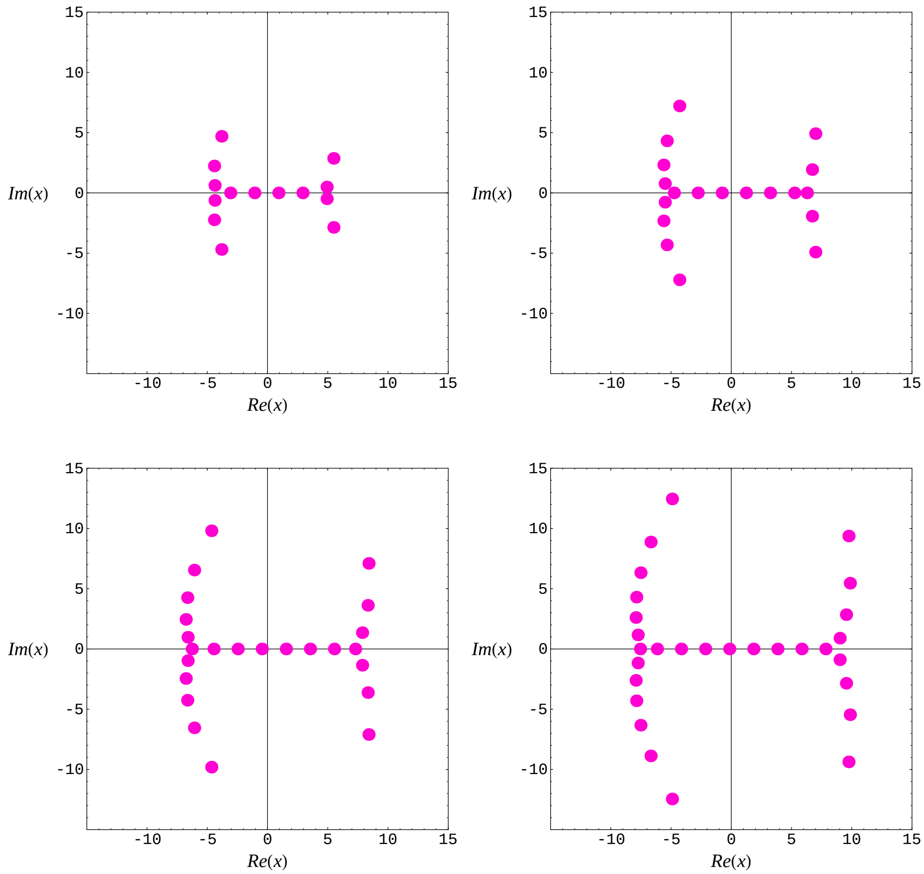

FIGURE 2. Zeros of $B_{n,\lambda}(x)$ for $n = 15, 20, 25, 30$

(bottom-left), we choose $n = 30$ and $\lambda = -20$. In Figure 3 (bottom-right), we choose $n = 30$ and $\lambda = -30$.

The real zeros of $B_{30,\lambda}(x)$ for $\lambda \to -1$ structure are presented (Figure 4). Stacks of zeros of $B_{n,-1/2}(x)$ for $1 \leq n \leq 30$ from a 3-D structure are presented (Figure 5). Plot of real zeros of $B_{n,\lambda}(x)$ for $1 \leq n \leq 30$ and $\lambda = -2, -1/2$ structure are presented (Figure 6). In Figure 6 (left), we choose $1 \leq n \leq 30$ and $\lambda = -2$. In Figure 6 (right), we choose $1 \leq n \leq 30$ and $\lambda = -1/2$. Plot of real zeros of $B_{n,\lambda}(x)$ for $\lambda \to -1$, $1 \leq n \leq 30$ structure are presented (Figure 7). Our numerical results for approximate solutions of real zeros of $B_{n,w}(x)$ are displayed (Tables 1, 2).

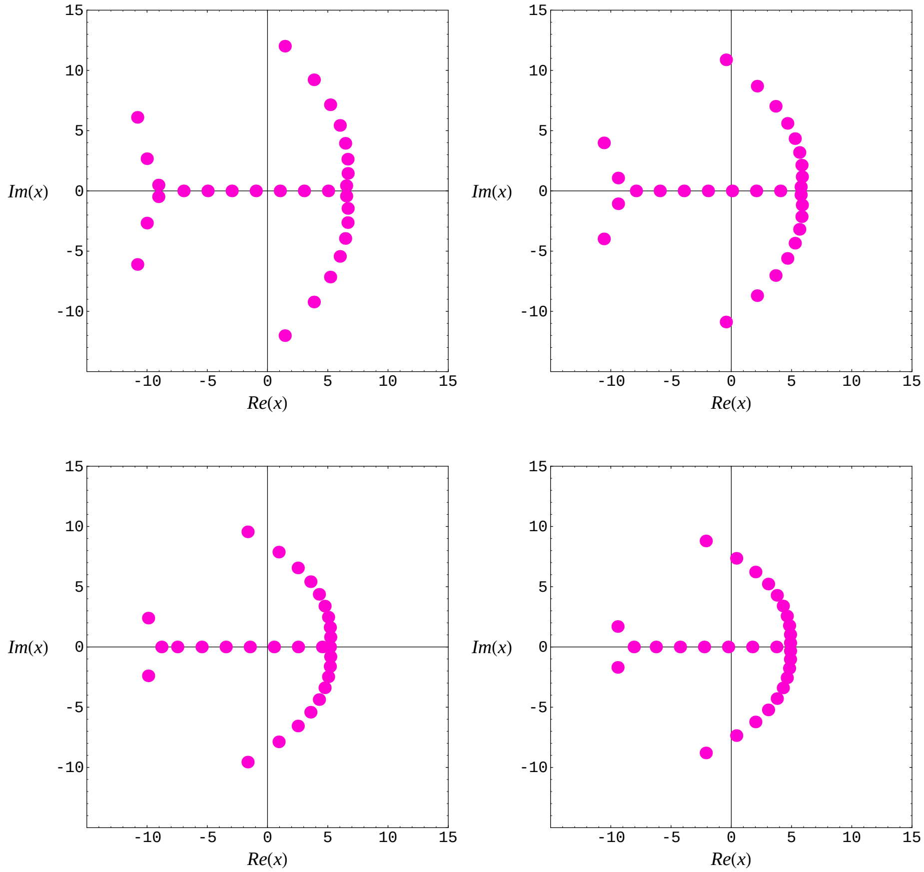

FIGURE 3. Zeros of $B_{n,\lambda}(x)$ for $\lambda = -5, -10, -20, -30$

Table 1. Numbers of real and complex zeros of $B_{n,\lambda}(x)$

n	$\lambda = -2$		$\lambda = -1/2$	
	real zeros	complex zeros	real zeros	complex zeros
2	1	0	1	0
3	2	0	2	0
4	3	0	3	0
5	2	2	2	2
6	3	2	3	2
7	4	2	4	2
8	3	4	3	4
9	4	4	4	4
10	3	6	3	6
11	4	6	4	6
12	5	6	5	6
13	6	6	6	6
14	5	8	5	8

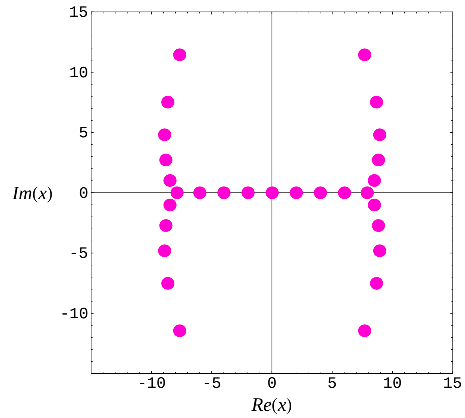

FIGURE 4. Real zeros of $B_{n,\lambda}(x)$ for $\lambda \to -1$

Table 2. Approximate solutions of $B_{n,\lambda}(x) = 0, x \in \mathbb{R}$

n	x
2	-0.33333
3	$-0.6095,\quad 1.276$
4	$-1.401,\quad 0.560,\quad 1.841$
5	$-2.055,\quad -0.309$
6	$-2.573,\quad -1.171,\quad 0.829$
7	$-2.92,\quad -2.056,\quad -0.0320,\quad 1.94$
8	$-0.894,\quad 1.106,\quad 2.68$
9	$-1.756,\quad 0.244,\quad 2.26,\quad 3.0$
10	$-2.61,\quad -0.618,\quad 1.38$
11	$-3.39,\quad -1.479,\quad 0.521,\quad 2.5$
12	$-4.01,\quad -2.34,\quad -0.341,\quad 1.66,\quad 3.4$

We observe a remarkably regular structure of the complex roots of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$. We hope to verify a remarkably regular structure of the complex roots of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ (Table 1). Next, we calculated an approximate solution satisfying $B_{n,\lambda}(x)$, $\lambda = -2, x \in \mathbb{R}$. The results are given in Table 2.

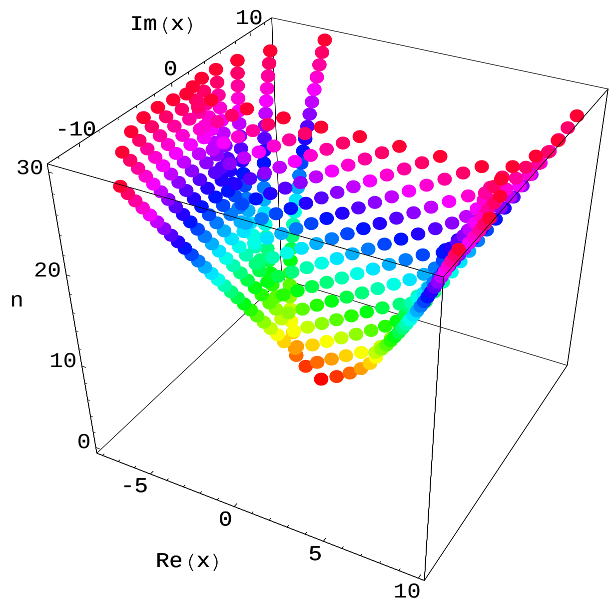

FIGURE 5. Stacks of zeros of $B_{n,-1/2}(x)$ for $1 \leq n \leq 30$

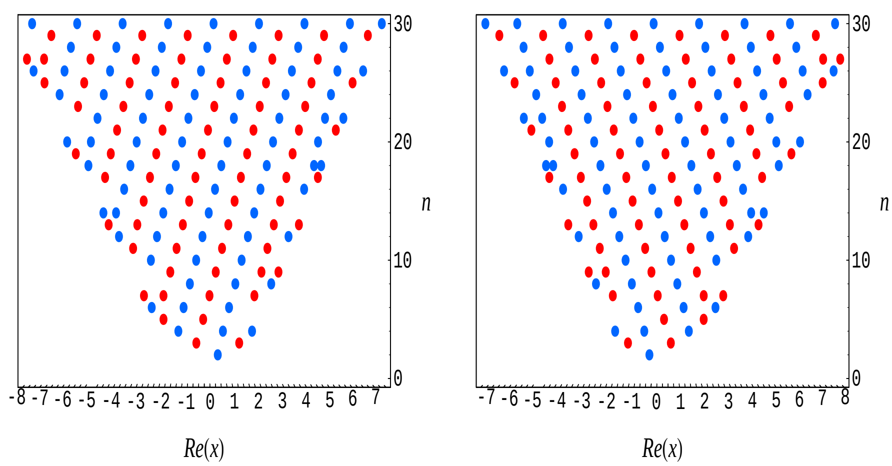

FIGURE 6. Real zeros of $B_{n,\lambda}(x)$ for $1 \leq n \leq 30$

The plot above shows $B_{n,\lambda}(x)$ for real $-7/10 \leq \lambda \leq 7/10$ and $-5 \leq x \leq 5$, with the zero contour indicated in black (Figure 8). In Figure 8 (top-left), we choose $n = 2$. In Figure 8 (top-right), we choose $n = 3$. In Figure 8 (bottom-left), we choose $n = 4$. In Figure 8 (bottom-right), we choose $n = 5$.

4. DIRECTIONS FOR FURTHER RESEARCH

In the special case, $\lambda = 1$, $B_n(x)$ are called the second Bernoulli polynomials (see [13]).

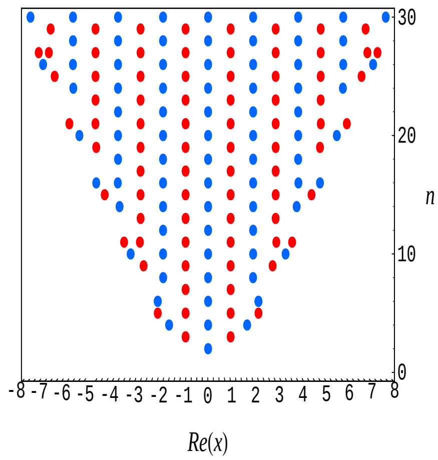

FIGURE 7. Real zeros of $B_{n,\lambda}(x)$ for $\lambda \to -1$ and $1 \leq n \leq 30$

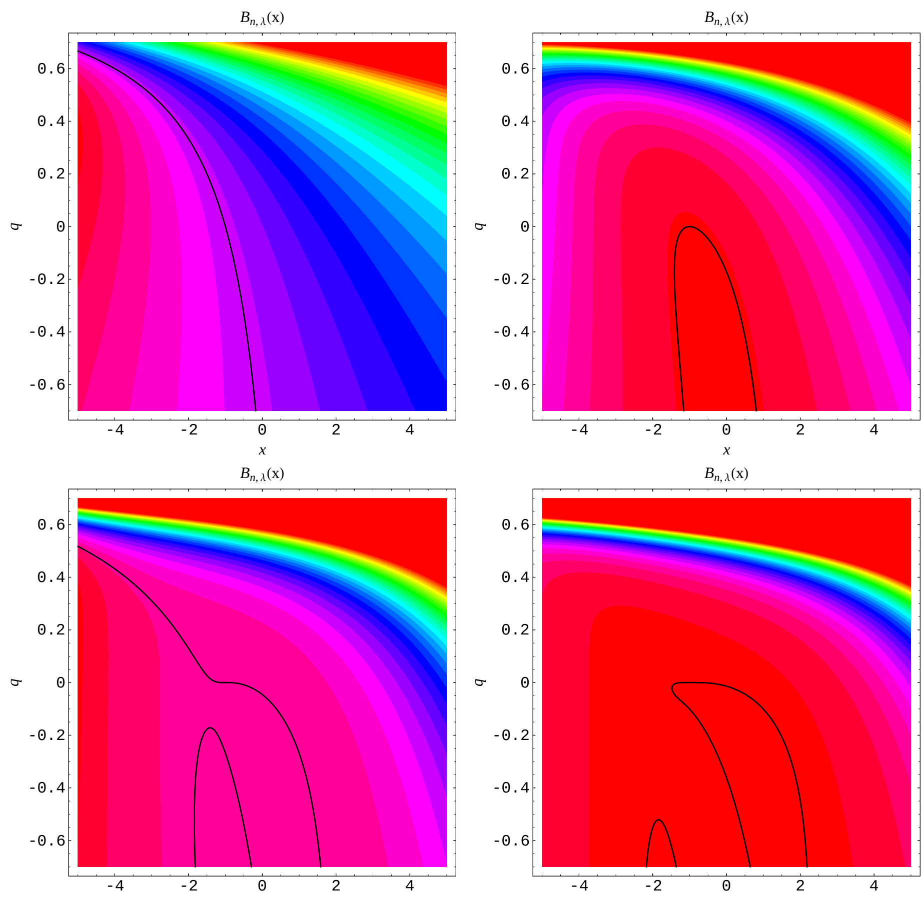

FIGURE 8. Zero contour of $B_{n,\lambda}(x)$

Since

$$\sum_{n=0}^{\infty} B_n(-x)\frac{(-t)^n}{n!} = \frac{-2te^{-t}}{e^{-2t}-1}e^{(-x)(-t)} = \frac{2te^t}{e^{2t}-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!},$$

we have

$$B_n(x) = (-1)^n B_n(-x) \text{ for } n \in \mathbb{N}.$$

Prove that $B_n(x)$, $x \in \mathbb{C}$, has $Re(x) = 0$ reflection symmetry in addition to the usual $\mathrm{Im}(x) = 0$ reflection symmetry analytic complex functions. The obvious corollary is that the zeros of $E_n(x)$ will also inherit these symmetries.

$$\text{If } B_n(x_0) = 0, \text{ then } B_n(-x_0) = 0 = B_n(x_0^*) = B_n(-x_0^*).$$

$*$ denotes complex conjugation.

The question is: what happens with the reflection symmetry (3.1), when one considers the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$?

Finally, we shall consider the more general problems. How many roots does $B_{n,\lambda}(x)$ have in general? This is an open problem. Prove or disprove: $B_{n,\lambda}(x) = 0$ has $n-1$ distinct solutions. Find the numbers of complex zeros $C_{B_{n,\lambda}(x)}$ of $B_{n,\lambda}(x)$, $\mathrm{Im}(x) \neq 0$. Since $n-1$ is the degree of the polynomial $B_{n,\lambda}(x)$, the number of real zeros $R_{B_{n,\lambda}(x)}$ lying on the real plane $\mathrm{Im}(x) = 0$ is then $R_{B_{n,\lambda}(x)} = n - 1 - C_{B_{n,\lambda}(x)}$, where $C_{B_{n,\lambda}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{B_{n,\lambda}(x)}$ and $C_{B_{n,\lambda}(x)}$. Observe that the structure of the zeros of the second kind Genocchi polynomials $G_n(x)$ resembles the structure of the zeros of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ as $\lambda \to -1$ (see Figures 3, 4, 6, 7, [12]). Find the equation of envelope curves bounding the real zeros lying on the plane. The theoretical prediction on the zeros of $B_{n,\lambda}(x)$ is await for further study. We plot the zeros of $B_{n,\lambda}(x)$, respectively (Figures 2–8). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $B_{n,\lambda}(x)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of the second kind λ-Bernoulli polynomials $B_{n,\lambda}(x)$ to appear in mathematics and physics. The reader may refer to [11–15] for the details.

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