THREE-DIMENSIONAL NONLINEAR CONTROL MODEL OF WASTEWATER BIOTREATMENT

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ABSTRACT. Biological treatment of waste water involves bacteria, which remove waste products from the treated water and kill pathogens. Thus, this problem is closely linked to both the population dynamics and chemical kinetics. Aerobic process is very efficient but rather costly, as it requires aeration of the treated water. An aim of this paper is to explore the possibility to apply the ideas and methods of the optimal control theory to the problem of optimizing the costs of this process. In this paper, we consider a three-dimensional nonlinear model of water treatment with includes a bounded control. In order to foretell possible outcomes of the control, we analytically investigate the corresponding attainable set, constructing a parametric description of this set by moments of switching of piecewise constant controls. This allows us to visualize the attainable set using MATLAB. These results lead to a number of practically relevant conclusions.

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1. Introduction

One of the most urgent problems that society is facing today is pollution of water, and in particularly the discharge of a sewage into the water. Biological treatment, which employs bacteria in order to eliminate pathogens and reduce the concentration of organic matter in wastewater to an acceptable level, is probably the most ecologically friendly approach for water treatment that we have to the date. A possible method of biotreatment is the so-called autotermal thermophilic aerobic digestion (ATAD) (Gomes et al., 2007; Rojas et al., 2010), which involves the treatment with aerobic thermophilic bacteria. The bacteria synthesize an organic matter and kill pathogens with a metabolic heat that is released in the result of synthesis. In ATAD, sludge, which previously passed filtering in a large refuse, enter a reactor, where it undergoes an intensive biological treatment with thermophilic aerobic bacteria. The

reactor is constantly aerated. The output is then filtered again. Figure 1 shows a schematic diagram of the reactor.
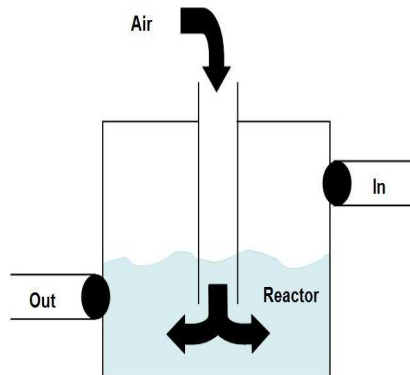

FIGURE 1. Scheme of the Device for Wastewater Treatment

ATAD is effective, but it can be also costly and energy demanding, as it requires a continuous aeration. Aeration, however, can be controlled, and optimizing the aeration can lead to significant reduction of the associated cost. In this paper, we apply the methods of the optimal control theory to the problem of optimizing the aeration in ATAD. This problem is essentially nonlinear, and hence it is highly non-trivial. One approach to this kind of problems is using the attainable set, which is the main characteristics of a control model. Construction of the parametric description of this set and studying its properties are the subject of this work.

2. Mathematical Model

In order to describe the process of aerobic biotreatment, we consider a simple mathematical model, which represent the process as a chemical reaction with three reagents, namely the concentration of oxygen $x(t)$, organic matter with concentration $y(t)$, and the thermophilic aerobic bacteria with concentration $z(t)$. It is assumed that the mass in the reactor is well stirred, and hence the reactant concentrations are homogeneous in the volume. The changes of concentrations of the reagents in the reaction is described by a three-dimensional nonlinear control system

$$(2.1) \quad \begin{cases} \dot{x}(t) = -x(t)y(t)z(t) + u(t)(m - x(t)), \ t \in [0, T], \\ \dot{y}(t) = -x(t)y(t)z(t), \\ \dot{z}(t) = x(t)y(t)z(t) - bz(t), \\ x(0) = x_0, \ y(0) = y_0, \ z(0) = z_0, \ x_0 \in (0, m), \ y_0 > 0, \ z_0 > 0. \end{cases}$$

Its nonlinearity is justified by the law of mass action (Krasnov et al., 1995) describing the dependence of the rate of chemical reaction from concentrations of initial

substances. The first equation of system (2.1) represents the evolution of oxygen concentration: the first term, $-x(t)y(t)z(t)$, describes the process of its absorption in the reaction, whereas the second term describes influx of oxygen (by pumping) into the reactor from outside. Here, $u(t)$ is the rate of aeration, which at the same time is the control function. The second equation describes a decrease of the organic matter in the reaction. The third equation of system (2.1) shows an evolution of the active biomass concentration; the bacteria mass grows at the rate $x(t)y(t)z(t)$ and decays at a rate b. The original system also includes positive initial conditions and a restriction on the rate of pumping air. We consider all possible Lebesgue measurable functions $u(t)$, which for almost all $t \in [0, T]$ satisfy the inequality $0 \leq u(t) \leq u_{\max}$, as the admissible controls $D(T)$.

The phase variables x, y, z of system (2.1) satisfy the following properties.

Lemma 2.1. *Let $u(\cdot) \in D(T)$ be an arbitrary control. Then the corresponding solution $w(t) = (x(t), y(t), z(t))^\top$ of the system (2.1) is defined on the interval $[0, T]$, the components $x(t)$, $y(t)$, $z(t)$ of which satisfy the following inequalities:*

$$0 < x(t) < x_{\max}, \ 0 < y(t) < y_{\max}, \ 0 < z(t) < z_{\max}, \ t \in [0, T],$$

where $x_{\max}$, $y_{\max}$, $z_{\max}$ - some positive constants depending on initial conditions x_0, y_0, z_0 and parameters m, b, $u_{\max}$, T of the original system.

Here and below, symbol $^\top$ means transpose.

Restrictions on the function $x(t)$ are specified as follows.

Lemma 2.2. *For an arbitrary control $u(\cdot) \in D(T)$ the corresponding solution $x(t)$ of the system (2.1) is subject to the inequality*

$$0 < x(t) < m, \ t \in [0, T].$$

Lemmas 2.1 and 2.2 imply that for any control $u(\cdot) \in D(T)$ the solutions $x(t)$, $y(t)$, $z(t)$ of the system (2.1) retain their physical meanings for all $t \in [0, T]$. Proofs of these Lemmas are given in (Bondarenko et al., 2010).

3. Attainable Set and Its Properties

Let $X(T) \subset \mathbb{R}^3$ be the attainable set for system (2.1) from an initial point $w_0 = (x_0, y_0, z_0)^\top$ at the moment of time T; that is, $X(T)$ is the set of all ends $w(T) = (x(T), y(T), z(T))^\top$ of trajectories $w(t) = (x(t), y(t), z(t))^\top$ of system (2.1) under all possible controls $u(\cdot) \in D(T)$. By (Lee & Markus, 1967) and Lemma 2.1, it follows that set $X(T)$ is a compact set in $\mathbb{R}^3$ located into the region

$$\left\{ w = (x, y, z)^\top \in \mathbb{R}^3 : x > 0, y > 0, z > 0 \right\}.$$

We denote by ∂Q and $\text{int}Q$ the boundary and the interior of the compact set, $Q \subset \mathbb{R}^3$, respectively.

Next, we study the boundary of the attainable set $X(T)$. Consider a point $w = (x, y, z)^\top$, such that $w \in \partial X(T)$. It corresponds to control $u(\cdot) \in D(T)$ and a trajectory $w(t) = (x(t), y(t), z(t))^\top$, $t \in [0, T]$, of system (2.1), such that $w = w(T)$. Then it follows from (Lee & Markus, 1967) that there exists a non-trivial solution $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3)^\top$, $t \in [0, T]$ of the adjoint system

(3.1)
$$\begin{cases} \dot{\psi}_1(t) = u(t)\psi_1(t) + y(t)z(t)(\psi_1(t) + \psi_2(t) - \psi_3(t)), \\ \dot{\psi}_2(t) = x(t)z(t)(\psi_1(t) + \psi_2(t) - \psi_3(t)), \\ \dot{\psi}_3(t) = x(t)y(t)(\psi_1(t) + \psi_2(t) - \psi_3(t)) + b\psi_3(t), \end{cases}$$

for which, by Lemma 2.2, the following relationship is valid:

(3.2)
$$u(t) = \begin{cases} 0, & \text{if } L(t) < 0, \\ \forall u \in [0, u_{\max}], & \text{if } L(t) = 0, \\ u_{\max}, & \text{if } L(t) > 0. \end{cases}$$

Here $L(t) = \psi_1(t)$. Function $L(t)$ is the so-called switching function and its behavior determines the type of control $u(t)$.

For convenience in subsequent arguments, we introduce the following auxiliary functions:

$$G(t) = \psi_1(t) + \psi_2(t) - \psi_3(t), \ P(t) = -\psi_3(t), \ d(t) = y(t)z(t) + z(t)x(t) - x(t)y(t).$$

Using the adjoint system (3.1), we write for functions $L(t)$, $G(t)$, $P(t)$ the following system of linear differential equations

(3.3)
$$\begin{cases} \dot{L}(t) = u(t)L(t) + y(t)z(t)G(t), \ t \in [0, T], \\ \dot{G}(t) = u(t)L(t) + d(t)G(t) + bP(t), \\ \dot{P}(t) = -x(t)y(t)G(t) + bP(t). \end{cases}$$

The validity of the following Lemma immediately follows from the non-triviality of solution $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))^\top$ of the adjoint system (3.1).

Lemma 3.1. *The switching function $L(t)$ and the auxiliary functions $G(t)$ and $P(t)$ are nonzero solutions of system* (3.3).

Lemma 3.1 allows us to rewrite the relationship (3.2) in the form

(3.4)
$$u(t) = \begin{cases} 0, & \text{if } L(t) < 0, \\ u_{\max}, & \text{if } L(t) > 0. \end{cases}$$

At points of discontinuity we will define function $u(t)$ by its limit from the left. Consequently, the control $u(t)$, $t \in [0, T]$, corresponding to a point $w \in \partial X(T)$, is a

piecewise constant function, taking values $\{0, u_{\max}\}$. Such type of control is usually called a bang-bang control.

Next, we estimate the number of switchings of control function $u(t)$, $t \in [0, T]$. It follows from (3.4) that it is sufficient to estimate the number of zeros of the function $L(t)$ on the interval $(0, T)$. The following important statement is valid.

Lemma 3.2. *The switching function $L(t)$ has at most two zeros on the interval $[0, T]$.*

A detailed proof of this Lemma is given in (Bondarenko et al., 2010). Based on the obtained results, we formulate the following statement.

Theorem 3.3. *Let point $w = (x, y, z)^\top$ belong to the boundary of the attainable set $X(T)$. Then the control $u(t)$, $t \in [0, T]$, which is associated with this point w, is a piecewise constant function taking values $\{0, u_{\max}\}$ and having at most two switchings on the interval $(0, T)$.*

4. Auxiliary Set and Its Properties

Using Theorem 3.3, we now can proceed to constructing a parametrization for the attainable set $X(T)$ with moments of switching of piecewise constant controls. For this task, we consider the set

$$\Lambda(T) = \left\{ \theta = (\theta_1, \theta_2, \theta_3)^\top \in \mathbb{R}^3 : 0 \le \theta_1 \le \theta_2 \le \theta_3 \le T \right\}.$$

For every point $\theta \in \Lambda(T)$ we form the control $u_\theta(\cdot) \in D(T)$ by formula

(4.1)
$$u_\theta(t) = \begin{cases} u_{\max}, & \text{if} \quad 0 \le t \le \theta_1, \\ 0, & \text{if} \quad \theta_1 < t \le \theta_2, \\ u_{\max}, & \text{if} \quad \theta_2 < t \le \theta_3, \\ 0, & \text{if} \quad \theta_3 < t \le T. \end{cases}$$

We denote by $w_\theta(t)$, $t \in [0, T]$ the solution of system (2.1) corresponding to the control $u_\theta(t)$. Finally, we define the mapping $F(\cdot, T) : \Lambda(T) \to \mathbb{R}^3$ as

$$F(\theta, T) = w_\theta(T), \ \theta \in \Lambda(T).$$

For this mapping we have the following proposition.

Lemma 4.1. *The mapping $F(\cdot, T)$ is continuous on the set $\Lambda(T)$.*

Proof. Let consider arbitrary values $\theta, \tau \in \Lambda(T)$. We extend controls $u_\theta(t)$, $u_\tau(t)$, defined by (4.1), to the interval $(T, T + \delta)$ for some $\delta > 0$ by value of zero. Corresponding trajectories $w_\theta(t)$, $w_\tau(t)$ of system (2.1) are also extended to this interval. Then we transform the Cauchy problems (2.1) for trajectories $w_\theta(t)$, $w_\tau(t)$ to the corresponding integral equations for all $t \in (T, T + \delta)$. Then, by Lemma 2.1, we evaluate

the difference $\|w_\theta(t) - w_\tau(t)\|$, $t \in (T, T + \delta)$. Applying the Gronwall's inequality (Robinson, 2004), we obtain as a result the relationship

$$\|w_\theta(t) - w_\tau(t)\| \leq L_w \|\theta - \tau\|, \ t \in [T, T + \delta),$$

where L_w is a positive constant. Assuming that $t = T$ in the inequality, we find that mapping $F(\cdot, T)$ satisfies the Lipschitz condition (Robinson, 2004) on the set $\Lambda(T)$. The required continuity of the mapping $F(\cdot, T)$ on the set $\Lambda(T)$ immediately follows from this fact. The proof is completed. □

Using the mapping $F(\cdot, T)$, we introduce the auxiliary set $Z(T) = F(\Lambda(T), T)$, which consists of all ends $w_\theta(T)$ of trajectories $w_\theta(t)$ of system (2.1) under all possible controls $u_\theta(t)$, $t \in [0, T]$, defined by formula (4.1). Every element of set $Z(T)$ is a result of a bang-bang control $u_\theta(t)$, $t \in [0, T]$, with at most three switchings on the interval $(0, T)$.

Now, we have to discuss some properties of auxiliary set $Z(T)$. Considering a point $\theta \in \text{int} \Lambda(T)$, its corresponding control $u_\theta(t)$ defined by (4.1) and a trajectory $w_\theta(t)$, $t \in [0, T]$, we can reformulate the Cauchy problem (2.1) in the form

(4.2)
$$\begin{cases} \dot{w}_\theta(t) = A w_\theta(t) + \varphi(w_\theta(t))c + u_\theta(t)g(w_\theta(t)), \ t \in [0, T], \\ w_\theta(0) = w_0 = (x_0, y_0, z_0)^\top, \end{cases}$$

where A is a 3×3 matrix, $c \in \mathbb{R}^3$, and functions $g(w)$ and $\varphi(w)$ are a vector and a scalar functions, respectively, such that

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -b \end{pmatrix}, \ c = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \ g(w) = \begin{pmatrix} m - x \\ 0 \\ 0 \end{pmatrix}, \ \varphi(w) = xyz.$$

For the system (4.2), we define a function $\Phi_\theta(t)$, $t \in [0, T]$, as a solution of the matrix Cauchy problem

(4.3)
$$\begin{cases} \dot{\Phi}_\theta(t) = \left(A + c \left(\frac{\partial \varphi}{\partial w}(w_\theta(t)) \right)^\top + u_\theta(t) \frac{\partial g}{\partial w}(w_\theta(t)) \right) \Phi_\theta(t), \ t \in [0, T], \\ \Phi_\theta(T) = E, \end{cases}$$

where E is the identity matrix. Let us evaluate the derivatives $\frac{\partial w_\theta}{\partial \theta_i}(T)$, $i = \overline{1, 3}$. Using an approach which is due to (Hajek, 1991), one can find that the derivatives satisfy the following equalities:

(4.4)
$$\frac{\partial w_\theta}{\partial \theta_i}(T) = (-1)^{i-1} u_{\max} \Phi_\theta^{-1}(\theta_i) g(w_\theta(\theta_i)), \ i = \overline{1, 3}.$$

Now we are in a position to state the following theorem.

Theorem 4.2. *The following equalities hold:*

$$(4.5) \qquad F(\mathrm{int}\Lambda(T), T) = \mathrm{int}Z(T), \ F(\partial\Lambda(T), T) = \partial Z(T),$$

and the restriction of mapping $F(\cdot, T)$ onto the interior of set $\Lambda(T)$ is one-to-one.

Proof. Firstly, we consider the set $\mathrm{int}\Lambda(T)$. The mapping $F(\cdot, T)$ is continuously differentiable on the set $\mathrm{int}\Lambda(T)$, and for every point $\theta \in \mathrm{int}\Lambda(T)$, by (4.4), the following equalities hold:

$$(4.6) \qquad \frac{\partial F}{\partial \theta_i}(\theta, T) = (-1)^{i-1} u_{\max} \Phi_\theta^{-1}(\theta_i) g(w_\theta(\theta_i)), \ i = \overline{1, 3}.$$

We have to show that the Jacobi matrix of the restriction of mapping $F(\cdot, T)$ onto $\mathrm{int}\Lambda(T)$ is nonsingular. Suppose the opposite. Then there is a point $\bar\theta \in \mathrm{int}\Lambda(T)$ for which vectors $\frac{\partial F}{\partial \theta_i}(\bar\theta, T)$, $i = \overline{1, 3}$, are linearly dependent. With respect to (4.6), it means the existence of a nonzero vector $q \in \mathbb{R}^3$ such that the equalities:

$$(4.7) \qquad (g(w_\theta(\bar\theta_i)), \eta(\bar\theta_i)) = 0, \ i = \overline{1, 3},$$

hold. Here $\eta(t) = (\Phi_\theta^{-1}(t))^\top q$. By (4.3), we can see that function $\eta(t)$, $t \in [0, T]$, satisfies the adjoint system (3.1), which is written as

$$\dot\psi(t) = -\left(A + c\left(\frac{\partial\varphi}{\partial w}(w_\theta(t))\right)^\top + u_\theta(t)\frac{\partial g}{\partial w}(w_\theta(t))\right)^\top \psi(t),$$

where $\psi(t) = (\psi_1(t), \psi_2(t), \psi_3(t))^\top$. Then, applying Lemmas 2.2 and 3.2 to the function $r(t) = (g(w_\theta(t)), \eta(t))$, we find that function $r(t)$ has two zeros on interval $(0, T)$ at most. This fact contradicts the equalities:

$$r(\bar\theta_i) = 0, \ i = \overline{1, 3},$$

resulting from (4.7). Therefore, the assumption is wrong, and hence the proposition is true. By this and by the Theorem on the invariance of interior points (Partasarathy, 1983), the first equality of (4.5) follows.

Furthermore, set $\mathrm{int}\Lambda(T)$ is a convex set, and the set $\mathrm{int}Z(T)$ is path connected. Indeed, the mapping $F(\cdot, T)$ transforms any segment of $\mathrm{int}\Lambda(T)$ into a curve located completely inside $\mathrm{int}Z(T)$. For every point of $\mathrm{int}\Lambda(T)$ the Local Theorem on an implicit function (Partasarathy, 1983) holds. Then the last statement of the proposition follows from the Global Theorem 3 on an implicit function (Shigeo, 1985). Hence the validity of the second equality of (4.5) follows. The proof is completed. $\square$

From the definitions of attainable set $X(T)$ and the auxiliary set $Z(T)$, and Theorems 3.3 and 4.2, the following inclusions hold:

$$(4.8) \qquad Z(T) \subseteq X(T), \ \partial X(T) \subseteq \partial Z(T).$$

These explain why the set $Z(T)$ plays such an important role in the study of the attainable set $X(T)$.

Further investigation of the auxiliary set $Z(T)$ involves the study of its supplement $\mathbb{R}^3 \setminus Z(T)$. The following statement is valid.

Theorem 4.3. *The set $\mathbb{R}^3 \setminus Z(T)$ is path connected.*

Proof. Let $F_i(\theta, T)$, $i = \overline{1,3}$, be the components of mapping $F(\cdot, T)$. We define the following values:

$$F_{\min}^i = \min_{\theta \in \Lambda(T)} F_i(\theta, T), \ F_{\max}^i = \max_{\theta \in \Lambda(T)} F_i(\theta, T), \ i = \overline{1,3}.$$

By Lemma 4.1 and the Extension Theorem of Brouwer-Urysohn (Hausdorff, 1962), we construct continuous mapping $\Pi(\cdot, T)$ defined on the whole space $\mathbb{R}^3$, which coincides with the mapping $F(\cdot, T)$ for all points of the set $\Lambda(T)$. In addition, for the components $\Pi_i(\theta, T)$, $i = \overline{1,3}$, of this mapping at each point $\theta \in \mathbb{R}^3$ the following inequalities hold:

$$F_{\min}^i \leq \Pi_i(\theta, T) \leq F_{\max}^i, \ i = \overline{1,3}.$$

Such bounded continuous mapping $\Pi(\cdot, T)$ is called an extension of the mapping $F(\cdot, T)$ from the set $\Lambda(T)$ on the whole space $\mathbb{R}^3$.

We now define continuous functions $\xi_i(\theta)$, $i = \overline{1,3}$, as

$$\xi_i(\theta) = \begin{cases} \theta_i + 1, & \text{if} \quad \theta_i < 0, \\ 1, & \text{if} \quad 0 \leq \theta_i \leq T, \ i = \overline{1,3}, \\ \theta_i - T + 1, & \text{if} \quad \theta_i > T. \end{cases}$$

With these functions, we define a continuous mapping $\Psi(\cdot, T)$ of the whole space $\mathbb{R}^3$ onto whole space $\mathbb{R}^3$ as

$$\Psi_i(\theta, T) = \xi_i(\theta) \Pi_i(\theta, T), \ i = \overline{1,3},$$

where $\Psi_i(\theta, T)$, $i = \overline{1,3}$, are the components of mapping $\Psi(\cdot, T)$.

Furthermore, the open set $\mathbb{R}^3 \setminus \Lambda(T)$ is path connected, and hence it is a connected set (Hall & Spencer, 1955). By Theorem 4.2, the continuous mapping $\Psi(\cdot, T)$ transfers the set $\mathbb{R}^3 \setminus \Lambda(T)$ onto the set $\mathbb{R}^3 \setminus Z(T)$, which is also open and connected (Hall & Spencer, 1955). Then the set $\mathbb{R}^3 \setminus Z(T)$ simultaneously is a path connected set. This completes the proof. $\square$

5. Parametric Description of Attainable Set

Finally, we now able to establish the validity of the main result of this paper.

Theorem 5.1. *For the attainable set $X(T)$ and the auxiliary set $Z(T)$, the equality $X(T) = Z(T)$ holds.*

Proof. It follows from the first inclusion in (4.8) that in order to prove the hypothesis it is sufficient to show the validity of the inclusion $X(T) \subseteq Z(T)$. Let us assume the opposite, i.e. assume that there exists a point $\widetilde{w}$ such that

$$\widetilde{w} \notin Z(T), \ \widetilde{w} \in X(T)$$

holds. Consider a point $\widehat{w} \notin X(T)$.

The arguments presented in Theorems 4.2 and 4.3 show that the boundary of set $Z(T)$ divides $\mathbb{R}^3$ into two path connected subsets $\mathrm{int}Z(T)$ and $\mathbb{R}^3 \setminus Z(T)$. The path connectedness of the second set ensures the existence of a continuous curve $\sigma(s)$, $s \in [0,1]$, as well as $\widetilde{w} = \sigma(0)$, $\widehat{w} = \sigma(1)$, and $\sigma(s) \notin Z(T)$ for all $s \in (0,1)$. By Theorem 36 on "transition through customs" in (Schwarts, 1967), there is a value $s_\star \in (0,1)$ such that $\sigma(s_\star) \in \partial X(T)$. Therefore, there is a defined point $\bar{w} = \sigma(s_\star)$, such that the relationships:

$$\bar{w} \in \partial X(T), \ \bar{w} \notin \partial Z(T),$$

simultaneously hold. This contradicts to the second inclusion in (4.8). Hence the assumption is incorrect, and the required inclusion holds. The proof is completed. $\square$

We have obtained analytically the properties of the attainable set $X(T)$. The moments of switching of controls $u_\theta(t)$ from (4.1), which form the set $\Lambda(T)$, together with the mapping $F(\cdot, T)$ play the role of parametrization for the set $X(T)$ (its interior and boundary). This implies that each point on the boundary of the attainable set $X(T)$ can be reached by a bang-bang control $u_\theta(t)$ with at most two switchings, and every point of the interior of the set $X(T)$ is the result of such control with precisely three switchings.

Remark 5.2. To establish these results we utilize an approach that was developed for another class of control systems by (Grigorieva & Khailov, 2001; Grigorieva & Khailov, 2005).

6. Numerical Simulations

Figures 2 to 8 show examples of attainable sets $X(T)$, constructed with MATLAB using Theorem 5.1.

Example 6.1. (Figure 2) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 1.0000, \quad y_0 = 1.0000, \quad z_0 = 1.0000,$$
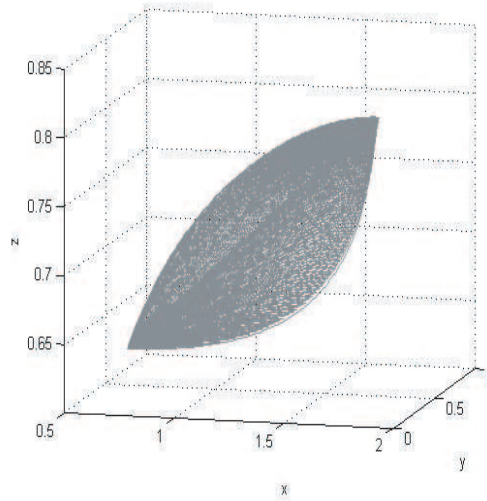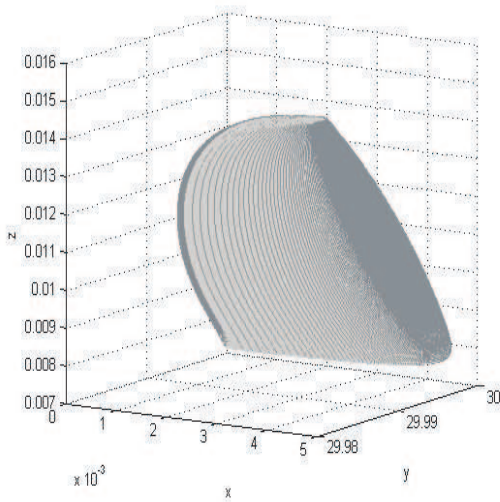$$m = 2.0000, \quad b = 1.0000, \quad u_{\max} = 4.0000, \quad T = 1.0000.$$
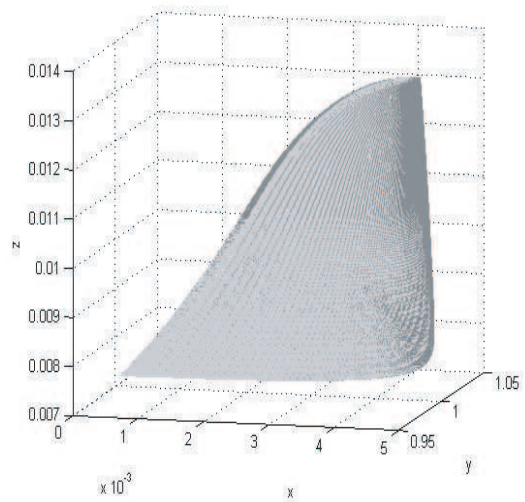
FIGURE 2. Set $X(T)$ for Example 6.1

Example 6.2. (Figure 3) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0002, \quad y_0 = 30.0000, \quad z_0 = 0.0300,$$
$$m = 0.0050, \quad b = 0.2400, \quad u_{\max} = 4.0000, \quad T = 6.0000.$$

Example 6.3. (Figure 4) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0019, \quad y_0 = 2.4980, \quad z_0 = 0.0874,$$
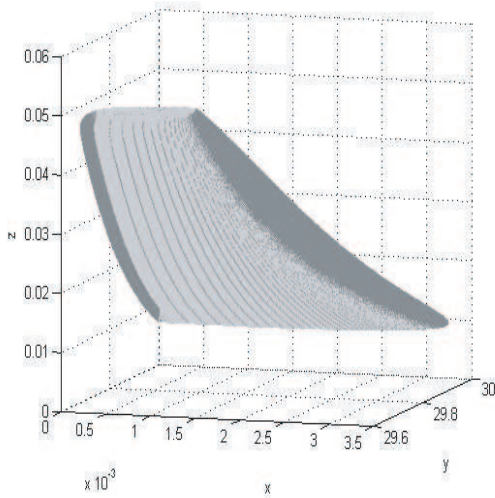$$m = 0.0480, \quad b = 0.2400, \quad u_{\max} = 4.0000, \quad T = 6.0000.$$

FIGURE 3. Set $X(T)$ for Example 6.2

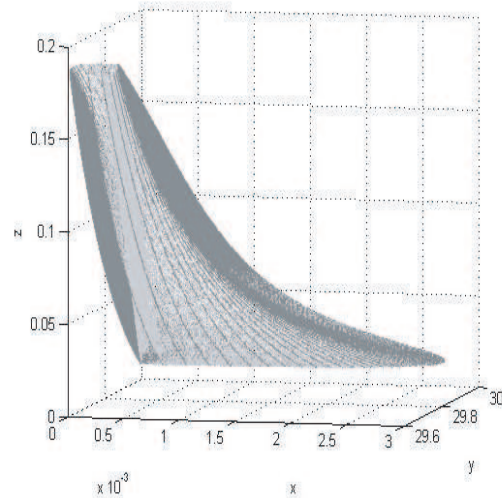
FIGURE 4. Set $X(T)$ for Example 6.3

Example 6.4. (Figure 5) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0011, \quad y_0 = 38.3406, \quad z_0 = 0.1643,$$
$$m = 0.0274, \quad b = 0.2400, \quad u_{\max} = 4.0000, \quad T = 6.0000.$$

Example 6.5. (Figure 6) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0192, \quad y_0 = 74.9400, \quad z_0 = 0.0874,$$
$$m = 0.0480, \quad b = 0.2400, \quad u_{\max} = 4.0000, \quad T = 6.0000.$$

FIGURE 5. Set $X(T)$ for
Example 6.4

FIGURE 6. Set $X(T)$ for
Example 6.5

Example 6.6. (Figure 7) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0019, \quad y_0 = 74.9400, \quad z_0 = 0.0874,$$
$$m = 0.0480, \quad b = 1.0000, \quad u_{\max} = 4.0000, \quad T = 12.0000.$$

Example 6.7. (Figure 8) The initial conditions and parameters of the system (2.1) are:

$$x_0 = 0.0010, \quad y_0 = 146.9694, \quad z_0 = 0.1715,$$
$$m = 0.0245, \quad b = 0.5000, \quad u_{\max} = 4.0000, \quad T = 20.0000.$$

7. Conclusions

The ultimate objective of this study was constructing an optimal control for the aerobic biotreatment process with the aim to rise the energy efficiency (and hence low the operational cost) of the process. The results, reported in this paper, are of apparent practical significance and can be straightforwardly applied to practice.
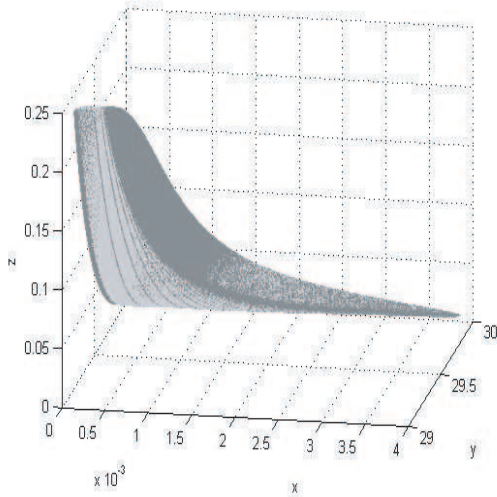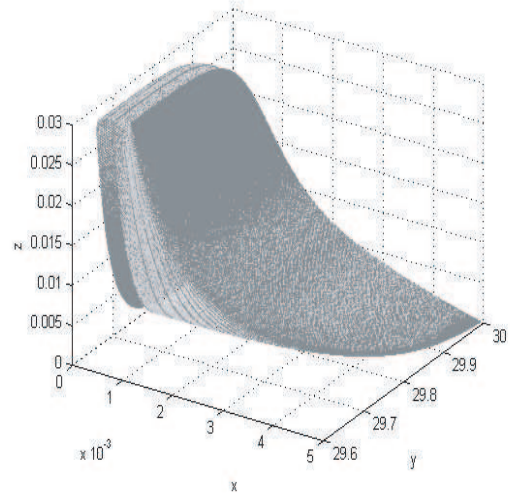
FIGURE 7. Set $X(T)$ for Example 6.6

FIGURE 8. Set $X(T)$ for Example 6.7

In this paper, we have analytically obtained the detailed structure of an attainable set $X(T)$ for the model of the process. It was rigorously proved that the optimal control for this particular system is a bang-bang process, with at most two switchings. The moments of switching of the controls $u_\theta(t)$ in (4.1), which form the set $\Lambda(T)$, together with the mapping $F(\cdot, T)$, play the role of parametrization for the set $X(T)$ (for both its interior and boundary). It was proved that each point on the boundary of set $X(T)$ can be reached by a control from the above mentioned class (bang-bang control with at most two switchings), and every point of the interior of $X(T)$ is the result of a bang-bang control with precisely three switchings. An original computer program (written in MATLAB) allows us to construct attainable sets for a variety of initial conditions and the system parameters. These results are of immediate practical importance, as they, firstly, tremendously narrow the class of functions, which should be considered as candidates for optimal control, and hence enable us to construct an optimal control for a real-life situation numerically. This result is highly nontrivial, taking into consideration the nonlinearity of the considered model and it three dimensions.

In this study we consider a particularly simple mathematical model of the process, which is composed of three variables and postulate that the reaction is governed by the mass action law. However, it can be expected that the same results will be valid for a more complex system, such as a model with reactions of more complicated form, e.g. given by Michaelis-Menten kinetics (functional responses with saturation), or a model with a larger number of variables.

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