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MULTIPLE SYMMETRIC POSITIVE SOLUTIONS FOR STURM-LIOUVILLE TWO-POINT BOUNDARY VALUE PROBLEMS ON TIME SCALES

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ABSTRACT. This paper is concerned with the existence of symmetric positive solutions for the even order boundary value problems on time scales satisfying Sturm-Liouville two-point boundary conditions. We establish the existence of at least three symmetric positive solutions for two-point boundary value problem by using Avery generalization of the Leggett-Williams fixed point theorem. Also, we establish the existence of at least $2k - 1$ symmetric positive solutions for an arbitrary positive integer k.

Key words. Symmetric time scale, Boundary value problem, Cone, Green's function, Symmetric positive solution

AMS (MOS) Subject Classification. 39A10, 34B15, 34A40

1. Introduction

Recently, there has been an increasing interest in obtaining positive solutions for the boundary value problems on time scales. The theory of time scales was introduced and developed by Hilger [20] to unify not only continuous and discrete theory, but also provide accurate information of phenomena that manifest themselves partly in continuous time and partly in discrete time. The time scale calculus would allow to explore a variety of situations like economic, biological, heat transfer, stock market and epidemic models etc. Recent results indicates that considerable achievement has been made in the existence of positive solutions of the boundary value problems on time scales. However they did not further provide characteristics of positive solutions such as symmetry. Symmetry have been widely used in science and engineering. The reason is that the symmetry not only has its theoretical value in studying the metric manifolds and symmetric graph and so forth, but also has its practical value, for

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example, we can apply this characteristic to study graph structures and chemistry structures.

The primary purpose of this investigation is to study the symmetry properties of the solutions of even order boundary value problems on time scales. For recent discussions on symmetry properties of solutions of boundary value problems associated to differential equations or finite difference equations or time scales, to mention a few papers along these lines are Davis and Henderson [9], Avery, Davis and Henderson [4], Davis, Henderson and Wong [10], Henderson [16], Henderson and Thompson [18], Henderson and Wong [19], Eloe, Henderson and Sheng [13] and Avery and Henderson [5, 6]. Recently, Henderson, Murali and Prasad [17] studied the multiple symmetric positive solutions for two-point even order boundary value problems on time scales,

$$(-1)^n y^{(\Delta\nabla)^n}(t) = f(y(t), y^{\Delta\nabla}(t), \ldots, y^{(\Delta\nabla)^{(n-1)}}(t)),\ t \in [a,b],$$

satisfying the boundary conditions,

$$y^{(\Delta\nabla)^i}(a) = 0 = y^{(\Delta\nabla)^i}(b),\ 0 \le i \le n-1.$$

Motivated by the papers mentioned above, in this paper we are concerned with the existence of multiple symmetric positive solutions for the even order boundary value problems on time scales,

$$(-1)^n y^{(\Delta\nabla)^n}(t) = f(y(t), y^{\Delta\nabla}(t), \ldots, y^{(\Delta\nabla)^{(n-1)}}(t)),\ t \in [a,b], \tag{1.1}$$

satisfying Sturm-Liouville type two-point boundary conditions,

$$\left.\begin{aligned} \alpha_{i+1} y^{(\Delta\nabla)^i}(a) - \beta_{i+1} y^{(\Delta\nabla)^i\Delta}(a) = 0,\\ \alpha_{i+1} y^{(\Delta\nabla)^i}(b) + \beta_{i+1} y^{(\Delta\nabla)^i\Delta}(b) = 0, \end{aligned}\right\} \tag{1.2}$$

for $0 \le i \le n-1$, where $n \ge 1$, $f : \mathbb{R}^n \to \mathbb{R}^+$ is continuous with $a \in \mathbb{T}_{\kappa^n},\ b \in \mathbb{T}^{\kappa^n}$ for a time scale $\mathbb{T}$, $\alpha_j > 0, \beta_j \ge 0$ $(1 \le j \le n)$ and $\sigma^n(a) < \rho^n(b)$.

To establish the symmetric positive solutions for the boundary value problem, we are dealing with symmetric time scales. By an interval time scale, we mean the intersection of a real interval with a given time scale. An interval time scale $\mathbb{T} = [a,b]$ is said to be a symmetric time scale, if

$$t \in \mathbb{T} \Leftrightarrow b + a - t \in \mathbb{T}.$$

By a symmetric solution $y(t)$ of the BVP (1.1)–(1.2), we mean $y(t)$ is a solution of the BVP (1.1)–(1.2) and satisfies

$$y(t) = y(b + a - t),\ \text{for } t \in [a,b].$$

The rest of the paper is organized as follows. In Section 2, we briefly describe some salient features of time scales. In Section 3, we construct the Green's function for the homogeneous problem corresponding to the BVP (1.1)–(1.2) and estimate bounds for the Green's function. In Section 4, we establish a criteria for the existence of at least three symmetric positive solutions of the BVP (1.1)–(1.2) by using the Avery generalization of the Leggett-Williams fixed point theorem. We also establish the existence of at least $2k-1$ symmetric positive solutions of the BVP (1.1)–(1.2) for an arbitrary positive integer k. Finally as an application, we give an example to illustrate our result.

2. **Preliminaries About Time Scales**

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For an excellent introduction to the overall area of dynamic equations on time scales, we refer the recent text books by Bohner and Peterson [7, 8], from which we cull the following definitions. The functions $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are jump operators given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t, \rho(t) < t, \sigma(t) = t, \sigma(t) > t$, respectively. If $\mathbb{T}$ has a right- scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative of f at t, denoted $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] | \leq \epsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$.

For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative of f at t, denoted $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] | \leq \epsilon \mid \rho(t) - s \mid,$$

for all $s \in U$. Define $f^{\Delta\nabla}(t)$ to be the nabla derivative of $f^\Delta(t)$, i.e., $f^{\Delta\nabla}(t) = (f^\Delta(t))^\nabla$.

A function $f : \mathbb{T} \to \mathbb{R}$ is left-dense continuous or ld-continuous on $[a, b]$, denoted $f \in C_{ld}[a, b]$, provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. It is known that if f is ld-continuous, then there is a function $F(t)$ such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

3. **Green's Function and Bounds**

In this section, we construct the Green's function for the homogeneous BVP corresponding to (1.1)–(1.2) and estimate bounds for the Green's function. We prove certain lemmas which are needed to establish our main result.

For $1 \leq j \leq n$, let $G_j(t,s)$ be the Green's function for the homogeneous BVP,

$$-y^{\Delta\nabla}(t) = 0,\ t \in [a,b],$$

$$\alpha_j y(a) - \beta_j y^{\Delta}(a) = 0, \quad \alpha_j y(b) + \beta_j y^{\Delta}(b) = 0.$$

Then, for $1 \leq j \leq n$

$$G_j(t,s) = \begin{cases} \frac{1}{d_j}\{\alpha_j(t-a)+\beta_j\}\{\alpha_j(b-s)+\beta_j\}, & t \leq s, \\ \frac{1}{d_j}\{\alpha_j(s-a)+\beta_j\}\{\alpha_j(b-t)+\beta_j\}, & s \leq t, \end{cases}$$

where $d_j = 2\alpha_j\beta_j + \alpha_j^2(b-a) > 0$.

For $1 \leq j \leq n$, the Green's function $G_j(t,s)$ is positive and satisfies the following inequality,

$$G_j(t,s) \leq G_j(s,s),\ \text{for all } t,s \in [a,b].$$

Let $I = \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]$. Then

$$G_j(t,s) \geq m_j G_j(s,s) \text{ for all } (t,s) \in I \times [a,b],$$

where

$$m_j = \frac{\alpha_j(b-a)+4\beta_j}{4[\alpha_j(b-a)+\beta_j]} < 1, \tag{3.1}$$

for $1 \leq j \leq n$. Let $H_1(t,s) = G_1(t,s)$ and recursively define

$$H_j(t,s) = \int_a^b H_{j-1}(t,r)G_j(r,s)\nabla r,\ \text{for } 2 \leq j \leq n. \tag{3.2}$$

Then $H_n(t,s)$ is the Green's function for the homogeneous BVP corresponding to (1.1)–(1.2).

Lemma 3.1. *If we define*

$$K = \prod_{j=1}^{n-1} K_j \text{ and } L = \prod_{j=1}^{n-1} m_j L_j,$$

then the Green's function $H_n(t,s)$ in (3.2) satisfies

$$0 \leq H_n(t,s) \leq K G_n(s,s),\ \textit{for all } (t,s) \in [a,b] \times [a,b]$$

and

$$H_n(t,s) \geq m_n L G_n(s,s),\ \textit{for all } (t,s) \in I \times [a,b],$$

where m_n is given as in (3.1),

$$K_j = \int_a^b G_j(s,s)\nabla s > 0, \ for \ 1 \le j \le n$$

and

$$L_j = \int_{s\in I} G_j(s,s)\nabla s > 0, \ for \ 1 \le j \le n.$$

Let $D = \{v | v : C[a,b] \to \mathbb{R}\}$. For each $1 \le j \le n-1$, define the operator $T_j : D \to D$ by

$$T_j v(t) = \int_a^b H_j(t,s)v(s)\nabla s, \ t \in [a,b]$$

and these integrals are converges. By the construction of T_j and the properties of $H_j(t,s)$, it is clear that

$$(-1)^j (T_j v)^{(\Delta\nabla)^j}(t) = v(t), \ t \in [a,b],$$

$$\alpha_{i+1}(T_j v)^{(\Delta\nabla)^i}(a) - \beta_{i+1}(T_j v)^{(\Delta\nabla)^i\Delta}(a) = 0,$$
$$\alpha_{i+1}(T_j v)^{(\Delta\nabla)^i}(b) + \beta_{i+1}(T_j v)^{(\Delta\nabla)^i\Delta}(b) = 0,$$

for $0 \le i \le j-1$. Hence, we see that the BVP (1.1)–(1.2) has a solution if and only if the following BVP has a solution,

$$v^{\Delta\nabla}(t) + f(T_{n-1}v(t), T_{n-2}v(t), \ldots, T_1 v(t), v(t)) = 0, \ t \in [a,b] \tag{3.3}$$

$$\alpha_{i+1}v(a) - \beta_{i+1}v^{\Delta}(a) = 0, \quad \alpha_{i+1}v(b) + \beta_{i+1}v^{\Delta}(b) = 0, \tag{3.4}$$

for $0 \le i \le n-1$. Indeed, if y is a solution of the BVP (1.1)–(1.2), then $v(t) = y^{(\Delta\nabla)^{(n-1)}}(t)$ is a solution of the BVP (3.3)–(3.4). Conversely, if v is a solution of the BVP (3.3)–(3.4), then $y(t) = T_{n-1}v(t)$ is a solution of the BVP (1.1)–(1.2). In fact, $y(t)$ is represented as

$$y(t) = \int_a^b H_{n-1}(t,s)v(s)\nabla s,$$

where

$$v(s) = \int_a^b G_n(s,\tau) f(T_{n-1}v(\tau), T_{n-2}v(\tau), \ldots, T_1 v(\tau), v(\tau))\nabla\tau.$$

Lemma 3.2. *For $t, s \in [a,b]$, the Green's function $H_j(t,s)$ satisfies the symmetric property,*

$$H_j(t,s) = H_j(b+a-t, \ b+a-s). \tag{3.5}$$

Proof. By the definition of $H_j(t,s)$ $(2 \le j \le n)$,

$$H_j(t,s) = \int_a^b H_{j-1}(t,r)G_j(r,s)\nabla r, \text{ for all } t,s \in [a,b].$$

Clearly, we can see that $G_j(t,s) = G_j(b+a-t, b+a-s)$. Now, the proof is by induction. Next, assume that (3.5) is true, for fixed $j \geq 2$. Then from (3.2) and using the transformation $r_1 = b+a-r$, we have

$$\begin{aligned} H_{j+1}(t,s) &= \int_a^b H_j(t,r)G_{j+1}(r,s)\nabla r \\ &= \int_a^b H_j(b+a-t, b+a-r)G_{j+1}(b+a-r, b+a-s)\nabla r \\ &= \int_a^b H_j(b+a-t, r_1)G_{j+1}(r_1, b+a-s)\nabla r_1 \\ &= H_{j+1}(b+a-t, b+a-s). \end{aligned}$$

□

Lemma 3.3. *For $t, s \in [a,b]$, the operator T_j satisfies the symmetric property,*

$$T_j v(t) = T_j v(b+a-t).$$

Proof. By the definition of T_j and using the transformation $s_1 = b+a-s$, we have

$$\begin{aligned} T_j v(t) &= \int_a^b H_j(t,s)v(s)\nabla s \\ &= \int_a^b H_j(b+a-t, b+a-s)v(s)\nabla s \\ &= \int_a^b H_j(b+a-t, s_1)v(s_1)\nabla s_1 \\ &= T_j v(b+a-t). \end{aligned}$$

□

4. **Multiple Symmetric Positive Solutions**

In this section, we establish the existence of at least three symmetric positive solutions for the BVP (1.1)–(1.2), by using Avery generalization of the Leggett-Williams fixed point theorem [3]. And then, we establish the existence of at least $2k-1$ symmetric positive solutions for an arbitrary positive integer k.

Let B be a real Banach space with cone P. A map $\alpha : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P if α is continuous and

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Similarly, we say that a map $\beta : P \to [0, \infty)$ is said to be a nonnegative continuous convex functional on P if β is continuous and

$$\beta(\lambda x + (1-\lambda)y) \leq \lambda\beta(x) + (1-\lambda)\beta(y),$$

for all $x, y \in P$ and $\lambda \in [0,1]$. Let γ, β, θ be nonnegative continuous convex functional on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers $h',\ a',\ b',\ d'$ and c', we define the following convex sets

$$P(\gamma, c') = \{y \in P | \gamma(y) < c'\},$$

$$P(\gamma, \alpha, a', c') = \{y \in P | a' \leq \alpha(y),\ \gamma(y) \leq c'\},$$

$$Q(\gamma, \beta, d', c') = \{y \in P | \beta(y) \leq d',\ \gamma(y) \leq c'\},$$

$$P(\gamma, \theta, \alpha, a', b', c') = \{y \in P | a' \leq \alpha(y),\ \theta(y) \leq b',\ \gamma(y) \leq c'\},$$

$$Q(\gamma, \beta, \psi, h', d', c') = \{y \in P | h' \leq \psi(y),\ \beta(y) \leq d',\ \gamma(y) \leq c'\}.$$

In obtaining multiple symmetric positive solutions of the BVP (1.1)–(1.2), the following Avery generalization of the Leggett-Williams fixed point theorem, so called Five Functionals Fixed Point Theorem will be fundamental.

Theorem 4.1. [3] *Let P be a cone in a real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β and θ are nonnegative continuous convex functionals on P such that, for some positive numbers c' and k,*

$$\alpha(y) \leq \beta(y) \text{ and } \parallel y \parallel \leq k\gamma(y),\ \text{for all } y \in \overline{P(\gamma, c')}.$$

Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants $h',\ d',\ a',\ b' \geq 0$ with $0 < d' < a'$ such that each of the following is satisfied.

(B1) $\{y \in P(\gamma, \theta, \alpha, a', b', c') | \alpha(y) > a'\} \neq \emptyset$ *and* $\alpha(Ty) > a'$, *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,
(B2) $\{y \in Q(\gamma, \beta, \psi, h', d', c') | \beta(y) < d'\} \neq \emptyset$ *and* $\beta(Ty) < d'$, *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,
(B3) $\alpha(Ty) > a'$, *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,
(B4) $\beta(Ty) < d'$, *provided* $y \in Q(\gamma, \beta, d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that

$$\beta(y_1) < d',\ a' < \alpha(y_2) \text{ and } d' < \beta(y_3) \text{ with } \alpha(y_3) < a'.$$

Let

$$M = m_n \prod_{j=1}^{n-1} \frac{m_j L_j}{K_j} \tag{4.1}$$

Let $B = \{v | v : C[a, b] \to \mathbb{R}\}$ be the Banach space equipped with the norm

$$\parallel v \parallel = \max_{t \in [a,b]} |v(t)|.$$

Define the cone $P \subset B$ by

$$P = \left\{ \begin{array}{l} v \in B : v(t) \geq 0 \text{ and } v^{\Delta\nabla}(t) \leq 0 \text{ on } [a, b], \\ v(t) = v(b + a - t),\ \forall\ t \in [a, b] \text{ and } \min_{t \in I} v(t) \geq M \parallel v \parallel \end{array} \right\},$$

where M is given as in (4.1). Now, let $I_1 = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right]$ and define the nonnegative continuous concave functionals $\alpha,\ \psi$ and the nonnegative continuous convex functionals $\beta,\ \theta,\ \gamma$ on P by

$$\begin{aligned}
&\gamma(v) = \max_{t\in[a,b]} |v(t)|,\ \psi(v) = \min_{t\in I_1} |v(t)|,\ \beta(v) = \max_{t\in I_1} |v(t)|,\\
&\alpha(v) = \min_{t\in I} |v(t)| \text{ and } \theta(v) = \max_{t\in I} |v(t)|.
\end{aligned}$$

We observe that for any $v \in P$,

$$\alpha(v) = \min_{t\in I} |v(t)| \leq \max_{t\in I_1} |v(t)| = \beta(v) \tag{4.2}$$

and

$$\| v \| \leq \frac{1}{M} \min_{t\in I} v(t) \leq \frac{1}{M} \max_{t\in[a,b]} |v(t)| = \frac{1}{M}\gamma(v). \tag{4.3}$$

We are now ready to present the main result of this section. We denote

$$M_j = \int_{s\in I_1} G_j(s,s)\nabla s, \text{ for } 1 \leq j \leq n.$$

Theorem 4.2. *Suppose there exist* $0 < a' < b' < \frac{b'}{M} \leq c'$ *such that* f *satisfies the following conditions:*

(A1) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{a'}{K_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j L M a' M_j, \frac{c' K K_j}{M}] \times [Ma', a']$,
(A2) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) > \frac{b'}{MK_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[m_j L b' L_j, \frac{c' K K_j}{M}] \times [b', \frac{b'}{M}]$,
(A3) $f(u_{n-1}, u_{n-2}, \dots, u_1, u_0) < \frac{c'}{K_n}$, *for all* $(|u_{n-1}|, |u_{n-2}|, \dots, |u_1|, |u_0|)$ *in* $\Pi_{j=n-1}^{1}[0, \frac{c' K K_j}{M}] \times [0, c']$.

Then the BVP (1.1)–(1.2) *has at least three symmetric positive solutions* y_1, y_2 *and* y_3 *such that*

$$\max_{t\in I_1} y_1(t) < a',\ b' < \min_{t\in I} y_2(t) \text{ and } a' < \max_{t\in I_1} y_3(t) \text{ with } \min_{t\in I} y_3(t) < b'.$$

Proof. Define the operator $T : P \to B$ by

$$Tv(t) = \int_a^b G_n(t,s) f(T_{n-1}v(s), T_{n-2}v(s), \dots, T_1 v(s), v(s))\nabla s. \tag{4.4}$$

It is obvious that a fixed point of T is the solution of the BVP (3.3)–(3.4). We seek three fixed points $v_1, v_2, v_3 \in P$ of T. First, we show that $T : P \to P$. Let $v \in P$. Clearly, $Tv(t) \geq 0$ and $(Tv)^{\Delta\nabla}(t) \leq 0$, for $t \in [a,b]$. Further, since $H_j(t,s) = H_j(b+a-t, b+a-s)$, we see that $T_j v(t) = T_j v(b+a-t)$, $1 \leq j \leq n-1$, for $t \in [a,b]$. Hence, it follows that $Tv(t) = Tv(b+a-t)$, for $t \in [a,b]$. Also, noting that

Tv satisfies the boundary conditions (3.4). Then, we have

$$\begin{aligned}\min_{t\in I} Tv(t) &= \min_{t\in I}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\int_a^b G_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M \parallel Tv \parallel .\end{aligned}$$

Hence, $Tv \in P$ and so $T : P \to P$. Moreover, T is completely continuous. From (4.2) and (4.3), for each $v \in P$, we have $\alpha(v) \leq \beta(v)$ and $\parallel v \parallel \leq \frac{1}{M}\gamma(v)$. To show that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $v \in \overline{P(\gamma, c')}$. This implies $\parallel v \parallel \leq \frac{c'}{M}$. Using Lemma 3.1, for $1 \leq j \leq n-1$ and $t \in [a, b]$, we have

$$T_jv(t) = \int_a^b H_j(t,s)v(s)\nabla s \leq \frac{c'}{M}\int_a^b H_j(t,s)\nabla s \leq \frac{c'}{M}K\int_a^b G_j(s,s)\nabla s = \frac{c'KK_j}{M}.$$

We may now use condition $(A3)$ to obtain

$$\begin{aligned}\gamma(Tv) &= \max_{t\in[a,b]}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &< \frac{c'}{K_n}\int_a^b G_n(s,s)\nabla s = c'.\end{aligned}$$

Therefore, $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$.

We first verify that conditions $(B1), (B2)$ of Theorem 4.1 are satisfied. It is obvious that

$$\{v \in P(\gamma,\theta,\alpha,b',\frac{b'}{M},c')|\alpha(v) > b'\} \neq \emptyset$$

and

$$\{v \in Q(\gamma,\beta,\psi,Ma',a',c')|\beta(v) < a'\} \neq \emptyset.$$

Next, let $v \in P(\gamma,\theta,\alpha,b',\frac{b'}{M},c')$ or $v \in Q(\gamma,\beta,\psi,Ma',a',c')$. Then, for $1 \leq j \leq n-1$,

$$T_jv(t) = \int_a^b H_j(t,s)v(s)\nabla s \leq \frac{c'}{M}\int_a^b H_j(t,s)\nabla s \leq \frac{c'}{M}K\int_a^b G_j(s,s)\nabla s = \frac{c'KK_j}{M}$$

and for $v \in P(\gamma,\theta,\alpha,b',\frac{b'}{M},c')$,

$$T_jv(t) = \int_a^b H_j(t,s)v(s)\nabla s \geq m_jLb'\int_{s\in I} G_j(s,s)\nabla s = m_jLb'L_j.$$

and also for $v \in Q(\gamma,\beta,\psi,Ma',a',c')$,

$$T_jv(t) = \int_a^b H_j(t,s)v(s)\nabla s \geq m_jLMa'\int_{s\in I_1} G_j(s,s)\nabla s = m_jLMa'M_j.$$

Now, we may apply condition $(A2)$ to get

$$\begin{aligned}\alpha(Tv) &= \min_{t\in I}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\int_a^b G_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &> \frac{b'}{K_n}\int_a^b G_n(s,s)\nabla s = b'.\end{aligned}$$

Clearly, by condition $(A1)$, we have

$$\begin{aligned}\beta(Tv) &= \max_{t\in I_1}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &< \frac{a'}{K_n}\int_a^b G_n(s,s)\nabla s = a'.\end{aligned}$$

To see that $(B3)$ is satisfied, let $v\in P(\gamma,\alpha,b',c')$ with $\theta(Tv) > \frac{b'}{M}$. Then, we have

$$\begin{aligned}\alpha(Tv) &= \min_{t\in I}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\int_a^b G_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\max_{t\in[a,b]}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\geq M\max_{t\in I}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &= M\theta(Tv) > b'.\end{aligned}$$

Finally, we show that $(B4)$ holds. Let $v\in Q(\gamma,\beta,a',c')$ with $\psi(Tv) < Ma'$. Then, we have

$$\begin{aligned}\beta(Tv) &= \max_{t\in I_1}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\leq \max_{t\in[a,b]}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\leq \int_a^b G_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &= \frac{1}{M}\int_a^b MG_n(s,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\leq \frac{1}{M}\min_{t\in I}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &\leq \frac{1}{M}\min_{t\in I_1}\int_a^b G_n(t,s)f(T_{n-1}v(s),T_{n-2}v(s),\ldots,T_1v(s),v(s))\nabla s\\ &= \frac{1}{M}\psi(Tv) < a'.\end{aligned}$$

We have proved that all the conditions of Theorem 4.1 are satisfied and so there exist at least three symmetric positive solutions $v_1, v_2, v_3 \in \overline{P(\gamma, c')}$ for the BVP (3.3)–(3.4). Therefore, the BVP (1.1)–(1.2) has at least three symmetric positive solutions y_1, y_2, y_3 of the form,

$$y_i(t) = T_{n-1}v_i(t) = \int_a^b H_{n-1}(t,s)v_i(s)\nabla s, \; i = 1, 2, 3,$$

such that

$$\beta(y_1) < a', \; b' < \alpha(y_2) \text{ and } a' < \beta(y_3) \text{ with } \alpha(y_3) < b'.$$

This completes the proof of the theorem. □

Now we prove the existence of at least $2k-1$ symmetric positive solutions for the BVP (1.1)–(1.2) by using induction on k.

Theorem 4.3. *Let k be an arbitrary positive integer. Assume that there exist numbers $a_r(r = 1, 2, \ldots, k)$ and $b_s(s = 1, 2, \ldots, k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \cdots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{M} < a_k$ such that*

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) < \frac{a_r}{K_n}, \; \textit{for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \textit{in } \Pi_{j=n-1}^{1}[m_j L M a_r M_j, \frac{a_k K K_j}{M}] \times [M a_r, a_r], \; r = 1, 2, \ldots, k, \end{array}\right\} \tag{4.5}$$

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \frac{b_s}{M K_n}, \; \textit{for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \textit{in } \Pi_{j=n-1}^{1}[m_j L b_s L_j, \frac{a_k K K_j}{M}] \times [b_s, \frac{b_s}{M}], \; s = 1, 2, \ldots, k-1. \end{array}\right\} \tag{4.6}$$

Then the BVP (1.1)–(1.2) has at least $2k-1$ symmetric positive solutions in $\overline{P}_{a_k}$.

Proof. We use induction on k. First, for $k = 1$, we know from (4.5) that $T : \overline{P}_{a_1} \to P_{a_1}$, then it follows from Schauder fixed point theorem that the BVP (1.1)–(1.2) has at least one symmetric positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = l$. In order to prove that this conclusion holds for $k = l+1$, we suppose that there exist numbers $a_r(r = 1, 2, \ldots, l+1)$ and $b_s(s = 1, 2, \ldots, l)$ with $0 < a_1 < b_1 < \frac{b_1}{M} < a_2 < b_2 < \frac{b_2}{M} < \cdots < a_l < b_l < \frac{b_l}{M} < a_{l+1}$ such that

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) < \frac{a_r}{K_n} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j L M a_r M_j, \frac{a_{l+1} K K_j}{M}] \times [M a_r, a_r], \; r = 1, 2, \ldots, l+1, \end{array}\right\} \tag{4.7}$$

$$\left.\begin{array}{l} f(u_{n-1}, u_{n-2}, \ldots, u_1, u_0) > \frac{b_s}{M K_n} \text{ for all } (|u_{n-1}|, |u_{n-2}|, \ldots, |u_1|, |u_0|) \\ \text{in } \Pi_{j=n-1}^{1}[m_j L b_s L_j, \frac{a_{l+1} K K_j}{M}] \times [b_s, \frac{b_s}{M}], \; s = 1, 2, \ldots, l. \end{array}\right\} \tag{4.8}$$

By assumption, the BVP (1.1)–(1.2) has at least $2l-1$ symmetric positive solutions $u_i(i = 1, 2, \ldots, 2l-1)$ in $\overline{P}_{a_l}$. At the same time, it follows from Theorem 4.2, (4.7) and (4.8) that the BVP (1.1)–(1.2) has at least three symmetric positive solutions u, v and w in $\overline{P}_{a_{l+1}}$ such that

$$\max_{t\in I_1} u(t) < a_l,\ b_l < \min_{t\in I} v(t) \text{ and } a_l < \max_{t\in I_1} w(t) \text{ with } \min_{t\in I} w(t) < b_l.$$

Obviously, v and w are different from $u_i (i = 1, 2, \ldots, 2l-1)$. Therefore, the BVP (1.1)–(1.2) has at least $2l+1$ symmetric positive solutions in $\overline{P}_{a_{l+1}}$, which shows that this conclusion also holds for $k = l+1$. □

5. **Example**

Let us consider an example to illustrate the usage of Theorem 4.2. Let $n = 2$, $\mathbb{T} = [0, 1.5] \cup [2, 3]$, $a = 0$, $b = 3$, $\alpha_1 = 0.5$, $\beta_1 = 0.6$, $\alpha_2 = 0.8$, $\beta_2 = 1.5$. Now consider the BVP,

$$y^{(\Delta\nabla)^2}(t) = f(y(t), y^{\Delta\nabla}(t)),\ t \in [0, 3] \tag{5.1}$$

satisfying the boundary conditions,

$$\left.\begin{aligned} 0.5y(0) - 0.6y^{\Delta}(0) = 0,\ 0.5y(3) + 0.6y^{\Delta}(3) = 0, \\ 0.8y^{\Delta\nabla}(0) - 1.5y^{(\Delta\nabla)\Delta}(0) = 0,\ 0.8y^{\Delta\nabla}(3) + 1.5y^{(\Delta\nabla)\Delta}(3) = 0, \end{aligned}\right\} \tag{5.2}$$

and

$$f(u, v) = \begin{cases} \frac{sinu}{100} + \frac{13}{100}v^6, & v \le 2, \\ \frac{sinu}{100} + \frac{208}{25}, & v \ge 2. \end{cases}$$

Then the Green's functions $G_1(t, s)$ and $G_2(t, s)$ are given by

$$G_1(t, s) = \begin{cases} \frac{(5t+6)(21-5s)}{135}, & t \le s, \\ \frac{(5s+6)(21-5t)}{135}, & s \le t, \end{cases}$$

and

$$G_2(t, s) = \begin{cases} \frac{(8t+15)(39-8s)}{432}, & t \le s, \\ \frac{(8s+15)(39-8t)}{432}, & s \le t. \end{cases}$$

Clearly, the Green's functions $G_1(t, s)$ and $G_2(t, s)$ are positive. By using MATLAB, we can compute the values, $m_1 = 0.4642857143$, $m_2 = 0.5384615385$,

$$K_1 = \int_0^3 G_1(s, s)\nabla s = 2.966049383,\ L_1 = \int_{s\in I} G_1(s, s)\nabla s = 1.305632716,$$

$$M_1 = \int_{s\in I_1} G_1(s, s)\nabla s = 0.6672839506,\ K_2 = \int_0^3 G_2(s, s)\nabla s = 3.891589506,$$

Therefore,

$$K = 2.966049383,\ L = 0.6061866182,\ M = 0.1100481269.$$

Clearly f is continuous and increasing on $[0, \infty)$. If we choose $a' = 1.01$, $b' = 2$ and $c' = 150$ then $0 < a' < b' < \frac{b'}{M} \le c'$ and f satisfies

(i) $f(u,v) < 0.2595340537 = \frac{a'}{K_2}$, for $(u,v) \in [0.0208740334, 11991.2749] \times [0.1111486082, 1.01]$,

(ii) $f(u,v) > 4.670036954 = \frac{b'}{MK_2}$, for $(u,v) \in [0.7349244321, 11991.2749] \times [2, 18.1738668]$,

(iii) $f(u,v) < 38.54466145 = \frac{c'}{K_2}$, for $(u,v) \in [0, 11991.2749] \times [0, 150]$.

Then all the conditions of Theorem 4.2 are satisfied. Thus by Theorem 4.2 the BVP (5.1)–(5.2) has at least three symmetric positive solutions y_1, y_2 and y_3 satisfying

$$\max_{t\in[1,2]} y_1(t) < 1.01,\ 2 < \min_{t\in[\frac{3}{4},\frac{9}{4}]} y_2(t) \text{ and } 1.01 < \max_{t\in[1,2]} y_3(t) \text{ with } \min_{t\in[\frac{3}{4},\frac{9}{4}]} y_3(t) < 2.$$

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