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SPLINE IN TENSION METHODS FOR SINGULARLY PERTURBED ONE SPACE DIMENSIONAL PARABOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

[1]R.K. MOHANTY, [2]RAJIVE KUMAR and [2]VIJAY DAHIYA

[1]Department of Mathematics, Faculty of Mathematical Sciences,
University of Delhi, Delhi – 110 007, INDIA
E-mail: rmohanty@maths.du.ac.in
[2]Department of Mathematics, Deenbandhu Chottu Ram University of Science & Technology,
Murthal, Haryana-131039, INDIA
E-mail: rajeev_kansal@yahoo.com; vijay_15dahiya@yahoo.com

Abstract. In this article, we report a class of two-level implicit difference schemes; using spline in tension for the numerical solution of singularity perturbed one space dimensional parabolic equations with singular coefficients. The proposed methods are of $O(k^2+h^2)$ accurate and applicable to problems both in singular and non-singular cases. Stability theory of a proposed method is discussed and numerical experiments carried out on the new schemes supplement the analytical results.

***Keywords*:** Spline in tensions, parabolic equations, two-level implicit schemes, singular perturbation, singular coefficients, RMS errors.

1. INTRODUCTION

Consider the following singularly perturbed one space dimensional parabolic equation

$$\epsilon u_{xx} = u_t + a(x)u_x + b(x)u + f(x,t), \qquad 0 < x < 1, t > 0 \tag{1}$$

where $0 < \epsilon \ll 1$ and $a(x), b(x)$ and $f(x,t)$ are continuous bounded functions defined in the semi-infinite region $\Omega = \{(x,t) \big| 0 < x < 1, t > 0\}$.

The initial and boundary conditions associated with equation (1) are given by

$$u(x,0) = a_0(x)\,,\ 0 \le x \le 1 \tag{2}$$

$$u(0,t) = b_0(t), \quad u(1,t) = b_1(t), \quad t \ge 0 \tag{3}$$

We assume that the functions, $a_0(x)$, $b_0(t)$ and $b_1(t)$ are sufficiently smooth and their required high-order derivatives exist in the solution space Ω.

The initial-boundary value problems (1)-(3) model many physical problems in various areas of science and engineering. In many applications equation (1) represents boundary or interior layers and has been studied by many authors. Earlier, the theory of splines functions and their applications were studied by (Ahlberg et el., 1967; Greville, 1969).

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Later, (O'Malley, 1974; Abrahamsson et al., 1974) have introduced singular perturbation technique to solve two point boundary value problems. Various properties of splines and variational methods were discussed by (Prenter, 1975; Pruess, 1976; Boor, 1978). Various numerical methods for singular perturbation problems have been discussed by (Hemker & Miller, 1979; Kreiss & Kreiss, 1982; Segal, 1982; Abrahamsson & Osher, 1982). Using adaptive spline function approximation (Jain & Aziz, 1983) have proposed a numerical method for stiff and convection-diffusion equation. Further, using lower order accurate upwind and central difference approximations (Miller et al., 1995) have developed a stable numerical method for solving singularly perturbed problems. During last decade, (Marusic & Rogina, 1996; Kadalbajoo & Patidar, 2002) have derived second order accurate numerical methods for the solution of singularly perturbed two point linear boundary value problems by spline in tension. Recently, (Khan et al., 2005) have surveyed various spline function approximations. All these methods are only applicable to problems in rectangular coordinates. Recently, using collocation and tension splines, (Beros & Marusic, 2005) have solved singularly perturbed heat conduction problems. Difficulties were experienced in the past for the numerical solution of singularly perturbed one space dimensional parabolic problems in polar coordinates. The solution usually deteriorates in the vicinity of singularity. In this piece of work, we have discussed the approximation method based on spline in tension to solve the problems of type (1). We have refined our procedure in such a way that the solution retains its order and accuracy even in the vicinity of the singularity $x = 0$. It is well known that the most classical methods fail when ϵ is small relative to the mesh length h >0, that is used for discretization of the differential equation (1) in the x-direction. Our aim is to show that tension splines can furnish accurate numerical approximations of equation (1), when all or any of the coefficients $a(x), b(x)$ and $f(x,t)$ contain singularity at $x = 0$ and when ϵ is either small or large as compared with h. We consider three types of problems. In the first case, we analyze the problems in which the second derivative term $u_{xx}(x,t)$ and the function term $u(x,t)$ are present, whereas the term containing the first term derivative $u_x(x,t)$ is absent. The problems having the second derivative $u_{xx}(x,t)$ term and first derivative term $u_x(x,t)$ but lacking the function term $u(x,t)$ are considered in the second case. Finally, the third case deals with the most general problems. In all cases, we use the continuity of first derivative of the spline function. The resulting spline difference methods are two-level implicit schemes (see Fig. 1) and of $O(k^2 + h^2)$ accurate and are tri-diagonal system of equations at each advanced time level, which can be solved by using a tri-diagonal solver. The main attraction of our work is that the proposed tension spline difference schemes are applicable to both singular and non-singular problems. In section 2, we have discussed the derivation of the spline methods and their application to singular problems. In section 3, we have discussed stability analysis of a method. In section 4, numerical results of three different singular problems are given to demonstrate the utility of the proposed method. The numerical results confirmed that the proposed tension spline methods produce an oscillation-free solution for $0 < \epsilon \ll 1$ everywhere in the solution region $0< x<1$, $t>0$.

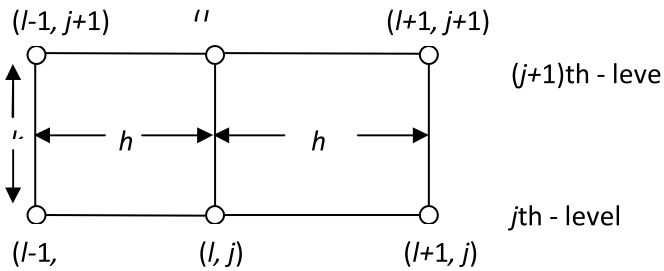

Figure 1 (Schematic representation of two-level scheme)

2. DESCRIPTION OF THE TENSION SPLINE METHOD

The solution domain $[0,1]\times[t>0]$ is divided into $(N+1)\times J$ mesh with the spatial step size $h = 1/(N+1)$ in x-direction and the time step size $k>0$ in t-direction respectively, where N and J are positive integers. The mesh ratio parameter is given by $\lambda = (k/h^2) > 0$.

Grid points are defined by $(x_l, t_j) = (lh, jk)$, $l = 0(1)N+1$ and $j = 0, 1, 2,\ldots J$. The notations u_l^j and U_l^j are used for the discrete approximation and the exact solution of $u(x,t)$ at the grid point (x_l, t_j), respectively.

Let

$$a_l = a(x_l),\quad b_l = b(x_l) \text{ and } f_l = f(x_l)$$

For $x \in [x_{l-1}, x_l]$, we denote

$$\hat{a}_l = \frac{1}{2}(a_{l-1} + a_l),\qquad \hat{b}_l = \frac{1}{2}(b_{l-1} + b_l),\qquad \hat{f}_l = \frac{1}{2}(f_{l-1} + f_l).$$

We consider the following three cases:

Case 1: First we consider the differential equation

$$\varepsilon u_{xx} = u_t + b(x)u + f(x,t)\ ,\ 0 < x < 1,\ t>0 \tag{4}$$

which is a particular case of equation (1) in which the first derivative term $u_x(x,t)$ is absent.

For the derivation of the method, we follow the approaches given by (Kadalbajoo & Patidar , 2005), and (Mohanty et al., 2005).

Now we consider the ordinary differential equation

$$-\varepsilon\frac{d^2u}{dx^2} + b(x)u = f(x),\ 0<x<1 \tag{5}$$

The numerical solution of this equation is sought in the form of the spline function $S(x)$, which on each interval $[x_{l-1}, x_l]$, denoted by $S_l(x)$ satisfies the differential equation

$$-\epsilon S_l''(x) + \hat{b}_l S_l(x) = \hat{f}_l \tag{6}$$

The interpolating conditions:

$$S_l(x_{l-1}) = u_{l-1}, \qquad S_l(x_l) = u_l \tag{7}$$

and the continuity condition:

$$S_l'(x_l^+) = S_l'(x_l^-) \tag{8}$$

Solving the equation (6) and using the interpolating conditions (7), we get

$$S_l(x) = \frac{-1}{\sinh(hp_l)}[A_l \sinh(p_l(x_{l-1} - x)) + B_l \sinh(p_l(x - x_l))] + \frac{\hat{f}_l}{\hat{b}_l}, x \in [x_{l-1}, x_l] \tag{9}$$

Where

$$A_l = u_l - \frac{\hat{f}_l}{\hat{b}_l}, \qquad B_l = u_{l-1} - \frac{\hat{f}_l}{\hat{b}_l}, \qquad p_l = \sqrt{\frac{\hat{b}_l}{\epsilon}},$$

Equation (9) is known as spline in tension (Pruess, 1976). Replacing l by $l+1$ in equation (9), we can obtain the spline function $S_{l+1}(x)$ defined in $[x_l, x_{l+1}]$.

As $S(x) \in C^2[0,1]$, we have

$$S_l'(x_l) = S_{l+1}'(x_l) \tag{10}$$

Differentiating equation (9) with respect to 'x' and using the continuity condition (8), we obtain the spline in tension scheme for the numerical solution of equation (5) as:

$$\begin{aligned}\frac{hp_l}{\sinh(hp_l)} u_{l-1} - [hp_l \coth(hp_l) + hp_{l+1} \coth(hp_{l+1})]u_l + \frac{hp_{l+1}}{\sinh(hp_{l+1})} u_{l+1} \\ = \left[\frac{hp_l}{\sinh(hp_l)} - hp_l \coth(hp_l)\right]\frac{\hat{f}_l}{\hat{b}_l} \\ + \left[\frac{hp_{l+1}}{\sinh(hp_{l+1})} - hp_{l+1} \coth(hp_{l+1})\right]\frac{\hat{f}_{l+1}}{\hat{b}_{l+1}}, l = 1(1)N\end{aligned} \tag{11}$$

Note that, the scheme (11) is of $O(h^2)$ accurate for the numerical solution of (5), however, the scheme fails to compute at l=1. We overcome this difficulty by using the following approximations:

$$a_{l\pm1} = a_l \pm h a_{xl} + O(h^2), \tag{12a}$$
$$b_{l\pm1} = b_l \pm h b_{xl} + O(h^2), \tag{12b}$$
$$f_{l\pm1} = f_l \pm h f_{xl} + O(h^2)\,. \tag{12c}$$

Now using the approximation (12a)-(12c) in equation (11) and neglecting high order terms we obtain the tension spline scheme for the equation (5) in compact form:

$$\left[1-\frac{h^2}{12\varepsilon}\left(2b_l - hb_{xl}\right)\right]u_{l-1} - \left[2+\frac{2h^2}{3\varepsilon}b_l\right]u_l + \left[1-\frac{h^2}{12\varepsilon}\left(2b_l + hb_{xl}\right)\right]u_{l+1} = \frac{-h^2}{\varepsilon}f_l\,,$$
$$l=1(1)N \tag{13}$$

In order to obtain the tension spline scheme for parabolic equation (4), we replace

u_l by $\frac{1}{2}(u_l^{j+1}+u_l^j)$, $u_{l\pm1}$ by $\frac{1}{2}(u_{l\pm1}^{j+1}+u_{l\pm1}^j)$, and f_l by $-(\overline{u}_{tl}^{\,j}+\overline{f}_l^{\,j})$ (where $\overline{f}_l^{\,j} = f(x_l, t_j+\frac{k}{2})$, and $\overline{u}_{tl}^{\,j} = (u_l^{j+1}-u_l^j)/k$) in (13) and we obtain

$$\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l - hb_{xl}\right)\right]u_{l-1}^{j+1} - \left[1+\frac{h^2}{3\varepsilon}b_l+\frac{h^2}{\varepsilon k}\right]u_l^{j+1} + \left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l + hb_{xl}\right)\right]u_{l+1}^{j+1}$$
$$= -\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l - hb_{xl}\right)\right]u_{l-1}^{j} + \left[1+\frac{h^2}{3\varepsilon}b_l-\frac{h^2}{\varepsilon k}\right]u_l^{j} - \left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l + hb_{xl}\right)\right]u_{l+1}^{j} + \frac{h^2}{\varepsilon}\overline{f}_l^{\,j}$$
$$l=1(1)N,\ j=0,1,2,\ldots \tag{14}$$

Case 2: In this case, we consider the differential equation of the form

$$\epsilon u_{xx} = u_t + a(x)u_x + f(x,t), \qquad 0 < x < 1, t > 0 \tag{15}$$

This is a particular case of equation (1), in which the function term $u(x,t)$ is absent.

For the derivation of the method, we follow the same ideas given by (Kadalbajoo & Patidar , 2005), and (Mohanty et al., 2005).

We consider the ordinary differential equation

$$-\varepsilon\frac{d^2u}{dx^2} + a(x)\frac{du}{dx} = f(x)\ , \quad 0 < x < 1, \tag{16}$$

which is a steady-state case of equation (15). As in case 1, we seek $S(x)$ as a solution of the above differential equation

$$-\epsilon S_l''(x) + \hat{a}_l S_l(x) = \hat{f}_l \tag{17}$$

This satisfies the interpolating conditions (7) and the continuity condition (8).

Solving equation (16) by the help of conditions (7), we obtain

$$S_l(x)=\frac{1}{F_l}\left(u_l e^{L_l x_{l-1}}-u_{l-1}e^{L_l x_l}\right)+\frac{1}{F_l}\left(x_{l-1}e^{L_l x_l}-x_l e^{L_l x_{l-1}}\right)\frac{\hat{f}_l}{\hat{a}_l}$$
$$+\frac{1}{F_l}\left(u_{l-1}-u_l+h\frac{\hat{f}_l}{\hat{a}_l}\right)e^{L_l x}+\frac{\hat{f}_l}{\hat{a}_l}x$$

(18)

where $x\in\left[x_{l-1},x_l\right]$, $L_l=\dfrac{\hat{a}_l}{\varepsilon}$, and $F_l=e^{L_l x_{l-1}}-e^{L_l x_l}$.

Similarly, replacing l by $l+1$ in equation (18), we can get the spline function $S_{l+1}(x)$ valid in $[x_l,x_{l+1}]$.

Differentiating equation (18) with respect to 'x' and using the continuity condition (8), we may obtain the spline in tension method for the approximate solution of equation (16) as:

$$p_l^- u_{l-1}+p_l^0 u_l+p_l^+ u_{l+1}=q_l^- f_{l-1}+q_l^0 f_l+q_l^+ f_{l+1} \qquad l=1(1)N \tag{19}$$

where

$$p_l^-=\frac{L_l}{e^{-hL_l}-1},\qquad p_l^+=\frac{L_{l+1}}{1-e^{hL_{l+1}}},\qquad p_l^0=-(p_l^++p_l^-),$$

$$q_l^-=\frac{-1}{2\hat{a}_l}(1+hp_l^-),\qquad q_l^+=\frac{1}{2\hat{a}_{l+1}}(1+hp_l^+),\qquad q_l^0=(q_l^++q_l^-).$$

Note that, the scheme (19) is of $O(h^2)$ accurate for the numerical solution of (16), however, the scheme fails to compute at l=1. We overcome this difficulty by using the approximations defined by (12) and we obtain

$$\left[1+\frac{h}{4\varepsilon}\left(2a_l-ha_{xl}\right)\right]u_{l-1}-\left[2-\frac{h^2}{2\varepsilon}a_{xl}\right]u_l+\left[1-\frac{h}{4\varepsilon}\left(2a_l+ha_{xl}\right)\right]u_{l+1}=\frac{-h^2}{\varepsilon}f_l,\ l=1(1)N \tag{20}$$

In order to obtain the tension spline method for the parabolic equation (15), we replace u_l by $\frac{1}{2}(u_l^{j+1}+u_l^j)$, $u_{l\pm1}$ by $\frac{1}{2}(u_{l\pm1}^{j+1}+u_{l\pm1}^j)$, and f_l by $-(\overline{u}_{tl}^{\,j}+\overline{f}_l^{\,j})$ (where $\overline{f}_l^{\,j}=f(x_l,t_j+\frac{k}{2})$, and $\overline{u}_{tl}^{\,j}=(u_l^{j+1}-u_l^j)/k$) in (20) and we obtain

$$\left[\frac{1}{2}+\frac{h}{8\varepsilon}\left(2a_l-ha_{xl}\right)\right]u_{l-1}^{j+1}-\left[1-\frac{h^2}{4\varepsilon}a_{xl}+\frac{h^2}{\varepsilon k}\right]u_l^{j+1}+\left[\frac{1}{2}-\frac{h}{8\varepsilon}\left(2a_l+ha_{xl}\right)\right]u_{l+1}^{j+1}$$
$$=-\left[\frac{1}{2}+\frac{h}{8\varepsilon}\left(2a_l-ha_{xl}\right)\right]u_{l-1}^j+\left[1-\frac{h^2}{4\varepsilon}a_{xl}-\frac{h^2}{\varepsilon k}\right]u_l^j-\left[\frac{1}{2}-\frac{h}{8\varepsilon}\left(2a_l+ha_{xl}\right)\right]u_{l+1}^j+\frac{h^2}{\varepsilon}\overline{f}_l^{\,j},$$

$$l=1(1)N,\ j=0,1,2,\ldots \tag{21}$$

Case 3: Finally we consider the most general problem (1), where both $u_x(x,t)$ and $u(x,t)$ are present.

For the derivation of the method, we now follow the techniques given by (Kadalbajoo & Patidar , 2005), and (Mohanty et al., 2005).

For this purpose, we consider the ordinary differential equation

$$-\varepsilon\frac{d^2u}{dx^2}+a(x)\frac{du}{dx}+b(x)u=f(x)\ ,\quad 0<x<1 \tag{22}$$

which is a steady-state case of (1). In this case the spline function $S(x)$ satisfies

$$-\varepsilon S_l''(x)+\hat{a}_l S_l'(x)+\hat{b}_l S_l(x)=\hat{f}_l \tag{23}$$

This also satisfies the conditions (7) and (8). Solving equation (23) with the help of interpolating conditions (7), we obtain

$$S_l(x)=\frac{e^{\hat{c}_l x}}{-\sinh(h\hat{d}_l)}\left[G_l\sinh(\hat{d}_l(x_{l-1}-x))+H_l\sinh(\hat{d}_l(x-x_l))\right]+\frac{\hat{f}_l}{\hat{b}_l},\ \ x\in[x_{l-1},x_l] \tag{24}$$

where

$$G_l=\left(u_l-\frac{\hat{f}_l}{\hat{b}_l}\right)e^{-\hat{c}_l x_l},\quad H_l=\left(u_{l-1}-\frac{\hat{f}_l}{\hat{b}_l}\right)e^{-\hat{c}_l x_{l-1}},\ \ \hat{c}_l=\frac{\hat{a}_l}{2\epsilon},\ \hat{d}_l=\frac{1}{2\epsilon}\sqrt{\hat{a}_l^2+4\epsilon\hat{b}_l},$$

Replacing l by $l+1$ in equation (24), we can obtain the spline function $S_{l+1}(x)$ for the equation (23) in the interval $[x_l, x_{l+1}]$.

Using the continuity condition (8), from equation (24) we obtain the difference scheme based on spline in tension for the approximate solution of equation (22) as

$$p_l^- u_{l-1}+p_l^0 u_l+p_l^+ u_{l+1}=q_l^- f_{l-1}+q_l^0 f_l+q_l^+ f_{l+1} \tag{25}$$

where

$$p_l^-=\frac{h\hat{d}_l}{\sinh(h\hat{d}_l)}e^{h\hat{c}_l},$$

$$p_l^+=\frac{h\hat{d}_{l+1}}{\sinh(h\hat{d}_{l+1})}e^{-h\hat{c}_{l+1}},$$

$$p_l^0=-h\hat{d}_l\coth(h\hat{d}_l)-h\hat{d}_{l+1}\coth(h\hat{d}_{l+1})+h\hat{c}_{l+1}-h\hat{c}_l,$$

$$q_l^-=\frac{1}{2\hat{b}_l}[p_l^- -h\hat{c}_l-h\hat{d}_l\coth(h\hat{d}_l)],$$

$$q_l^+=\frac{1}{2\hat{b}_{l+1}}[p_l^+ +h\hat{c}_{l+1}-h\hat{d}_{l+1}\coth(h\hat{d}_{l+1})],$$

$$q_l^0 = q_l^- + q_l^+ .$$

Note that the tension spline scheme (25) is of $O(h^2)$ for the approximate solution of the equation (22). However, these scheme fails when the coefficients $a(x), b(x)$ and $f(x)$ contain singularities and the solution is to be determined at l=1. We overcome this difficulty by modifying the scheme (25) in such a manner that the solution retains its order and accuracy even in the vicinity of the singularity x=0.

As discussed in case 1 and case 2, using the approximations (12) and neglecting high order terms, we obtain the following tension spline scheme for the solution of parabolic equation (1) in compact form:

$$\left[\frac{1}{2}+\frac{h}{4\varepsilon}\left(a_l-\frac{h}{2}a_{xl}\right)-\frac{h^2b_l}{12\varepsilon}\right]u_{l-1}^{j+1}-\left[1+\frac{h^2}{12\varepsilon}\left(4b_l-3a_{xl}\right)+\frac{h^2}{\varepsilon k}\right]u_l^{j+1}+\left[\frac{1}{2}-\frac{h}{4\varepsilon}\left(a_l+\frac{h}{2}a_{xl}\right)-\frac{h^2b_l}{12\varepsilon}\right]u_{l+1}^{j+1}$$
$$=-\left[\frac{1}{2}+\frac{h}{4\varepsilon}\left(a_l-\frac{h}{2}a_{xl}\right)-\frac{h^2b_l}{12\varepsilon}\right]u_{l-1}^{j}+\left[1+\frac{h^2}{12\varepsilon}\left(4b_l-3a_{xl}\right)-\frac{h^2}{\varepsilon k}\right]u_l^{j}-\left[\frac{1}{2}-\frac{h}{4\varepsilon}\left(a_l+\frac{h}{2}a_{xl}\right)-\frac{h^2b_l}{12\varepsilon}\right]u_{l+1}^{j}+\frac{h^2}{\varepsilon}\overline{f}_l^{\,j}$$
$$l=1(1)N,\quad j=0,1,2,\ldots \quad (26)$$

Note that the tension spline schemes (14), (21) and (26) are of $O(k^2+h^2)$accurate for the numerical solution of singularly perturbed parabolic partial differential equations (4), (15) and (1), respectively and free from the terms $(1/x_{l\pm1})$, hence very easily computed for l=1(1)N, j = 0,1,2,..in the solution region Ω.

3. STABILITY ANALYSIS

Now we discuss the stability analysis for the scheme (14).

In this case the exact solution U_l^j satisfies

$$\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l-hb_{xl}\right)\right]U_{l-1}^{j+1}-\left[1+\frac{h^2}{3\varepsilon}b_l+\frac{h^2}{\varepsilon k}\right]U_l^{j+1}+\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l+hb_{xl}\right)\right]U_{l+1}^{j+1}$$
$$=-\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l-hb_{xl}\right)\right]U_{l-1}^{j}+\left[1+\frac{h^2}{3\varepsilon}b_l-\frac{h^2}{\varepsilon k}\right]U_l^{j}-\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l+hb_{xl}\right)\right]U_{l+1}^{j} \quad (27)$$
$$+\frac{h^2}{\varepsilon}\overline{f}_l^{\,j}+O(k^2h^2+h^4).$$

We assume that there exists an error $e_l^j = U_l^j - u_l^j$ at each grid point (x_l, t_j), then subtracting (14) from (27), we obtain the error equation

$$\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l-hb_{xl}\right)\right]e_{l-1}^{j+1}-\left[1+\frac{h^2}{3\varepsilon}b_l+\frac{h^2}{\varepsilon k}\right]e_l^{j+1}+\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l+hb_{xl}\right)\right]e_{l+1}^{j+1}$$

$$=-\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l-hb_{xl}\right)\right]e_{l-1}^{j}+\left[1+\frac{h^2}{3\varepsilon}b_l-\frac{h^2}{\varepsilon k}\right]e_l^{j}-\left[\frac{1}{2}-\frac{h^2}{24\varepsilon}\left(2b_l+hb_{xl}\right)\right]e_{l+1}^{j} \tag{28}$$

$$+\frac{h^2}{\varepsilon}\overline{f}_l^{\,j}+O(k^2h^2+h^4).$$

To establish stability for the scheme (14), it is necessary to assume that the solution of the homogeneous part of the error equation is of the form $e_l^j=\xi^j e^{i\beta l}$, where ξ is in general complex, $i=\sqrt{-1}$, β is real and we obtain the amplification factor

$$\xi=\frac{\left[\frac{h^2b_l}{2\varepsilon}-\frac{h^2}{k\varepsilon}+2\left(1-\frac{h^2b_l}{6\varepsilon}\right)\sin^2\left(\frac{\beta}{2}\right)+i\frac{h^3b_{xl}}{12\varepsilon}\sin\beta\right]}{\left[-\frac{h^2b_l}{2\varepsilon}-\frac{h^2}{k\varepsilon}-2\left(1-\frac{h^2b_l}{6\varepsilon}\right)\sin^2\left(\frac{\beta}{2}\right)-i\frac{h^3b_{xl}}{12\varepsilon}\sin\beta\right]} \tag{29}$$

For stability it is required that $\left|\xi\right|\le 1$. Since max $\sin^2\left(\frac{\beta}{2}\right)=1$ and $\varepsilon\propto h$, it is easy to verify from (29) that $\left|\xi\right|\le 1$ for all variable angle β. Hence the scheme (14) is unconditionally stable.

4. EXPERIMENTAL RESULTS

Numerical results presented in this section are concerned with the application of tension spline methods to singular perturbation problems. We have solved the following singularly perturbed singular problems for a fixed value of $\lambda=1.6$. The homogeneous functions, initial and boundary conditions may be obtained using the exact solutions as a test procedure. In all cases, we have used Gauss-elimination method (Saad, 2003; Hageman & Young, 2004). All computations were carried out using double precision.

Example 1: $\epsilon u_{xx}=u_t+\frac{1}{x}u+f(x,t),\quad 0<x<1, t>0$ (30)

The exact solution is given by $u(x,t)=e^{-t}\cosh x$. The root mean square (RMS) errors at t = 1.0 are tabulated in Table 1 for various values of ϵ $(0<\epsilon\ll 1)$.

Example 2: $\epsilon u_{xx}=u_t+\frac{\alpha}{x}u_x+f(x,t),\quad 0<x<1, t>0$ (31)

The exact solution is given by $u(x,t)=e^{-t}\sinh x$. For α =1 and 2, the equation above represents singularly perturbed linear singular parabolic equation in cylindrical and spherical symmetry, respectively. The RMS errors at t = 1 are tabulated in Table 2 for different values of ϵ $(0<\epsilon\ll 1)$ and for α=1 and 2, respectively.

Table 1: The RMS errors

h	$\varepsilon=2^{-3}$	$\varepsilon=2^{-4}$	$\varepsilon=2^{-5}$	$\varepsilon=2^{-6}$	$\varepsilon=2^{-7}$	$\varepsilon=2^{-8}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-10}$
2^{-3}	0.6156(-04)	0.9773(-04)	0.1271(-03)	0.1492(-03)	0.1649(-03)	0.1751(-03)	0.1812(-03)	0.1846(-03)
2^{-4}	0.1132(-04)	0.1931(-04)	0.2608(-04)	0.3132(-04)	0.3517(-04)	0.3793(-04)	0.3982(-04)	0.4105(-04)
2^{-5}	0.2574(-05)	0.4495(-05)	0.6140(-05)	0.7419(-05)	0.8363(-05)	0.9041(-05)	0.9524(-05)	0.9863(-05)
2^{-6}	0.6256(-06)	0.1099(-05)	0.1506(-05)	0.1822(-05)	0.2056(-05)	0.2224(-05)	0.2343(-05)	0.2428(-05)
2^{-7}	0.1549(-06)	0.2727(-06)	0.3739(-06)	0.4527(-06)	0.5109(-06)	0.5527(-06)	0.5824(-06)	0.6034(-06)
2^{-8}	0.3862(-07)	0.6799(-07)	0.9324(-07)	0.1129(-06)	0.1274(-06)	0.1378(-06)	0.1452(-06)	0.1504(-06)
2^{-9}	0.9642(-08)	0.1697(-07)	0.2328(-07)	0.2819(-07)	0.3181(-07)	0.3442(-07)	0.3627(-07)	0.3758(-07)
2^{-10}	0.2408(-08)	0.4241(-08)	0.5817(-08)	0.7043(-08)	0.7948(-08)	0.8599(-08)	0.9063(-08)	0.9390(-08)

Table 2: The RMS errors

h	$\varepsilon=2^{-3}$	$\varepsilon=2^{-4}$	$\varepsilon=2^{-5}$	$\varepsilon=2^{-6}$	$\varepsilon=2^{-7}$	$\varepsilon=2^{-8}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-10}$
$\alpha=1$								
2^{-3}	0.2567(-02)	0.5594(-02)	0.1119(-01)	0.1888(-01)	0.2321(-01)	0.2236(-01)	0.2097(-01)	0.2015(-01)
2^{-4}	0.6167(-03)	0.1344(-02)	0.2829(-02)	0.5750(-02)	0.1106(-01)	0.1857(-01)	0.2283(-01)	0.2182(-01)
2^{-5}	0.1512(-03)	0.3283(-03)	0.6889(-03)	0.1415(-02)	0.2860(-02)	0.5681(-02)	0.1088(-01)	0.1832(-01)
2^{-6}	0.3747(-04)	0.8124(-04)	0.1701(-03)	0.3488(-03)	0.7072(-03)	0.1422(-02)	0.2840(-02)	0.5626(-02)
2^{-7}	0.9330(-05)	0.2021(-04)	0.4231(-04)	0.8667(-04)	0.1755(-03)	0.3535(-03)	0.7090(-03)	0.1417(-02)
2^{-8}	0.2327(-05)	0.5044(-05)	0.1055(-04)	0.2161(-04)	0.4375(-04)	0.8806(-04)	0.1767(-03)	0.3540(-03)
2^{-9}	0.5813(-06)	0.1259(-05)	0.2635(-05)	0.5396(-05)	0.1092(-04)	0.2198(-04)	0.4410(-04)	0.8836(-04)
2^{-10}	0.1452(-06)	0.3147(-06)	0.6585(-06)	0.1348(-05)	0.2729(-05)	0.5492(-05)	0.1101(-04)	0.2207(-04)
$\alpha=2$								
2^{-3}	0.4959(-02)	0.9911(-02)	0.1695(-01)	0.2110(-01)	0.2117(-01)	0.2043(-01)	0.1988(-01)	0.1952(-01)
2^{-4}	0.1206(-02)	0.2514(-02)	0.5066(-02)	0.9760(-02)	0.1665(-01)	0.2074(-01)	0.2066(-01)	0.1985(-01)
2^{-5}	0.2959(-03)	0.6174(-03)	0.1262(-02)	0.2538(-02)	0.5007(-02)	0.9596(-02)	0.1642(-01)	0.2049(-01)
2^{-6}	0.7332(-04)	0.1528(-03)	0.3126(-03)	0.6326(-03)	0.1268(-02)	0.2522(-02)	0.4961(-02)	0.9512(-02)
2^{-7}	0.1825(-04)	0.3803(-04)	0.7777(-04)	0.1573(-03)	0.3166(-03)	0.6342(-03)	0.1264(-02)	0.2510(-02)
2^{-8}	0.4554(-05)	0.9489(-05)	0.1939(-04)	0.3924(-04)	0.7896(-04)	0.1583(-03)	0.3170(-03)	0.6332(-03)
2^{-9}	0.1137(-05)	0.2369(-05)	0.4844(-05)	0.9800(-05)	0.1971(-04)	0.3954(-04)	0.7921(-04)	0.1584(-03)
2^{-10}	0.2842(-06)	0.5921(-06)	0.1210(-05)	0.2448(-05)	0.4926(-05)	0.9881(-05)	0.1979(-04)	0.3961(-04)

Example 3: $\epsilon u_{xx} = u_t + \frac{1}{x}u_x + (1+x^2)u + f(x,t),\ \ 0 < x < 1,\ t > 0$ (32)

The exact solution is given by $u(x,t) = e^{-t}\sin\pi x$ .The RMS errors at $t = 1.0$ are tabulated in Table 3 for different values of $\epsilon\ (0 < \epsilon \ll 1)$.

Table 3: The RMS errors

h	$\varepsilon=2^{-3}$	$\varepsilon=2^{-4}$	$\varepsilon=2^{-5}$	$\varepsilon=2^{-6}$	$\varepsilon=2^{-7}$	$\varepsilon=2^{-8}$	$\varepsilon=2^{-9}$	$\varepsilon=2^{-10}$
2^{-3}	0.8376(-02)	0.1002(-01)	0.1133(-01)	0.1262(-01)	0.1395(-01)	0.1507(-01)	0.1584(-01)	0.1629(-01)
2^{-4}	0.1995(-02)	0.2344(-02)	0.2575(-02)	0.2747(-02)	0.2921(-02)	0.3149(-02)	0.3425(-02)	0.3675(-02)
2^{-5}	0.4889(-03)	0.5715(-03)	0.6227(-03)	0.6539(-03)	0.6760(-03)	0.6984(-03)	0.7304(-03)	0.7809(-03)
2^{-6}	0.1211(-03)	0.1414(-03)	0.1537(-03)	0.1607(-03)	0.1648(-03)	0.1676(-03)	0.1705(-03)	0.1746(-03)
2^{-7}	0.3016(-04)	0.3520(-04)	0.3824(-04)	0.3994(-04)	0.4087(-04)	0.4139(-04)	0.4174(-04)	0.4210(-04)
2^{-8}	0.7525(-05)	0.8782(-05)	0.9539(-05)	0.9961(-05)	0.1018(-04)	0.1030(-04)	0.1037(-04)	0.1041(-04)
2^{-9}	0.1879(-05)	0.2193(-05)	0.2382(-05)	0.2487(-05)	0.2543(-05)	0.2572(-05)	0.2587(-05)	0.2595(-05)
2^{-10}	0.4696(-06)	0.5480(-06)	0.5953(-06)	0.6215(-06)	0.6355(-06)	0.6427(-06)	0.6464(-06)	0.6483(-06)

5. FINAL DISCUSSION

The traditional lower order methods of accuracy of $O(k^2+h^2)$ have their inherent difficulties to handle singularly perturbed singular parabolic initial boundary value problems, although some correction techniques may be used to yield stable tension spline methods for $0<\varepsilon<<1$. The stability analysis of a tension spline method has been discussed and it has been shown that the method is unconditionally stable. Some text problems have been solved to demonstrate the efficiency of the proposed method when $\varepsilon>0$ is either small or large as compared to the corresponding mesh sizes $h>0$ and $k>0$. The technique used in this paper may be extended to derive other numerical methods, not necessarily limited to tension spline methods.

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