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# ANALYSIS OF $\mathcal{L}_2 - \mathcal{L}_\infty$ STABILITY FOR MULTILAYER HOPFIELD NEURAL NETWORKS

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**ABSTRACT.** In this paper, we propose some new conditions on $\mathcal{L}_2 - \mathcal{L}_\infty$ stability of multilayer Hopfield neural networks. These sufficient conditions are represented based on matrix norm and linear matrix inequality (LMI). Under these conditions, multilayer Hopfield neural networks reduce the effect of external input on the state vector to a predefined level. Moreover, the proposed conditions ensure asymptotic stability for multilayer Hopfield neural networks without external input.

**AMS (MOS) Subject Classification.** 92B20, 34D20.

## 1. Introduction

In this paper, we consider the following multilayer Hopfield neural network:

$$\dot{x}(t) = Ax(t) + W\phi(\bar{W}x(t)) + J(t), \tag{1.1}$$

$$z(t) = Hx(t), \tag{1.2}$$

where $x(t) = [x_1(t) \ ... \ x_n(t)]^T \in R^n$ is the state vector, $z(t) \in R^p$ is a linear combination of the states, $A = diag\{-a_1, \ldots, -a_n\} \in R^{n\times n}$ $(a_k > 0, k = 1, \ldots, n)$ is the self-feedback matrix, $W \in R^{n\times n}$ and $\bar{W} \in R^{n\times n}$ are the connection weight matrices, $\phi(x(t)) = [\phi_1(x(t)) \ ... \ \phi_n(x(t))]^T : R^n \to R^n$ is the nonlinear function vector satisfying the global Lipschitz condition with Lipschitz constant $L_\phi > 0$, $J(t) \in R^n$ is an external input vector, and $H \in R^{p\times n}$ is a known constant matrix. Hopfield [9, 10] has introduced Hopfield neural networks, which have found applications in several disciplines where the targeted problems can reduce to optimization problems. Recently, Hopfield neural networks and their several generalizations have attracted the great attention in many scientific fields due to their potential for the tasks of associative memory, classification, parallel computation and their ability to solve difficult optimization problems [8, 13, 11, 3].

In real physical systems, there always exist the effects of noise disturbances and model uncertainties. In order to reduce these effects, recently, some researchers have presented the $\mathcal{L}_2 - \mathcal{L}_\infty$ approach in filtering problems [7, 16, 14, 6, 5, 12, 15, 17]. The $\mathcal{L}_2 - \mathcal{L}_\infty$ approach has been regarded as an important method to design filters for

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various dynamical systems. At this point, the following question arises: Can we obtain an $\mathcal{L}_2 - \mathcal{L}_\infty$ stability condition for multilayer Hopfield neural networks? This paper answers this question. To the best of our knowledge, the $\mathcal{L}_2 - \mathcal{L}_\infty$ stability analysis of multilayer Hopfield neural networks has not yet been reported in the literature.

In this paper, we propose new $\mathcal{L}_2-\mathcal{L}_\infty$ stability conditions for multilayer Hopfield neural networks. The conditions proposed in this paper are a new contribution to the stability analysis of Hopfield neural networks. Under these new conditions, multilayer Hopfield neural networks attenuate the effect of external input on the state vector to a predefined level. These conditions are represented in terms of matrix norm and linear matrix inequality (LMI). This paper is organized as follows. In Section 2, new $\mathcal{L}_2-\mathcal{L}_\infty$ stability conditions are derived. Finally, conclusions are presented in Section 3.

## 2. New Conditions

Let $\gamma > 0$ be a predefined level of disturbance attenuation. The main aim of this paper is to find new conditions, under which the multilayer Hopfield neural network (1.1)-(1.2) with $J(t) = 0$ is asymptotically stable ($\lim_{t\to\infty} x(t) = 0$) and

$$\sup_{t\geq 0}\{z^T(t)z(t)\} < \gamma^2 \int_0^\infty J^T(t)J(t)dt, \tag{2.1}$$

under zero-initial conditions for all nonzero $J(t) \in L_2[0,\infty)$, where $L_2[0,\infty)$ is the space of square integrable vector functions over $[0,\infty)$.

Now, we propose a new $\mathcal{L}_2 - \mathcal{L}_\infty$ stability condition of the multilayer Hopfield neural network (1.1)-(1.2).

**Theorem 2.1.** *For a given level $\gamma > 0$, the multilayer Hopfield neural network (1.1)-(1.2) is $\mathcal{L}_2 - \mathcal{L}_\infty$ stable if*

$$\|\bar{W}\| < \frac{\sqrt{k - \|P\|^2 - \|W\|^2\|P\|^2}}{L_\phi}, \tag{2.2}$$

$$\|W\| < \frac{\sqrt{k - \|P\|^2}}{\|P\|}, \tag{2.3}$$

$$\|P\| < \sqrt{k}, \qquad k > 0, \qquad P = P^T > 0, \tag{2.4}$$

$$\|H\| \leq \gamma\sqrt{\lambda_{\min}(P)}, \tag{2.5}$$

*where $\lambda_{\min}(\cdot)$ is the minimum eigenvalue of the matrix and $P$ satisfies the Lyapunov equation $A^TP + PA = -kI$.*

*Proof.* Consider the quadratic Lyapunov function $V(t) = x^T(t)Px(t)$. The time derivative of this function along the trajectory of (1.1) is

$$\dot{V}(t) = -kx^T(t)x(t) + 2x^T(t)PW\phi(\bar{W}x(t)) + 2x^T(t)PJ(t). \tag{2.6}$$

If we apply Young's inequality [1], we have

$$\begin{aligned}
2x^T(t)PW\phi(x(t)) &\le x^T(t)PWW^TPx(t) + \phi^T(\bar{W}x(t))\phi(\bar{W}x(t)) \\
&\le \|P\|^2\|W\|^2\|x(t)\|^2 + L_\phi^2\|\bar{W}\|^2\|x(t)\|^2, \\
2x^T(t)PJ(t) &\le x^T(t)PP^Tx(t) + J^T(t)J(t) \\
&\le \|P\|^2\|x(t)\|^2 + \|J(t)\|^2.
\end{aligned}$$

Substituting these two inequalities into (2.6), we have

$$\dot{V}(t) \le -\left(k - \|P\|^2 - \|W\|^2\|P\|^2 - L_\phi^2\|\bar{W}\|^2\right)\|x(t)\|^2 + \|J(t)\|^2. \tag{2.7}$$

If the following condition is satisfied:

$$k - \|P\|^2 - \|W\|^2\|P\|^2 - L_\phi^2\|\bar{W}\|^2 > 0, \tag{2.8}$$

we have

$$\dot{V}(t) < \|J(t)\|^2. \tag{2.9}$$

The following three inequalities

$$\begin{aligned}
&\|\bar{W}\|^2 < \frac{k - \|P\|^2 - \|W\|^2\|P\|^2}{L_\phi^2}, \\
&\|W\|^2 < \frac{k - \|P\|^2}{\|P\|^2}, \\
&\|P\|^2 < k,
\end{aligned}$$

imply the condition (2.8). Thus, we obtain (2.2), (2.3), and (2.4). Under the zero-initial condition, we have $V(t)|_{t=0} = 0$ and $V(t) \ge 0$. Define

$$\Phi(t) = V(t) - \int_0^t J^T(\sigma)J(\sigma)d\sigma. \tag{2.10}$$

Then, for any nonzero $J(t)$, we obtain

$$\begin{aligned}
\Phi(t) &= V(t) - V(t)|_{t=0} - \int_0^t J^T(\sigma)J(\sigma)d\sigma \\
&= \int_0^t \left[\dot{V}(\sigma) - J^T(\sigma)J(\sigma)\right]d\sigma.
\end{aligned}$$

From (2.9), we have $\Phi(t) < 0$. It means

$$V(t) < \int_0^t J^T(\sigma)J(\sigma)d\sigma.$$

The condition (2.5) implies

$$
\begin{aligned}
z^T(t)z(t) &= x^T(t)H^THx(t) \\
&\le \|H\|^2\|x(t)\|^2 \\
&\le \gamma^2\lambda_{\min}(P)\|x(t)\|^2 \\
&\le \gamma^2 x^T(t)Px(t) \\
&= \gamma^2 V(t) \\
&< \gamma^2\int_0^t J^T(\sigma)J(\sigma)d\sigma \\
&\le \gamma^2\int_0^\infty J^T(\sigma)J(\sigma)d\sigma. \qquad (2.11)
\end{aligned}
$$

Taking the supremum over $t > 0$ leads to (2.1). This completes the proof. □

**Corollary 2.2.** *When $J(t) = 0$, the condition (2.2)-(2.5) ensures that the multilayer Hopfield neural network (1.1)-(1.2) is asymptotically stable.*

*Proof.* When $J(t) = 0$, from (2.9), we have $\dot{V}(t) < 0, \forall x(t) \ne 0$. This relation ensures that the multilayer Hopfield neural network (1.1)-(1.2) is asymptotically stable from Lyapunov stability theory. This completes the proof. □

Next, we find a new LMI based condition for the $\mathcal{L}_2$−$\mathcal{L}_\infty$ stability of the multilayer Hopfield neural network (1.1)-(1.2). This LMI based condition can be check easily via standard numerical algorithms [2, 4].

**Theorem 2.3.** *For a given level $\gamma > 0$, the multilayer Hopfield neural network (1.1)-(1.2) is $\mathcal{L}_2 - \mathcal{L}_\infty$ stable if there exist a positive symmetric matrix $P$ and a positive scalar $\epsilon$ such that*

$$
\begin{bmatrix} A^TP + PA + \epsilon L_\phi^2 \bar{W}^T\bar{W} & PW & P \\ W^TP & -\epsilon I & 0 \\ P & 0 & -I \end{bmatrix} < 0, \qquad (2.12)
$$

$$
\begin{bmatrix} P & H^T \\ H & \gamma^2 I \end{bmatrix} > 0. \qquad (2.13)
$$

*Proof.* Consider the quadratic Lyapunov function $V(t) = x^T(t)Px(t)$. If we Young's inequality [1], we have $\epsilon[L_\phi^2 x^T(t)\bar{W}^T\bar{W}x(t) - \phi^T(\bar{W}x(t))\phi(\bar{W}x(t))] \ge 0$. If we use this inequality, the time derivative of $V(t)$ along the trajectory of (1.1) is

$$\begin{aligned}
\dot{V}(t) &= x^T(t)[A^TP + PA]x(t) + 2x^T(t)PW\phi(\bar{W}x(t)) + 2x^T(t)PJ(t) \\
&\le x^T(t)[A^TP + PA]x(t) + 2x^T(t)PW\phi(\bar{W}x(t)) + 2x^T(t)PJ(t) \\
&+ \epsilon[L_\phi^2 x^T(t)\bar{W}^T\bar{W}x(t) - \phi^T(\bar{W}x(t))\phi(\bar{W}x(t))] \\
&= \begin{bmatrix} x(t) \\ \phi(\bar{W}x(t)) \\ J(t) \end{bmatrix}^T \begin{bmatrix} A^TP + PA + \epsilon L_\phi^2 \bar{W}^T\bar{W} & PW & P \\ W^TP & -\epsilon I & 0 \\ P & 0 & -I \end{bmatrix} \begin{bmatrix} x(t) \\ \phi(\bar{W}x(t)) \\ J(t) \end{bmatrix} \\
&+ J^T(t)J(t). \qquad (2.14)
\end{aligned}$$

If the LMI (2.12) is satisfied, we have

$$\dot{V}(t) < J^T(t)J(t). \qquad (2.15)$$

Under the zero-initial condition, one has $V(t)|_{t=0} = 0$ and $V(t) \ge 0$. Define

$$\Phi(t) = V(t) - \int_0^t J^T(\sigma)J(\sigma)d\sigma. \qquad (2.16)$$

Then, for any nonzero $J(t)$, we obtain

$$\begin{aligned}
\Phi(t) &= V(t) - V(t)|_{t=0} - \int_0^t J^T(\sigma)J(\sigma)d\sigma \\
&= \int_0^t \left[\dot{V}(\sigma) - J^T(\sigma)J(\sigma)\right] d\sigma.
\end{aligned}$$

From (2.15), we have $\Phi(t) < 0$. It means

$$V(t) < \int_0^t J^T(\sigma)J(\sigma)d\sigma.$$

The LMI (2.13) implies

$$\begin{aligned}
z^T(t)z(t) &= x^T(t)H^THx(t) \\
&< \gamma^2 x^T(t)Px(t) \\
&= \gamma^2 V(t) \\
&< \gamma^2 \int_0^t J^T(\sigma)J(\sigma)d\sigma \\
&\le \gamma^2 \int_0^\infty J^T(\sigma)J(\sigma)d\sigma. \qquad (2.17)
\end{aligned}$$

Taking the supremum over $t > 0$ leads to (2.1). This completes the proof. $\square$

**Corollary 2.4.** *When $J(t) = 0$, the LMI conditions (2.12)-(2.13) ensure that the multilayer Hopfield neural network (1.1)-(1.2) is asymptotically stable.*

*Proof.* When $J(t) = 0$, from (2.15), we have $\dot{V}(t) < 0, \forall x(t) \neq 0$. This inequality ensures that the multilayer Hopfield neural network (1.1)-(1.2) is asymptotically stable from Lyapunov stability theory. This completes the proof. $\square$

**Example 2.5.** Consider the multilayer Hopfield neural network (1.1)-(1.2), where

$$A = \begin{bmatrix} -2.8 & 0 \\ 0 & -4.1 \end{bmatrix}, \ \phi(x(t)) = \begin{bmatrix} \tanh(x_1(t)) \\ \tanh(x_2(t)) \end{bmatrix},$$
$$W = \begin{bmatrix} -0.3 & 0.4 \\ 0 & -0.1 \end{bmatrix}, \ \bar{W} = \begin{bmatrix} 0.2 & -0.1 \\ 0.4 & 0.3 \end{bmatrix}, \ H = \begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}.$$

By applying Theorem 2.3 via the Matlab LMI Control Toolbox [4], we have the following feasible solution:

$$P = \begin{bmatrix} 3.6620 & -0.3408 \\ -0.3408 & 6.0665 \end{bmatrix}, \ \epsilon = 20.1188,$$

with the $\mathcal{L}_2 - \mathcal{L}_\infty$ performance index $\gamma = 0.6$.

## 3. **Conclusion**

This paper has proposed some new conditions on $\mathcal{L}_2 - \mathcal{L}_\infty$ stability for multilayer Hopfield neural networks. These stability conditions were represented in terms of matrix norm and LMI. Under the proposed conditions, multilayer Hopfield neural networks reduced the effect of external input to a predefined level. In addition, these conditions ensured asymptotic stability for multilayer Hopfield neural networks without external input.

## REFERENCES


[1] V.I. Arnold. *Mathematical Methods of Classical Mechanics*. Springer, 1989.

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishinan. *Linear matrix inequalities in systems and control theory*. SIAM, Philadelphia, PA, 1994.

[3] V. Covachev, H. Akca, and M. Sarr. Discrete-time counterparts of impulsive Cohen-Grossberg neural networks of neutral type. *Neural, Parallel, and Scientific Computations*, 19:345-360, 2011.

[4] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali. *LMI Control Toolbox*. The Mathworks Inc., 1995.

[5] H. Gao and C. Wang. Delay-dependent robust $\mathcal{H}_\infty$ and $\mathcal{L}_2 - \mathcal{L}_\infty$ fitering for a class of uncertain nonlinear time-delay systems. *IEEE Trans. Automat. Contr.*, 48:1661–1666, 2003.

[6] H. Gao and C. Wang. Robust $\mathcal{L}_2 - \mathcal{L}_\infty$ fitering for uncertain systems with multiple time-varying state delays. *IEEE Trans. Trans. Circ. Syst. I*, 50:594–599, 2003.

[7] K.M. Grigoriadis and J.T. Watson. Reduced-order $\mathcal{H}_\infty$ and $\mathcal{L}_2 - \mathcal{L}_\infty$ filtering via linear matrix inequalities. *IEEE Trans. Aerosp. Electron. Syst.*, 33:1326–1338, 1997.

[8] M.M. Gupta, L. Jin, and N. Homma. *Static and Dynamic Neural Networks*. Wiley-Interscience, 2003.

[9] J.J. Hopfield. Neural networks and physical systems with emergement collective computational abilities. *Proc. Nat. Acad. Sci.*, 79:2554–2558, 1982.

[10] J.J. Hopfield. Neurons with grade response have collective computational properties like those of two-state neurons. *Proc. Nat. Acad. Sci.*, 81:3088–3092, 1984.

[11] X. Liu, K.L. Teo, and B. Xu. Exponential stability of impulsive high-oreder hopfield type neural networks with time varying delays. *IEEE Trans. Neural Networks*, 16:1329–1339, 2005.

[12] M.S. Mahmoud. Resilient $\mathcal{L}_2 - \mathcal{L}_\infty$ filtering of polytopic systems with state delays. *IET Control Theory & Applications*, 1(1):141–154, 2007.

[13] R. McEliece, E. Posner, E. Rodemich, and S. Venkatesh. The capacity of the hopfield associative memory. *IEEE Trans. Inform. Theory*, 33:461–482, 1987.

[14] R.M. Palhares and P.L.D. Peres. Robust filtering with guaranteed energy-to-peak performance - an LMI approach. *Automatica*, 36:851–858, 2000.

[15] J. Qiu, G. Feng, and J. Yang. New results on robust energy-to-peak filtering for discrete-time switched polytopic linear systems with time-varying delay. *IET Control Theory & Applications*, 2(9):795–806, 2008.

[16] J.T. Watson and K.M. Grigoriadis. Optimal unbiased filtering via linear matrix inequalities. *Systems and Control Letters*, 35:111–118, 1998.

[17] Y. Zhou and J. Li. Energy-to-peak filtering for singular systems: the discrete-time case. *IET Control Theory & Applications*, 2(9):773–781, 2008.