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GENERALIZED MONOTONE METHOD FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH APPLICATIONS TO POPULATION MODELS

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ABSTRACT. Monotone method combined with method of upper and lower solutions is a productive technique to prove existence of extremal solutions in dynamic systems. However, this method is applicable when the forcing function is increasing or can be made increasing by adding a linear term. Monotone method also works when the forcing function is decreasing in dynamic systems. In this work, we prove existence of coupled minimal and maximal solutions by using generalized monotone method for Caputo fractional differential equation with initial condition. Also, we consider the case when the forcing function is the sum of an increasing and decreasing function. In general, this is true for many mathematical models, including population models and chemical combustion models. Finally, we obtain numerical results to demonstrate an application of our theoretical results of the Logistic equation.

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1. INTRODUCTION

Nonlinear problems (nonlinear dynamic systems) occur naturally as mathematical models in many branches of science, engineering, finance, economics, etc. So far, in literature, most models are differential equations with integer derivative. For example, the Logistic model of population dynamics is given by,

$$({}^cD^q u)(t) = au(t) - bu^2(t), \quad u(0) = u_0, \tag{1.1}$$

where a, b are positive constants.

In reference [8] from 1967, the interest (in fractional derivatives and integrals) of researchers began when they realized half-order derivatives and integrals led to a formulation (of the particular electrochemical problems) that are more economical

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and useful than the classical diffusion equation. Fractional differential equations are implementable in many applications. See references, [4, 5], for details.

In solving nonlinear problems, monotone method combined with method of upper and lower solutions is a popular choice, because existence of solution by monotone method is both theoretical and computational. Monotone method for various nonlinear problems has been developed in reference [2]. Monotone method (monotone iterative technique) combined with method of lower and upper solutions yields monotone sequences, which converges to minimal and maximal solutions of nonlinear differential equation.

In many nonlinear problems (nonlinear dynamic systems), the nonlinear term is the sum of an increasing and decreasing functions. Monotone method extended to such systems is called generalized monotone method. Generalized monotone method for first order nonlinear initial value problems has been developed in [7]. In this work, we extend generalized monotone method for Caputo fractional differential equation of order q (where $0 < q < 1$) with initial condition. Also, we prove existence of coupled minimal and maximal solutions of Caputo fractional differential equation with initial condition. Further, under uniqueness assumption, we prove existence of unique solution of Caputo fractional differential equation with initial condition. Finally, we provide a numerical example as an application of our theoretical results.

2. PRELIMINARIES

In this section, we recall known results, which are needed for our main results. Initially, we recall some definitions.

Definition 2.1. Caputo fractional derivative of order q is given by equation

$$({}^cD^q u)(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s)ds,$$

where $0 < q < 1$.

Also, consider nonlinear Caputo fractional differential equation with initial condition,

$$({}^cD^q u)(t) = f(t, u(t)), \quad u(0) = u_0. \tag{2.1}$$

The integral representation of (2.1) is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s))ds, \tag{2.2}$$

where $\Gamma(q)$ is the Gamma function.

Definition 2.2. Mittag Leffler function is given by equation

$$E_{\alpha,\beta}(\lambda(t-t_0)^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^{\alpha})^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha, \beta > 0$. Also, for $t_0 = 0$, $\alpha = q$ and $\beta = 1$, we get equation

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation,

$$({}^cD^q u)(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \tag{2.3}$$

on J, where $J = [0, T]$, λ is a constant and $f(t) \in C[J, R]$. The solution of (2.1) exists and is unique. The explicit solution of (2.1) is given by the equation

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda t^q) f(s) ds.$$

See [3] for details. In particular, if $\lambda = 0$, the solution $u(t)$ is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \tag{2.4}$$

where $\Gamma(q)$ is the Gamma function.

Next, we recall a lemma from [3] without proof. However, we observe this lemma can be proved without the assumption that $m(t)$ is Hölder continuous of order $\lambda > q$.

Lemma 2.3. *Let $J = [0, T]$. $m(t) \in C^1[J, R]$. If $m(t_1) = 0$ (for $t_1 \in J$) and $m(t) \leq 0$ (for $t \in J$), then $({}^cD^q m)(t_1) \geq 0$.*

Definition 2.4. If $v \in C^1[J, R]$ such that $({}^cD^q v)(t) \leq f(t, v(t))$ and $v(0) \leq u_0$ on J, then $v(t)$ is called lower solution of (2.1). If the inequalities are reversed, then $v(t)$ is called upper solution of (2.1).

Furthermore, we recall the following theorem without proof.

Theorem 2.5. *Let $v, w \in C^1[J, R]$ be lower and upper solutions of (2.1), respectively. Furthermore, let $f(t, u)$ satisfies the following one-side Lipschitz condition,*

$$f(t, u_1) - f(t, u_2) \leq L(u_1 - u_2), \tag{2.5}$$

where $u_1 \geq u_2$ and $L > 0$. Then $v(t) \leq w(t)$ on J, provided that $v(0) \leq w(0)$.

See [3] for details of the proof.

Next, we recall a corollary of Theorem 2.5, which we will use often in our main result.

Corollary 2.6. *Let $p \in C^1[J,R]$. $({}^cD^q p)(t) \le Lp(t)$, where $L \ge 0$ and $p(0) \le 0$. Then $p(t) \le 0$ on J.*

Furthermore, we will develop generalized monotone method for Caputo fractional differential equation,

$$({}^cD^q u)(t) = f(t,u(t)) + g(t,u(t)), \quad u(0) = u_0, \tag{2.6}$$

where $0 < q < 1$, $f(t,u)$, $g(t,u) \in C[J \times R, R]$, $f(t,u)$ is increasing in u on J, and $g(t,u)$ is decreasing in u on J.

Now, we state the following definition.

Definition 2.7. Let $v_0, w_0 \in C^1[J,R]$. v_0, w_0 are said to be

(a) Natural lower and upper solutions of (2.6) if

$$\begin{aligned} ({}^cD^q v)(t) &\le f(t,v(t)) + g(t,v(t)), \quad v(0) \le u_0, \ \textit{and} \\ ({}^cD^q w)(t) &\ge f(t,w(t)) + g(t,w(t)), \quad w(0) \ge u_0; \end{aligned}$$

(b) Coupled lower and upper solutions of type I of (2.6) if

$$\begin{aligned} ({}^cD^q v)(t) &\le f(t,v(t)) + g(t,w(t)), \quad v(0) \le u_0, \ \textit{and} \\ ({}^cD^q w)(t) &\ge f(t,w(t)) + g(t,v(t)), \quad w(0) \ge u_0; \end{aligned}$$

(c) Coupled lower and upper solutions of type II of (2.6) if

$$\begin{aligned} ({}^cD^q v)(t) &\le f(t,w(t)) + g(t,v(t)), \quad v(0) \le u_0, \ \textit{and} \\ ({}^cD^q w)(t) &\ge f(t,v(t)) + g(t,w(t)), \quad w(0) \ge u_0; \end{aligned}$$

(d) Coupled lower and upper solutions of type III of (2.6) if

$$\begin{aligned} ({}^cD^q v)(t) &\le f(t,w(t)) + g(t,w(t)), \quad v(0) \le u_0, \ \textit{and} \\ ({}^cD^q w)(t) &\ge f(t,v(t)) + g(t,v(t)), \quad w(0) \ge u_0. \end{aligned}$$

Furthermore, corresponding to each type of lower and upper solutions, we can develop four types of sequences. However, it was discovered in [7] that only the following two types of sequences, type (ii) and (iii) sequences, provide meaningful results:

(ii)

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t,v_n(t)) + g(t,w_n(t)), \quad v_{n+1}(0) = u_0, \quad \text{and} \\ ({}^cD^q w_{n+1})(t) &= f(t,w_n(t)) + g(t,v_n(t)), \quad w_{n+1}(0) = u_0; \end{aligned}$$

(iii)

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t,w_n(t)) + g(t,v_n(t)), \quad v_{n+1}(0) = u_0, \quad \text{and} \\ ({}^cD^q w_{n+1})(t) &= f(t,v_n(t)) + g(t,w_n(t)), \quad w_{n+1}(0) = u_0. \end{aligned}$$

3. MAIN RESULTS

In this section, we develop generalized monotone method for Caputo fractional differential equation,

$$({}^cD^q u)(t) = f(t, u(t)) + g(t, u(t)), \quad u(0) = u_0, \tag{3.1}$$

where $0 < q < 1$ and $f(t,u)$, $g(t,u) \in C[J \times R, R]$. $f(t,u)$ is increasing in u on J, and $g(t,u)$ is decreasing in u on J. Generalized monotone method yields natural or intertwined monotone sequences. Existence of natural or intertwined monotone sequences depends on how the sequences are developed. Further, we prove these sequences converge uniformly and monotonically to coupled minimal and maximal solutions. Finally, we obtain a sufficient condition for the solution of (3.1) to be unique.

For our first result, we use coupled lower and upper solutions of type I and sequences that are developed using type (ii) kind of sequences.

Theorem 3.1. *Assume that*

(i) $v_0, w_0 \in C^1[J, R]$. v_0, w_0 *are coupled lower and upper solutions of type I, with* $v_0(t) \le w_0(t)$ *on* J.

(ii) $f(t,u)$, $g(t,u) \in C[J \times R, R]$, *where* $f(t,u)$ *is increasing in* u *on* J, *and* $g(t,u)$ *is decreasing in* u *on* J.

Then there exist monotone sequences, $v_n(t)$ *and* $w_n(t)$, *such that* $v_n(t) \to \rho(t)$ *and* $w_n(t) \to r(t)$ *uniformly and monotonically, where* $\rho(t)$ *and* $r(t)$ *are coupled minimal and maximal solutions of equation* (3.1) *on* J. *That is, for any solution,* $u(t)$, *of* (3.1) *with* $v_0 \le u \le w_0$ *on* J, *we get natural sequences,* $\{v_n\}$ *and* $\{w_n\}$, *satisfying the following,*

$$v_0(t) \le v_1(t) \le v_2(t) \le \cdots \le v_n(t) \le u(t) \le w_n(t) \le \cdots \le w_2(t) \le w_1(t) \le w_0(t),$$

for each $n \ge 1$ *on* J. *This requires using type (ii) iterative schemes,*

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t, v_n(t)) + g(t, w_n(t)), \quad v_{n+1}(0) = u_0, \text{ and} \\ ({}^cD^q w_{n+1})(t) &= f(t, w_n(t)) + g(t, v_n(t)), \quad w_{n+1}(0) = u_0. \end{aligned} \tag{3.2}$$

Further, using (3.2), *we prove* $\rho(t)$ *and* $r(t)$ *satisfy the coupled system,*

$$\begin{aligned} ({}^cD^q \rho)(t) &= f(t, \rho(t)) + g(t, r(t)), \quad \rho(0) = u_0, \text{ and} \\ ({}^cD^q r)(t) &= f(t, r(t)) + g(t, \rho(t)), \quad r(0) = u_0. \end{aligned} \tag{3.3}$$

Also, $\rho(t) \le u(t) \le r(t)$ *on* J.

Proof. We can see that the solutions of the linear initial value problems (3.2), exist and are unique for $k \in \mathbb{N}$. We will prove that $v_k(t), w_k(t) \in [v_0, w_0] = \Omega$, where

$\Omega = [u \in [J, R] : v_0(t) \le u \le w_0(t), t \in J]$, with $v_k \le w_k$ for each $k \ge 1$. Our aim is to show

$$v_0(t) \le v_1(t) \le \cdots \le v_n(t) \le w_n(t) \le \cdots \le w_1(t) \le w_0(t). \tag{3.4}$$

Initially, we will show $v_0(t) \le v_1(t)$.

Let $p(t) = v_0(t) - v_1(t)$. Then, we have $p(0) \le 0$. Also, let

$$\begin{aligned}({}^cD^q p)(t) &= ({}^cD^q v_0)(t) - ({}^cD^q v_1)(t)\\ &\le f(t, v_0(t)) + g(t, w_0(t)) - f(t, v_0(t)) - g(t, w_0(t))\\ &= 0\end{aligned}$$

From Lemma 2.3, we get that $p(t) \le 0$ implies that $v_0(t) \le v_1(t)$ on J. Next, likewise, we can show that $w_1(t) \le w_0(t)$.

Last, we will show $v_1(t) \le w_1(t)$. Thus, let $p(t) = v_1(t) - w_1(t)$. Then, we have that $p(0) = 0$. Also, let

$$\begin{aligned}({}^cD^q p)(t) &= ({}^cD^q v_1)(t) - ({}^cD^q w_1)(t)\\ &= f(t, v_0(t)) + g(t, w_0(t)) - f(t, w_0(t)) - g(t, v_0(t))\\ &\le 0.\end{aligned}$$

From Lemma 2.3, we get $p(t) \le 0$ implies $v_1(t) \le w_1(t)$ on J. Thus, (3.4) holds true for $n = 1$.

Further, assume (3.4) is true for $n = k$. Then, we need to show (3.4) holds for $n = k+1$. Thus, first, we will show $v_k(t) \le v_{k+1}(t)$. Let $p(t) = v_k(t) - v_{k+1}(t)$. Then, we have $p(0) = 0$. Also, let

$$\begin{aligned}({}^cD^q p)(t) &= ({}^cD^q v_k)(t) - ({}^cD^q v_{k+1})(t)\\ &= f(t, v_{k-1}(t)) + g(t, w_{k-1}(t)) - f(t, v_k(t)) - g(t, w_k(t))\\ &\le 0.\end{aligned}$$

From Lemma 2.3, we get that $p(t) \le 0$ implies that $v_k(t) \le v_{k+1}(t)$ on J. Next, likewise, we can show $w_{k+1}(t) \le w_k(t)$ on J. Last, we will show $v_{k+1}(t) \le w_{k+1}(t)$ on J. Thus, let $p(t) = v_{k+1}(t) - w_{k+1}(t)$. Then, we have that $p(0) = 0$. Also, let

$$\begin{aligned}({}^cD^q p)(t) &= ({}^cD^q v_{k+1})(t) - ({}^cD^q w_{k+1})(t)\\ &= f(t, v_k(t)) + g(t, w_k(t)) - f(t, w_k(t)) - g(t, v_k(t))]\\ &\le 0.\end{aligned}$$

From Lemma 2.3, we get $p(t) \le 0$ implies $v_{k+1}(t) \le w_{k+1}(t)$ on J. This proves (3.4) holds for $n = k + 1$. Thus, (3.4) is valid for all $n \ge 1$.

Further, suppose u is any solution of (3.1) such that $v_0(t) \leq u(t) \leq w_0(t)$ on J. We will show that

$$v_k(t) \leq u(t) \leq w_k(t), \tag{3.5}$$

for all k. First, we prove that (3.5) is true for $k = 1$. For that purpose, let $p(t) = v_1(t) - u(t)$. Then, we have $p(0) = 0$. Also, let

$$\begin{aligned} ({}^cD^q p)(t) &= ({}^cD^q v_1)(t) - ({}^cD^q u)(t) \\ &= f(t, v_0(t)) + g(t, w_0(t)) - f(t, u(t)) - g(t, u(t)) \\ &\leq 0. \end{aligned}$$

From Lemma 2.3, we get $p(t) \leq 0$ implies $v_1(t) \leq u(t)$ on J. Next, likewise, we can show $u(t) \leq w_1(t)$. This proves (3.5) holds for $k = 1$. By induction argument, we can prove

$$v_0(t) \leq v_1(t) \leq \cdots \leq v_n(t) \leq u(t) \leq w_n(t) \leq \cdots \leq w_1(t) \leq w_0(t). \tag{3.6}$$

Furthermore, the sequences, $\{v_n(t)\}, \{w_n(t)\}$, can be shown to be equicontinuous and uniformly bounded. Thus, by Ascoli-Arzela′s theorem, the subsequences, $\{v_{n_k}(t)\}, \{w_{n_k}(t)\}$, converge to $\rho(t)$ and $r(t)$, respectively, on J. Since the sequences, $\{v_k(t)\}, \{w_k(t)\}$, are monotone, the entire sequences converge uniformly and monotonically to $\rho(t)$ and $r(t)$, respectively, on J. Thus, $\rho(t)$ and $r(t)$ satisfy the initial value problems (3.3).

Finally, we claim $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of (3.1). From (3.6), we can see $v_0(t) \leq \rho(t) \leq u(t) \leq r(t) \leq w_0(t)$ on J is true. This completes this proof. □

For our next result, we use coupled lower and upper solutions of type I of (3.1) and type (iii) sequences. In this case, we obtain intertwined sequences. Since the proof of Theorem 3.2 follows on the same lines as Theorem 3.1, we only state Theorem 3.2 without showing its proof.

Theorem 3.2. *Assume hypotheses, (i) and (ii), of Theorem* 3.1 *hold. Then for any solution, $u(t)$, of equation* (3.1) *with $v_0 \leq u \leq w_0$ on J, we get alternating sequences, $\{v_{2n}, w_{2n+1}\}$ and $\{v_{2n+1}, w_{2n}\}$, satisfying intertwined sequences,*

$$\begin{aligned} v_0(t) \leq w_1(t) \leq \cdots \leq v_{2n}(t) \leq w_{2n+1}(t) \leq u(t) \\ \leq v_{2n+1}(t) \leq w_{2n}(t) \leq \cdots \leq v_1(t) \leq w_0(t), \end{aligned} \tag{3.7}$$

for each $n \geq 1$ on J. This requires using type (iii) iterative schemes,

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t, w_n(t)) + g(t, v_n(t)), \quad v_{n+1}(0) = u_0, \ \textit{and} \\ ({}^cD^q w_{n+1})(t) &= f(t, v_n(t)) + g(t, w_n(t)), \quad w_{n+1}(0) = u_0. \end{aligned}$$

Further, monotone sequences, $\{v_{2n}, w_{2n+1}\}$ and $\{w_{2n}, v_{2n+1}\}$, converge to $\rho(t)$ and $r(t)$, respectively, on J. $\rho(t)$ and $r(t)$ are coupled minimal and maximal solutions of (3.1), respectively. That is, $\rho(t)$ and $r(t)$ satisfies the coupled system,

$$\begin{aligned}({}^cD^q\rho)(t) &= f(t,\rho(t)) + g(t,r(t)), \quad \rho(0) = u_0, \text{ and}\\ ({}^cD^q r)(t) &= f(t,r(t)) + g(t,\rho(t)), \quad r(0) = u_0.\end{aligned}$$

Also, $\rho(t) \le u(t) \le r(t)$ on J.

Our next result provides existence of unique solution of (3.1) while requiring suitable conditions.

Theorem 3.3. *If in addition to hypotheses of Theorem 3.1 or Theorem 3.2, let $f(t,u)$ satisfies left-hand-side Lipschitz condition,*

$$f(t,u_2) - f(t,u_1) \le L_1(u_2 - u_1), \quad u_2 \ge u_1,$$

and $g(t,u)$ satisfies right-hand-side Lipschitz condition,

$$g(t,u_2) - g(t,u_1) \ge -L_2(u_2 - u_1), \quad u_2 \ge u_1,$$

where $v_0 \le u_1 \le u_2 \le w_0$ implies $\rho = r = u$, the unique solution of (3.1).

Proof. From Theorem 3.1 and 3.2, we already have $\rho \le r$. It is enough to prove $r \le \rho$. For that purpose, we set $p(t) = r - \rho$. Then $p(0) = 0$.

$$\begin{aligned}({}^cD^q m)(t) = ({}^cD^q r)(t) - ({}^cD^q\rho)(t) &= f(t,r) - f(t,\rho) + g(t,r) - g(t,\rho)\\ &\le L_1(r-\rho) + L_2(r-\rho)\\ &= (L_1 + L_2)p(t).\end{aligned}$$

Using Corollary 2.6, we get $r \le \rho$. Thus, we obtain $\rho = r = u$, the unique solution of (3.1). □

Our next result, Lemma 3.4, provides a methodology to construct coupled lower and solutions of type II.

Lemma 3.4. *Suppose $f(t,u)$, $g(t,u)$ are monotonically increasing and decreasing in u, respectively, for $t \in J$, then there exists coupled lower and upper solutions of type II,*

$$\begin{aligned}({}^cD^q v)(t) &\le f(t,w(t)) + g(t,v(t)), \quad v(0) \le u_0, \text{ and}\\ ({}^cD^q w)(t) &\ge f(t,v(t)) + g(t,w(t)), \quad w(0) \ge u_0,\end{aligned}$$

for (3.1) on J.

Proof. Consider a linear Caputo fractional differential equation,

$$({}^cD^q z)(t) = f(t,0) + g(t,0), \quad z(0) = u_0,$$

whose solution $z(t)$ can be computed easily.

Now, we choose an $R > 0$ such that

$$\begin{aligned} v &= z - R \leq 0, \quad \text{and} \\ w &= z + R \geq 0. \end{aligned}$$

We claim v, w are coupled lower and upper solutions of (3.1) of type II. To prove this, we consider

$$\begin{aligned} {}^cD^q v &= {}^cD^q z -^c D^q R =^c D^q z \Rightarrow^c D^q v = f(t,0) + g(t,0), v(0) = u_0, \quad \text{and} \\ {}^cD^q w &= {}^cD^q z +^c D^q R =^c D^q z \Rightarrow^c D^q w = f(t,0) + g(t,0), w(0) = u_0. \end{aligned}$$

Since f is increasing in u, g is decreasing in u, $v \leq 0$ and $0 \leq w$, we get

$$\begin{aligned} {}^cD^q v &= f(t,0) + g(t,0) \leq f(t,w) + g(t,v) \Rightarrow^c D^q v = f(t,w) + g(t,v), \quad \text{and} \\ {}^cD^q w &= f(t,0) + g(t,0) \geq f(t,v) + g(t,w) \Rightarrow^c D^q w = f(t,v) + g(t,w). \end{aligned}$$

This completes this proof. □

We merely state our next two results that relate to coupled lower and upper solutions of type II for (3.1). The proofs of Theorems 3.5 and 3.6 follow on the same lines as the proof of Theorem 3.1 while requiring extra assumptions.

Theorem 3.5. *Assume the hypothesis of Lemma* 3.4 *holds, and let v_0 and w_0 be coupled lower and upper solutions, respectively, of type II with $v_0 \leq w_0$ on J. Further, starting from v_0 and w_0, if type (ii) iterative schemes are constructed by*

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t, v_n(t)) + g(t, w_n(t)), \quad v_{n+1}(0) = u_0, \ \text{and} \\ ({}^cD^q w_{n+1})(t) &= f(t, w_n(t)) + g(t, v_n(t)), \quad w_{n+1}(0) = u_0, \end{aligned}$$

then the conclusions of Theorem 3.1 *hold, provided $v_0 \leq v_1$ and $w_1 \leq w_0$ on J.*

Theorem 3.6. *Assume hypothesis of Lemma* 3.4 *holds, and let v_0 and w_0 be coupled lower and upper solutions, respectively, of type II with $v_0 \leq w_0$ on J. Further, starting from v_0 and w_0, if type (iii) iterative schemes are constructed by*

$$\begin{aligned} ({}^cD^q v_{n+1})(t) &= f(t, w_n(t)) + g(t, v_n(t)), \quad v_{n+1}(0) = u_0, \ \text{and} \\ ({}^cD^q w_{n+1})(t) &= f(t, v_n(t)) + g(t, w_n(t)), \quad w_{n+1}(0) = u_0, \end{aligned}$$

then the conclusions of Theorem 3.2 *hold, provided $v_0 \leq w_1$ and $v_1 \leq w_0$ on J.*

4. NUMERICAL RESULTS

In this section we provide several examples that illustrate the theoretical results obtained in the previous section. We used Maple and Mathematica software for the computations and graph drawings.

We consider an example and develop a numerical result, which demonstrates the application to one of our main theoretical results, namely Theorem 3.5.

Example 4.1. Consider a special case of the Logistic model of the form

$$({}^cD^{\frac{1}{2}}u)(t) = 0.99u(t) - u^2(t), \quad u(0) = \tfrac{1}{2}. \tag{4.1}$$

We choose $v_0(t), w_0(t)$ as coupled lower and upper solutions of type II. Let $v_0(t) = 0.01$ and $w_0(t) = 1$. We claim that $v_0(t)$ and $w_0(t)$ are coupled lower and upper solutions of type II of (4.1). We justify $v_0 = 0.01$ and $w_0 = 1$ as below.

$$\begin{aligned} {}^cD^{\frac{1}{2}}(0.01) &\leq 0.99(1) - 0.01^2 \Rightarrow 0 \leq 0.9899,\ \ v_0(0) = 0.01 \leq \frac{1}{2} = u_0, \ \text{ and} \\ {}^cD^{\frac{1}{2}}(1) &\geq 0.99(0.01) - 1^2 \Rightarrow 0 \geq -0.9901,\ \ w_0(0) = 1 \geq \frac{1}{2} = u_0. \end{aligned}$$

Next, we compute $v_1(t)$ and $w_1(t)$ using Theorem 3.5 iterative schemes

$$\begin{aligned} {}^cD^{\frac{1}{2}}v_1 &= 0.99v_0 - {w_0}^2 = 0.99(0.01) - 1^2 = -0.9901, \quad v_1(0) = \tfrac{1}{2}, \ \text{ and} \\ {}^cD^{\frac{1}{2}}w_1 &= 0.99w_0 - {v_0}^2 = 0.99(1) - 0.01^2 = 0.9899, \quad w_1(0) = \tfrac{1}{2}. \end{aligned}$$

From Figure 1 below, we can computationally check $v_0(t) \leq v_1(t)$ on $[0, 0.19]$ and $w_1(t) \leq w_0(t)$ on [0, 0.19], which is needed to satisfy hypothesis of Theorem 3.5. Also, we compute all $v_n(t)$ and $w_n(t)$ iterations using Theorem 3.5 iterative schemes

$$\begin{aligned} ({}^cD^{\frac{1}{2}}v_{n+1})(t) &= 0.99v_n(t) - w_n^2(t), \quad v_{n+1}(0) = \tfrac{1}{2}, \ \text{ and} \\ ({}^cD^{\frac{1}{2}}w_{n+1})(t) &= 0.99w_n(t) - v_n^2(t), \quad w_{n+1}(0) = \tfrac{1}{2}. \end{aligned}$$

From Figure 1, for $n = 0, 1, 2, 3, 4$, we can see $v_0 \leq v_1 \leq v_2 \leq v_3 \leq v_4 \leq w_4 \leq w_3 \leq w_2 \leq w_1 \leq w_0$.

For $n = 0, 1, 2, 3, 4$, Figure 1 has five iterations.

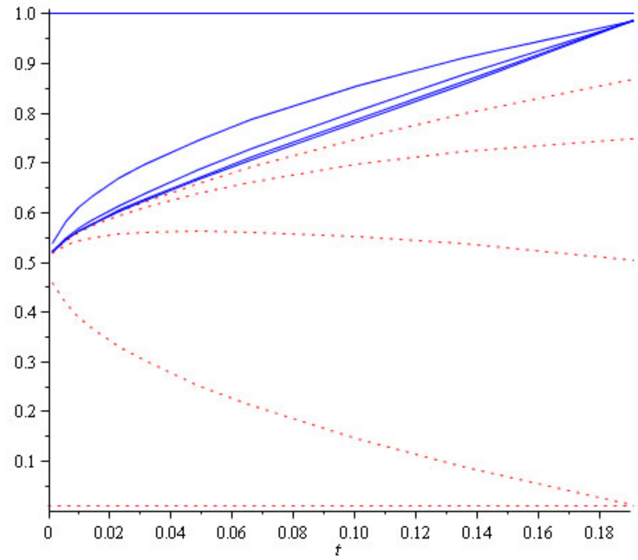

FIGURE 1. Dashed: $\{v_n\}$. Solid: $\{w_n\}$.

The difference $\max|v_n - w_n|$ can be made as small as possible by choosing an appropriate n. Computationally, $\max|v_{25} - w_{25}| = |(\frac{1}{2} + \frac{1}{\sqrt{\pi}})(14.25644704(0.19)^{\frac{23}{2}}) - (\frac{1}{2} + \frac{1}{\sqrt{\pi}})(14.25644704(0.19)^{\frac{23}{2}})| = 0$. Since $\max|v_{25} - w_{25}| = 0$, v_n and w_n converge to a unique solution of (4.1). That is, $v_{25} = w_{25} = u$. The unique solution of (4.1) can be seen in Figure 2.

For $n = 0, 1, 2, \ldots, 25$, Figure 2 has 26 iterations.

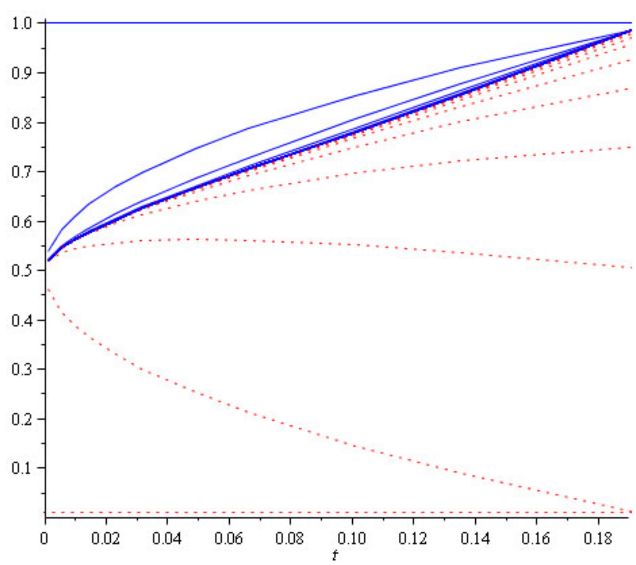

FIGURE 2. Dashed: $\{v_n\}$. Solid: $\{w_n\}$.

TABLE 1. Table of the Twenty Six Iterations of $v_n(t), w_n(t)$ for Figures 1 and 2.

n	t	$v_n(t)$	$w_n(t)$	n	t	$v_n(t)$	$w_n(t)$
0	0.19	0.01	1	13	0.19	0.98375	0.98395
1	0.19	0.01304	0.98686	14	0.19	0.98385	0.98395
2	0.19	0.50640	0.98538	15	0.19	0.98389	0.98395
3	0.19	0.74906	0.98465	16	0.19	0.98392	0.98395
4	0.19	0.86842	0.98429	17	0.19	0.98393	0.98395
5	0.19	0.92712	0.98411	18	0.19	0.98394	0.98395
6	0.19	0.95600	0.98403	19	0.19	0.98394	0.98395
7	0.19	0.97020	0.98398	20	0.19	0.98394	0.98395
8	0.19	0.97718	0.98396	21	0.19	0.98394	0.98395
9	0.19	0.98062	0.98395	22	0.19	0.98395	0.98395
10	0.19	0.98231	0.98395	23	0.19	0.98395	0.98395
11	0.19	0.98314	0.98395	24	0.19	0.98395	0.98395
12	0.19	0.98354	0.98395	25	0.19	0.98395	0.98395

Example 4.2. Now consider the following initial value problem,

$$\begin{aligned} {}^cD^{\frac{1}{2}}u &= \frac{u}{6} - \frac{u^2}{3}, \\ u(0) &= \tfrac{1}{2}. \end{aligned} \tag{4.2}$$

Clearly, $u \equiv \frac{1}{2}$ is a solution to (4.2). Moreover, $v_0 \equiv \frac{1}{10}$ and $w_0 \equiv 1$ are coupled lower and upper solutions of type II to (4.2).

In fact, $0 = {}^cD^{\frac{1}{2}}v_0 \le \frac{w_0}{6} - \frac{v_0^2}{3} = \frac{49}{300}$, and $0 = {}^cD^{\frac{1}{2}}w_0 \ge \frac{v_0}{6} - \frac{w_0^2}{3} = -\frac{19}{60}$.

We compute intertwined sequences according to Theorem 3.6 and show in Figure 3 five steps of $\{v_{2n}, w_{2n+1}\}$ and five steps of $\{w_{2n}, v_{2n+1}\}$. We also show a table of values at the endpoint $t = 1$.

TABLE 2. Table of five iterations of $v_n(t)$ and $w_n(t)$ for equation (4.2)

n	t	$v_n(t)$	$w_n(t)$
0	1	0.10000	1.00000
1	1	0.68430	0.14268
2	1	0.38426	0.60076
3	1	0.53922	0.45727
4	1	0.48614	0.51363

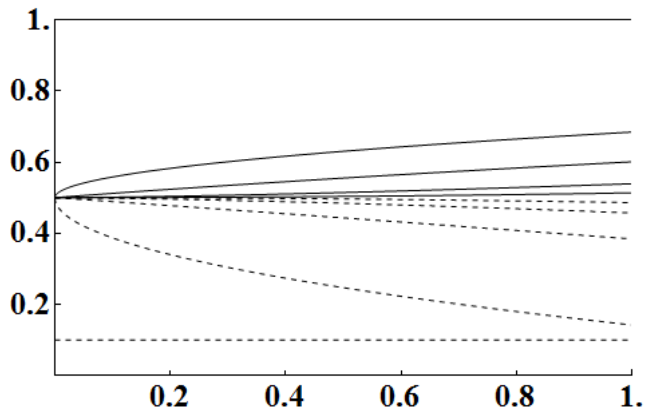

FIGURE 3. Dashed: $\{v_{2n}, w_{2n+1}\}$. Solid: $\{w_{2n}, v_{2n+1}\}$

Now, if we apply the Gauss Seidel method to (4.2) the sequences converge faster. We show the corresponding graph and table of values.

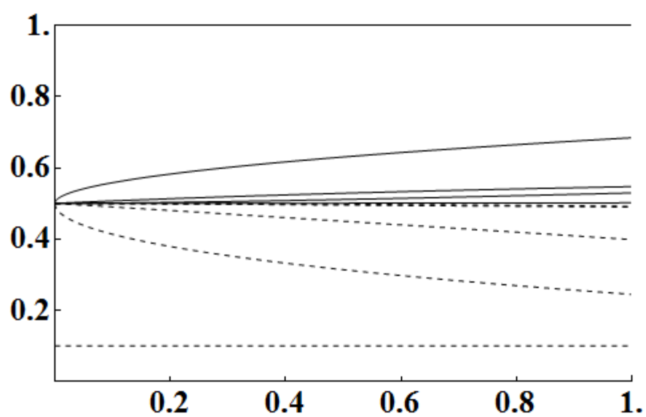

FIGURE 4. Dashed: $\{v_{2n}, w_{2n+1}\}$. Solid: $\{w_{2n}, v_{2n+1}\}$

TABLE 3. Table of five iterations of $v_n(t)$ and $w_n(t)$ for equation (4.2)

n	t	$v_n(t)$	$w_n(t)$
0	1	0.10000	1.00000
1	1	0.68430	0.24513
2	1	0.39879	0.54719
3	1	0.52961	0.49001
4	1	0.49194	0.50158

Conclusion. In this work, we developed generalized monotone method for Caputo fractional differential equation with initial condition. The computation of Mittag Leffler function is not required in each iteration—which is an advantage of generalized monotone method over the usual monotone method, specifically, for Caputo fractional differential equation. In addition, even though we have considered a scalar equation, because of the generalized method we could use the Gauss-Seidal method to accelerate the rate of convergence.

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