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NUMERICAL ANALYSIS FOR SOME SINGULAR INTEGRAL EQUATIONS

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ABSTRACT. We consider some simplest classes of multi-dimensional singular integral equations and their discrete analogues. Numerical solution for such equations is based on the fast Fourier transform. We give some error estimates and show certain advantage of this approach with respect to usual projection methods.

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1. Introduction

Multidimensional singular integral equation in the space $\mathbf{R}^m$ is called the equation of type

$$a(x)u(x)+\int\limits_{\mathbf{R}^m} K(x,x-y)u(y)dy=v(x),\ \ x\in\mathbf{R}^m, \tag{1.1}$$

where the kernel $K(x,y)$ is so-called Calderon-Zygmund kernel [6,8], and the integral in (1.1) is treated in a principal value sense

$$\int\limits_{\mathbf{R}^m} K(x,x-y)u(y)dy=\lim_{\substack{\varepsilon\to 0\\ N\to\infty}}\int\limits_{\varepsilon<|x-y|<N} K(x,x-y)u(y)dy.$$

Definition 1.1. The function $K(x,y)$, defined on $\mathbf{R}^m\times(\mathbf{R}^m\setminus\{0\})$, is called Calderon-Zygmund kernel, if it satisfies the following conditions:

1) $K(x,ty)=t^{-m}K(x,y),\ \ \forall x\in\mathbf{R}^m,\ \ \forall t>0;$
2) $\int\limits_{S^{m-1}} K(x,\omega)=0,\ \ \forall x\in\mathbf{R}^m;$
3) $|K(x,y)|\leq C$, $K(x,\omega)$ is differentiable on S^{m-1}, $\ \forall x\in\mathbf{R}^m$,

S^{m-1} is unit sphere in m-dimensional space, C is a constant.

For such equations of type (1.1) the solvability problem was studied in papers of many authors in different functional spaces. The equations of type (1.1) for example

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with bounded domain or surface instead of $\mathbf{R}^m$ arise often in different mathematical physics problems [9,10], and their solving is very important point. But theoretical studies based, as a rule, on a local principle, lead to Nöther conditions and calculating the index of the operator. Thus, in this paper we will try for simplest types of equations (1.1) to justify the digitization scheme for such equations, and finding the approximate solution, to obtain the approximation rate and to show, that fast Fourier transform can be applied to solving such equations.

We consider the equation (1.1) for the case, when the kernel $K(x,y)$ doesn't depend on the pole x, i.e. it has the form

$$au(x) + \int\limits_{\mathbf{R}^m} K(x-y)u(y)dy = v(x),\ x \in \mathbf{R}^m. \tag{1.2}$$

It seems, the equation (1.2) can be solved simply by the Fourier transform, but it is possible theoretically only. From computer point of view we need discrete (and finite else) sets of points for simulating the equation (1.2). Thus, first we suggest to change the equation (1.2) by the discrete system, and then to consider its finite approximations. Some pieces of this paper earlier were described in authors' papers [2–5].

2. Discrete Singular Integral Operator

For multidimensional singular integral operator

$$(Ku)(x) = \int\limits_{\mathbf{R}^m} K(x-y)u(y)dy$$

we suggest to consider the following discrete analogue:

$$(K_d u_d)(x) = \sum_{\tilde{y} \in \mathbf{Z}_h^m} K_d(\tilde{x}-\tilde{y}) \left[u_d(\tilde{y}) - u_d(\tilde{x})\right] h^m,\ \tilde{x} \in \mathbf{Z}_h^m, \tag{2.1}$$

where we take the following notations.

In m-dimensional space $\mathbf{R}^m$ we define integer point lattice $(\operatorname{mod} h)\mathbf{Z}_h^m$, take into account $K(0) = 0$ and denote by K_d the restriction of the kernel $K(x)$ on $\mathbf{Z}_h^m$, u_d is function of discrete variable, defined on the lattice $\mathbf{Z}_h^m$ and last, the sum of series (2.1) is treated as a limit of partial sums

$$\lim_{N\to\infty} \sum_{\tilde{y} \in \mathbf{Z}_h^m \cap Q_N} K_d(\tilde{x}-\tilde{y}) \left[u_d(\tilde{y}) - u_d(\tilde{x})\right] h^m,$$

where

$$Q_N = \left\{ x \in \mathbf{R}^m : \max_{1\le k\le m} |x_k| \le N \right\}.$$

We denote by ℓ_h^2 the Hilbert space of discrete variable functions $L_2(\mathbf{Z}_h^m)$ with inner product

$$(u_d, v_d) = \sum_{\tilde{x}\in\mathbf{Z}_h^m} u_d(\tilde{x})\overline{v_d(\tilde{x})}$$

and corresponding norm

$$\|u_d\|_{\ell_h^2} = \left(\sum_{\tilde{x}\in\mathbf{Z}_h^m} |u_d(\tilde{x})|^2 h^m\right)^{1/2}.$$

It's well-known, that under conditions for the kernel above the operator K continuously acts in the space $L_2(\mathbf{R}^m)$ [6,8]. This fact implies the following

Theorem 2.1. *The estimate*

$$\|K_d u_d\|_{\ell_h^2} \leq c\|u_d\|_{\ell_h^2},$$

holds, where constant c doesn't depend on h.

So, the family of discrete operators (2.1) is uniformly bounded on h.

3. Operator Symbols and Invertibility

Definition 3.1. Symbol of the operator K is called the Fourier transform of the kernel $K(x)$ in principal value sense

$$\sigma(\xi) = \lim_{\substack{\varepsilon\to 0\\ N\to\infty}} \int\limits_{\varepsilon<|x|<N} K(x)e^{i\xi\cdot x}dx.$$

If we apply the Fourier transform to the equation (1.2), then we obtain the equation

$$(a + \sigma(\xi))\tilde{u}(\xi) = \tilde{v}(\xi), \tag{3.1}$$

for which the necessary and sufficient condition for its solvability in the space $L_2(\mathbf{R}^n)$ is [6,8]

$$\inf |a + \sigma(\xi)| > 0, \quad \xi \in \mathbf{R}^m. \tag{3.2}$$

The function $a + \sigma(\xi)$ is called symbol of the operator $aI + K$, I is identity operator.

To the discrete operator K_d we attach the symbol $\sigma_d(\xi)$, $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$, defined by multivariable Fourier series

$$\sigma_d(\xi) = \sum_{\tilde{x}\in\mathbf{Z}_h^m} K(\tilde{x})e^{-i\tilde{x}\cdot\xi}h^m, \tag{3.3}$$

where the partial sums are chosen over discrete cubes $Q_N \cap \mathbf{Z}_h^m$, and it is periodic function in $\mathbf{R}^m$ with basic cube period $[-\pi h^{-1}, \pi h^{-1}]^m$ [11].

The function $a + \sigma_d(\xi)$, $\xi \in [-\pi h^{-1}, \pi h^{-1}]^m$ is called the symbol of discrete singular integral equation

$$(aI + K_d)u_d = v_d, \tag{3.4}$$

respectively.

In the paper [14] the result asserting the coincidence of the images of symbols $\sigma(\xi)$ and $\sigma_d(\xi)$ was described, and this implies immediately that the equation (1.2) and its discrete analogue (3.4) are both solvable or unsolvable. So, if we have the solution of infinite system of linear algebraic equation (3.4), then we can think, that for small $h > 0$ it will be near the solution of starting equation (1.2).

4. **Error Estimate**

Let's denote P_h the restriction operator on the lattice $\mathbf{Z}_h^m$, i.e. the operator, which an arbitrary function, defined on $\mathbf{R}^m$, maps to the set of its discrete values in lattice points $\mathbf{Z}_h^m$.

Following [15], we give

Definition 4.1. The approximation rate for the operators K and K_d in vector normed space X of functions defined on $\mathbf{R}^m$, is called the operator norm

$$\|P_h K - K_d P_h\|_{X_d},$$

where X_d is the normed space of functions defined on the lattice Z_h^m with norm, which is induced by the norm of the space X.

For the space X_d we will use (with the space ℓ_h^2) the space C_h, which is the space of functions u_d of discrete variable $\tilde{x} \in \mathbf{Z}_h^m$ with the norm

$$\|u_d\|_{C_h} = \max_{\tilde{x} \in \mathbf{Z}_h^m} |u_d(\tilde{x})|.$$

In other words, the space C_h is the space of functions $u \in C(\mathbf{R}^m)$ restricted on lattice points $\mathbf{Z}_h^m$. Here we remind, that the operator K isn't bounded in the space $C(\mathbf{R}^m)$, but it is bounded in the space $L_2(\mathbf{R}^m)$, and it is well-known, that if the right hand side of the equation (1.2) has some smoothness properties (for example, it satisfies the Hölder condition), then the solution of the equation (1.2) (if it exists in the space $L_2(\mathbf{R}^m)$) has the same smoothness property [6].

Let's define the discrete space $C_h(\alpha, \beta)$ as a functional space of discrete variable $\tilde{x} \in \mathbf{Z}_h^m$ with finite norm

$$\|u_d\|_{C_h(\alpha,\beta)} = \|u_d\|_{C_h} + \sup_{\tilde{x},\tilde{y} \in \mathbf{Z}_h^m} \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|,\, 1 + |\tilde{y}|\})^\beta},$$

and additional assumptions

$$|u_d(\tilde{x}) - u_d(\tilde{y})| \le c\, \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|,\, 1 + |\tilde{y}|\})^\beta},$$

$$|u_d(\tilde{x})| \le \frac{c}{(1 + |\tilde{x}|)^{\beta - \alpha}}, \qquad \forall \tilde{x}, \tilde{y} \in \mathbf{R}^m,\ \alpha, \beta > 0,\ 0 < \alpha < 1.$$

The continual analogue of such spaces is the space $H^\alpha_\beta(\mathbf{R}^m)$ of functions, which are continuous in $\mathbf{R}^m$ and satisfy the Hölder condition of order $0 < \alpha < 1$ and with weight $(1 + |x|)^\beta$ (see [1]). The results from [1] implies particularly, that the operator K is linear bounded operator $K : H^\alpha_\beta(\mathbf{R}^m) \to H^\alpha_\beta(\mathbf{R}^m)$ under the condition $m < \beta < \alpha + m$.

For the space $C_h(\alpha, \beta)$ we have

Theorem 4.2. *If $m < \beta < \alpha + m$, then the estimate*

$$\|K_d u_d\|_{C_h(\alpha,\beta)} \le c \|u_d\|_{C_h(\alpha,\beta)},$$

is valid, and c doesn't depend on h.

We will give the approximation rate for the operators K and K_d in the space $C_h(\alpha, \beta)$. It will permit to obtain the error estimate for approximate solution, if we will change the continual operator K by its discrete analogue K_d.

Theorem 4.3. *The approximate rate for the operators K and K_d is the following*

$$\|P_h K - K_d P_h\|_{C_h(\alpha,\beta)} \le c h^{\tilde{\alpha}},$$

where c doesn't depend on h, $\tilde{\alpha} < \alpha$, $\tilde{\beta} > \beta$.

Proof. We need the following two estimates:

$$|((P_h K - K_d P_h) u)(\tilde{x})| \le c_1 h^{\tilde{\alpha}}, \tag{4.1}$$

$$\begin{aligned} &|[(P_h K - K_d P_h) u](\tilde{x}) - [(P_h K - K_d P_h) u](\tilde{y})| \le \\ &\le c_2 h^{\tilde{\alpha}} \sup_{\tilde{x}, \tilde{y} \in \mathbf{Z}^m_h} \frac{|\tilde{x} - \tilde{y}|^\alpha}{(\max\{1 + |\tilde{x}|,\, 1 + |\tilde{y}|\})^{\tilde{\beta}}}. \end{aligned} \tag{4.2}$$

with constants c_1, c_2, non-depending on h.

We will start from the estimate (4.1).

$$((P_hK - K_dP_h)u)(\tilde{x}) =$$
$$= \int\limits_{\mathbf{R}^m} K(\tilde{x}-y)[u(y)-u(\tilde{x})]dy - \sum_{\tilde{y}\in\mathbf{Z}_h^m} K(\tilde{x}-\tilde{y}[u(\tilde{y})-u(\tilde{x})]h^m$$
$$= \int\limits_{\mathbf{R}^m\setminus Q_N} K(\tilde{x}-y)[u(y)-u(\tilde{x})]dy - \sum_{\tilde{y}\in\mathbf{Z}_h^m\setminus Q_N} K(\tilde{x}-\tilde{y})[u(\tilde{y})-u(\tilde{x})]h^m$$
$$+ \sum_{\tilde{y}\in\mathbf{Z}_h^m\cap Q_N} \int\limits_{Q_h(\tilde{y})} \left(K(\tilde{x}-y)[u(y)-u(\tilde{x})] - K(\tilde{x}-\tilde{y})[u(\tilde{y})-u(\tilde{x})]\right)dy$$
$$= I_1 + I_2 + I_3,$$

where $Q_h(\tilde{y})$ is the cube with the center $\tilde{y}\in\mathbf{Z}_h^m$ and the edge of length h.

The first two summands are the tailes of continual and discrete singular integrals, and the third summand only estimates the distance between a singular integral and corresponding cubature formula. Thus, we start from I_3.

1) If $\tilde{x}=\tilde{y}$, then

$$\left|\int\limits_{Q_h(\tilde{y})} K(\tilde{x}-y)[u(y)-u(\tilde{x})]dy\right| \le c \int\limits_{Q_h(\tilde{y})} \frac{|u(y)-u(\tilde{x})|}{|\tilde{x}-y|^m}dy$$
$$\le c \int\limits_{Q_h(\tilde{y})} \frac{dy}{|\tilde{x}-y|^{m-\alpha}(1+|y|)^\beta}. \tag{4.3}$$

2) If $\tilde{x}\ne\tilde{y}$, $\tilde{x}\in Q_N$, then denoting

$$I_{3,n} = \int\limits_{Q_h(\tilde{y})} \left(K(\tilde{x}-y)[u(y)-u(\tilde{x})] - K(\tilde{x}-\tilde{y})[u(\tilde{y})-u(\tilde{x})]\right)dy,$$

we will decompose it on two summands

$$I_{3,n} = \int\limits_{Q_h(\tilde{y})} [K(\tilde{x}-y)-K(\tilde{x}-\tilde{y})]\,[u(y)-u(\tilde{x})]dy$$
$$+ \int\limits_{Q_h(\tilde{y})} K(\tilde{x}-\tilde{y})[u(y)-u(\tilde{y})] = I_{3,n}^{(1)} + I_{3,n}^{(2)}.$$

We have, because $|\tilde{x}-y|\sim|\tilde{x}-\tilde{y}|$,

$$\left|I_{3,n}^{(2)}\right| \le ch^\alpha \int\limits_{Q_h(\tilde{y})} \frac{dy}{|\tilde{x}-y|^m(1+|y|)^\beta}. \tag{4.4}$$

For the estimate $I_{3,n}^{(1)}$ we need the following estimate for the Calderon-Zygmund kernel

$$|K(\tilde{x}-y)-K(\tilde{x}-\tilde{y}| \le c\frac{|y-\tilde{y}|}{|\tilde{x}-y|^{m+1}},$$

which can be obtained by simple calculations.

Taking into account this fact, we have

$$\left|I_{3,n}^{(1)}\right| \leq ch \int\limits_{Q_h(\tilde{y})} \frac{dy}{|\tilde{x}-y|^{m+1-\alpha}(1+|y|)^{\beta}}. \tag{4.5}$$

It is left to collect the estimates (4.3)–(4.5) summing over the cubes $Q_h(\tilde{y}) \subset Q_N$. Let's note the estimate (4.3) is single.

For the

$$R_N = \int\limits_{Q_N} \frac{dy}{|\tilde{x}-y|^{m}(1+|y|)^{\beta}}.$$

we have the following estimate (remind, $|\tilde{x}-y| \geq h/2$), dividing $\mathbf{R}^m$ by two sets

$$A = \left\{ y \in \mathbf{R}^m : \ |\tilde{x}-y| \geq \frac{1+|\tilde{x}|}{2} \right\}$$

$$B = \left\{ y \in \mathbf{R}^m : \ |\tilde{x}-y| < \frac{1+|\tilde{x}|}{2} \right\},$$

$$R_N \leq \left(\int\limits_A + \int\limits_B \right) \frac{dy}{|\tilde{x}-y|^{m}(1+|y|)^{\beta}}.$$

On the set A

$$\begin{aligned}\int\limits_A \frac{dy}{|\tilde{x}-y|^{m}(1+|y|)^{\beta}} &\leq \frac{c}{(1+|\tilde{x}|)^m} \int\limits_A \frac{dy}{(1+|y|)^{\beta}} \\ &\leq c(1+|\tilde{x}|)^{-\beta},\end{aligned}$$

because $\beta > m$.

On the set B, we introduce the spherical coordinates with the center $\tilde{x}$, and obtain

$$\int\limits_B \frac{dy}{|\tilde{x}-y|^{m}(1+|y|)^{\beta}} \leq c \int\limits_h^{\frac{1+|\tilde{x}|}{2}} \frac{dt}{t} \sim c \ln \frac{1+|\tilde{x}|}{h}.$$

Taking into account (4.4) we have

$$\left| \sum_n I_{3,n}^{(2)} \right| \leq c\ h^{\alpha} \ln \frac{1+|\tilde{x}|}{h}.$$

Further, summing the estimates (4.5) we need to bound the integral

$$r_N = \int\limits_{Q_N} \frac{dy}{|\tilde{x}-y|^{m+1-\alpha}(1+|y|)^{\beta}}.$$

Using the same partition $A+B$, we have

$$\int\limits_A (\cdots) \leq c,$$

$$\int_B \frac{dy}{|\tilde{x}-y|^{m+1-\alpha}(1+|y|)^\beta} \leq c \int_h^{\frac{1+|\tilde{x}|}{2}} \frac{dt}{t^{2-\alpha}}$$
$$= c\left(\frac{1}{(1+|x|)^{1-\alpha}} - \frac{1}{h^{1-\alpha}}\right) \leq ch^{-1+\alpha}.$$

Collecting the estimates for (4.5), we obtain

$$\left|\sum_n I_{3,n}^{(1)}\right| \leq ch^\alpha.$$

Taking into account all estimates obtained we have

$$|I_3| \leq c\ h^\alpha \ln \frac{1+|\tilde{x}|}{h}.$$

The estimates for the integrals I_1, I_2 are very like. Particularly,

$$|I_1| \leq c \int_{\mathbf{R}^m \setminus Q_N} \frac{dy}{|\tilde{x}-y|^{m-\alpha}\left(\max\{1+|y|,\, 1+|\tilde{x}|\}\right)^\beta}$$
$$\leq c \int_{\mathbf{R}^m \setminus Q_N} \frac{dy}{|\tilde{x}-y|^{m-\alpha}(1+|y|)^\beta} \leq c \int_{\mathbf{R}^m \setminus Q_N} \frac{dy}{|y|^{m-\alpha+\beta}} \leq \frac{c}{N^{\beta-\alpha}}$$

(N is enough large.)

Tending N to ∞, we have finally

$$|((P_hK - K_dP_h)u)\,(\tilde{x})| \leq \text{const} \cdot \ln \frac{1+|\tilde{x}|}{h}.$$

The second estimate (4.2) is proved by the same more complicated calculations, and we don't stay on this point here. Let's note that the estimate (4.1) proves the nearness of operators K and K_d for C_h-norm. □

5. **Computational Algorithms**

The results above imply, that theoretically one can obtain the convergence of discrete solution to continual ones varying the step of lattice. But (3.4) is infinite system of linear algebraic equations, and so in practice the evaluation of the solution for such system deals with the finding good finite approximation.

Such infinite systems of linear algebraic equations were considered earlier in mathematical papers [7], in which were suggested and justified projection methods for their solving. Concerning the equation (3.4) this scheme is looked as following. If we denote by P_N the restriction operator acting from $\mathbf{Z}_h^m$ to the discrete cube $\mathbf{Z}_h^m \cap Q_N$, then the equation (3.4) is replaced by the finite system of linear algebraic equations

(5.1) $$P_N(aI + K_d)u_{d,N} = P_N\, v_d.$$

The theoretical studying as a rule are restricted by justifying the transfer from (3.4) to (5.1), and it means that the following [7]

Assertion. *If the equation* (3.4) *is uniquely solvable in the space* ℓ_h^2, *then for enough large* N *the equation* (5.1) *is uniquely solvable on the sub-space* $P_N \ell_h^2$

is valid.

From practical point of view this assertion is not effective, because for small h and large N the system (5.1) can be very large, and from computational point of view is hardly realizable.

In our opinion, the following scheme for finite approximation is more pragmatic. Given discrete kernel K_d and right-hand side v_d one construct their periodic approximation by restriction on $Q_N \cap \mathbf{Z}_h^m$ and by periodic continuation on $\mathbf{Z}_h^m$. We denote these continuations by $K_{d,N}$ and $v_{d,N}$ respectively. We consider the equation

$$au_{d,N}(\tilde{x}) + \sum_{\tilde{y} \in Z_h^m} K_{d,N}(\tilde{x} - \tilde{y}) u_{d,N}(\tilde{x}) h^m = v_{d,N}(\tilde{x}), \quad \tilde{x} \in Z_h^m, \tag{5.2}$$

instead of the equation (3.4), and really it is finite system of linear algebraic equations with so-called cyclic convolution [12,13]. The theory of discrete Fourier transform and symbol properties of multidimensional singular integral permits to prove the solvability of the equation (5.2) for large N, fast Fourier transform permits to reject the solving system of linear algebraic equations and restricting himself by calculating the Fourier transform twice (direct transform and inverse ones). Moreover, the comparison of numerical results for simplest test equations (both regular integral equations and singular ones) obtained by projection methods and fast Fourier transform, showed their nearest location and large advantage in time for the last method even in one-dimensional case [4]. It seems, the difference in time will be more essential under enlarging space's dimension.

Example 5.1. If we consider the convolution equation

$$\int_{-\infty}^{+\infty} K(x-y) u(y) dy = v(x),$$

where $K(x) = \exp(-\frac{x^2}{2}),\ v(x) = exp(-\frac{x^2}{4})$, then the Fourier transform

$$\int_{-\infty}^{+\infty} \exp(-ix\xi) f(x) dx \equiv \tilde{f}(\xi) \quad \text{(direct)}$$

and

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(ix\xi) \tilde{f}(\xi) d\xi \equiv f(x) \quad \text{(inverse)}$$

permits to find exact solution immediately:

$$u(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2}).$$

The following table involves the approximate values of $u(x)$ obtained by three different ways. The fast Fourier transform works more rapidly than solving truncated system of linear algebraic equations (approximately ten times).

Type/X axis	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Analytical	0	0,0000003	0,00008	0.039	0.135	0.29	0.369	0.29	0.135	0.039	0,00008	0,0000003	0
System	0	0,0000003	0,000025	0.0251	0.1005	0.25	0.355	0.25	0.1005	0.0251	0,000025	0,0000003	0
Fourier	0	0,0000003	0,000025	0.025	0.1	0.247	0.3548	0.247	0.1	0.025	0,000025	0,0000003	0

6. **A Half-Space Case**

Here we consider more complicated equation, so called pair equation

$$(M_1P_+ + M_2P_-)U = F, \tag{6.1}$$

assuming M_1, M_2 are Calderon-Zygmund operators (like in equation (1.2)), and P_+, P_- we denote the restriction operators on half-space $\mathbf{R}^m_\pm = \{x = (x_1, \dots, x_m), \pm x_m > 0\}$.

For this case using the Fourier transform leads to the following constructions [18]:

$$FP_+ = Q_{\xi'}F, \;\; FP_- = P_{\xi'}F,$$

$$P = 1/2(I + H_{\xi'}), \;\; Q = 1/2(I - H_{\xi'}),$$

where F is the Fourier transform, $H_{\xi'}$ is the Hilbert transform on variable ξ_m, $\xi' = (\xi_1, \dots, \xi_{m-1})$ is fixed:

$$(H_{\xi'}u)(\xi', \xi_m) \equiv \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{u(\xi', \tau)}{\tau - \xi_m} d\tau.$$

The equation (6.1) for this case will take the form of the following one-dimensional singular integral equation with the parameter ξ' [17]:

$$\begin{aligned} &\frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2} \tilde{U}(\xi) \\ &+ \frac{\sigma_{M_1}(\xi', \xi_m) + \sigma_{M_2}(\xi', \xi_m)}{2\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{\tilde{U}(\xi', \eta)}{\eta - \xi_m} d\eta = \tilde{F}(\xi). \end{aligned} \tag{6.2}$$

It corresponds to the Riemann boundary problem (with parameter ξ' also) with coefficient [16]

$$G(\xi', \xi_m) = \sigma_{M_1}(\xi', \xi_m) \sigma^{-1}_{M_2}(\xi', \xi_m).$$

For unique solvability of the equation (6.2) we need the index of $G(\xi', \xi_m)$ on variable ξ_m is equal to 0.

The symbol of the Calderon-Zygmund operator is very specific, it is a function homogeneous of order 0, i.e. indeed it is defined on unit sphere S^{m-1}. Let $m \geq 3$. Fix $\xi' \in S^{m-2}$ and suppose $G(0,-1) = G(0,+1)$. Varying ξ_m from $-\infty$ to $+\infty$ the function $G(\xi)$ will vary along the arc of big half-circle across south pole $(0,-1)$ and north pole $(0,+1)$. At the same time the symbol's values will go along closed curve in a complex plane. These all curves for different ξ' will be homotopic, and therefore they have the same entire-valued index æ with respect to origin. The condition æ $= 0$ implies the unique solvability for the equation (6.2).

7. **Discrete Half-Space Case**

Here we return again to the equations in a discrete context assuming that in equation (6.1) $P_\pm$ are restriction operators on $\mathbf{Z}^m_{h,\pm}$, M_1, M_2 are discrete Calderon-Zygmund operators generated by kernels $M_1(x), M_2(x)$, which are bounded in the space $L_2(\mathbf{Z}^m_h)$.

For functions of discrete variable defined on the lattice $\mathbf{Z}^m_h$ its discrete Fourier transform is given by the formula

$$u(\tilde{x}) \longmapsto \frac{1}{(2\pi)^m} \sum_{\tilde{x} \in \mathbf{Z}^m_h} u(\tilde{x}) e^{-i\tilde{x}\cdot\xi} h^m \equiv \tilde{u}(\xi), \ \xi \in [-\pi, \pi]^m.$$

Such Fourier transform has the same properties as usual Fourier transform [11].

According to Sec. 6 we introduce periodic analogue of the Hilbert transform on variable ξ_m ($\xi \in [-\pi, \pi]^m$, ξ' is fixed) by the formula

$$(H^{per}_{\xi'} u)(\xi_m) = \frac{1}{2\pi i} \int\limits_{-\pi h^{-1}}^{\pi h^{-1}} u(t) \cot \frac{h(t - \xi_m)}{2} dt$$

and periodic analogues of projectors P, Q

$$P^{per}_{\xi'} = 1/2(I + H^{per}_{\xi'}), \ Q^{per}_{\xi'} = 1/2(I - H^{per}_{\xi'}).$$

Instead of the equation (6.2) we obtain its periodical analogue

$$\frac{\sigma_{1,h}(\xi', \xi_m) + \sigma_{2,h}(\xi', \xi_m)}{2} \tilde{U}(\xi) + \frac{\sigma_{1,h}(\xi', \xi_m) + \sigma_{2,h}(\xi', \xi_m)}{4\pi i}$$

$$\times \, v.p. \int\limits_{-\pi h^{-1}}^{\pi h^{-1}} \tilde{U}(\xi', \eta) \cot \frac{h(\eta - \xi_m)}{2} d\eta = \tilde{F}(\xi) \tag{7.1}$$

where $\sigma_{1,h}, \sigma_{2,h}$ are symbols of discrete operators M_1, M_2. Naturally, the equation (7.1) will be related to corresponding periodic Riemann boundary problem, for which its unique solvability condition is

$$Ind \ \sigma_{1,h}(\cdot, \xi_m) \sigma^{-1}_{2,h}(\cdot, \xi_m) = 0.$$

Further we remind, that images of symbols σ and σ_h are the same [14]. Moreover, the index is entire-valued characteristic both in continual case (if the transmission condition $\sigma(0,-1) = \sigma(0,+1)$ holds) and in discrete (periodic) ones. Looking for the variation $\sigma_h(\cdot, \xi_m)$ along arcs of big half-circles on S^{m-1} and taking into account that

$$\lim_{h \to 0} \sigma_h(\xi) = \sigma(\xi), \ \forall \xi \in S^{m-1},$$

we conclude, that for transmission condition we have

Theorem 7.1. *The equations* (6.2) *and* (7.1) *are both solvable or unsolvable.*

8. Conclusion

We intend to continue these studies and to obtain the same error estimate for multidimensional discrete singular integral in corresponding Hölder space with weight $H^\alpha_\beta(\mathbf{R}^m_+)$.

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