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STABILIZATION OF INVARIANT MANIFOLDS FOR NONLINEAR STOCHASTIC SYSTEMS

IRINA BASHKIRTSEVA AND LEV RYASHKO

Institute of Mathematics and Computer Science, Ural Federal University,
Ekaterinburg, Russia

ABSTRACT. An exponential mean square (EMS) stabilization of manifolds of stochastically forced nonlinear systems is considered. The necessary and sufficient stabilizability conditions based on spectral criterion of the EMS-stability of invariant manifolds are presented. We suggest methods for the design of the feedback stabilizing regulator for SDEs. A parametrical criterion of the stochastic stabilizability for limit cycles is given. This theoretical technique is applied to numerical simulations of the solution of control problem for the stochastically forced Hopf system.

AMS (MOS) Subject Classification. 37H20

1. INTRODUCTION

Many nonlinear dynamical phenomena observed under transition from order to chaos are frequently connected with a chain of bifurcations: a stationary regime (equilibrium point) - periodic regime (limit cycle) - quasiperiodic regime (torus) - chaotic regime (strange attractor). So, complicated oscillatory regimes are usually observed for such type bifurcations. Invariant manifold is a convenient general mathematical model for the stability analysis of these various nonlinear oscillations.

The stability investigation and control of stochastic systems are attractive from theoretical and engineering points of view. Indeed, even weak noise can result in qualitative changes in the system's dynamics. We consider the exponential mean square stability and stabilization problem for invariant manifolds of stochastic differential equations (SDEs).

One of the most important methods of the stability analysis is Lyapunov function technique (LFT) [1], [2]. LFT in the research of the stochastic stability of equilibria has been widely studied by many authors (see [3], [4]). A problem of the synthesis of stochastic attractors and controlling chaos was investigated in [5], [6].

The orbital Lyapunov functions were used in the stability and sensitivity analysis of stochastically forced limit cycles [7], [8], [9]. LFT for the stability analysis of the general invariant manifolds is considered for deterministic [10] and stochastic

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[11] systems. On the base of LFT, a general spectral criterion of EMS stability of manifolds has been proved [12].

The aim of this work is to apply this criterion to the solution of the control problem and show how it works by numerical simulations.

2. STOCHASTIC STABILITY OF INVARIANT MANIFOLDS

Consider a deterministic nonlinear system

$$dx = f(x)\, dt, \tag{2.1}$$

where x is n-vector, $f(x)$ is sufficiently smooth vector-function of the appropriate dimension. It is assumed that the system (2.1) has a smooth compact invariant manifold M [13], [14], [15].

Consider a function $\gamma(x)$ in a neighbourhood U of the manifold M. Here $\gamma(x)$ is a point of the manifold M that is nearest to x, $\Delta(x) = x - \gamma(x)$ is a vector of the deviation of the point x from the manifold M. It is assumed that the neighbourhood U is invariant for the system (2.1). For any $x \in M$, denote by T_x the tangent subspace to M at x. Denote by N_x the orthogonal complement to T_x and by P_x the operator of the orthogonal projection onto the subspace N_x.

In this paper, we consider a randomly forced deterministic system (2.1) as follows:

$$dx = f(x)dt + \sum_{r=1}^{m} \sigma_r(x) dw_r(t), \tag{2.2}$$

where $w_r(t)$ $(r = 1, \ldots, m)$ are independent standard Wiener processes, $\sigma_r(x)$ are sufficiently smooth vector-functions of the appopriate dimension. To ensure M is an invariant of the stochastic system (2.2) we assume that

$$\sigma_r|_M = 0. \tag{2.3}$$

Definition 1. The manifold M is called exponentially stable in the mean square sense (EMS-stable) for the system (2.2) in U if there exist $K > 0$, $l > 0$ such that

$$\mathrm{E}\|\Delta(x(t))\|^2 \le K e^{-lt} \mathrm{E}\|\Delta(x_0)\|^2,$$

where $x(t)$ is a solution of the system (2.2) with the initial condition $x(0) = x_0 \in U$.

Consider a space Σ of symmetrical $n \times n$ matrix functions defined and sufficiently smooth on M and satisfying the following singularity condition

$$\forall x \in M\ \forall z \in T_x \quad V(x)z = 0.$$

Definition 2. A matrix function $V(x) \in \Sigma$ is called P-positive definite if

$$\forall x \in M \ \ \forall z \qquad P_x z \neq 0 \Rightarrow (z, V(x)z) > 0.$$

On the space Σ, we shall consider operators:

$$\mathcal{A}[V] = \left(f, \frac{\partial V}{\partial x}\right) + F^\top V + VF, \quad \mathcal{S}[V] = \sum_{r=1}^{m} S_r^\top V S_r, \quad \mathcal{P} = -\mathcal{A}^{-1}\mathcal{S},$$

where

$$F(x) = \frac{\partial f}{\partial x}(x), \quad S_r(x) = \frac{\partial \sigma_r}{\partial x}(x).$$

Let $\rho(\mathcal{P})$ be a spectral radius of the operator $\mathcal{P}$.

Theorem 1. *The manifold M of the stochastic system* (2.2) *is EMS-stable if and only if*

(a) The manifold M of the deterministic system (2.1) *is exponentially stable,*

(b) The inequality $\rho(\mathcal{P}) < 1$ holds.

This theorem has been proved in [12] on the base of the spectral theory of the positive operators [16]. An analogous approach was used earlier in [17] for the stability analysis and stabilization of linear SDEs with periodic coefficients.

Example 1. Stability of the limit cycle for 2D-system. We assume that an invariant manifold M is a limit cycle corresponding to T-periodic solution $\xi(t)$. The function $\xi(t)$ gives us a natural parametrization of the cycle orbit and defines the one-to-one correspondence between cycle points and the time interval $[0, T)$.

Using this parametrization, we introduce functions

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \quad S_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t))$$

defined on $[0, T]$.

In the case $n = 2$ for the spectral radius of the operator $\mathcal{P}$, one can find the following explicit formula:

$$\rho(\mathcal{P}) = -\frac{\langle \beta \rangle}{\langle \alpha \rangle}.$$

Here

$$\alpha(t) = p(t)^\top [F^\top(t) + F(t)] p(t), \qquad \beta(t) = \operatorname{tr}\left(\sum_{r=1}^{m} S_r(t) S_r^\top(t)\right),$$

$p(t)$ is a vector orthonormal to the limit cycle M at the point $\xi(t)$, brackets $\langle \cdot \rangle$ mean an integral with the time averaging:

$$\langle \alpha \rangle = \frac{1}{T} \int_0^T \alpha(t) dt.$$

The inequality (famous Poincare criterion)

$$\langle \alpha \rangle < 0$$

is a necessary and sufficient condition of the exponential stability of the limit cycle M for the deterministic system (2.1). Thus, the inequality $\rho(\mathcal{P}) < 1$ written as

$$\langle \alpha + \beta \rangle = \left\langle \operatorname{tr} \left[2F(t) + \sum_{r=1}^{m} S_r(t) S_r^\top(t) \right] \right\rangle < 0$$

is a necessary and sufficient condition of EMS-stability of the cycle M for the stochastic system (2.2) in 2D-case.

3. STABILIZATION

Consider a stochastic system with a control in the form

$$dx = f(x,u)dt + \sum_{r=1}^{m} \sigma_r(x,u)dw_r(t), \tag{3.1}$$

where x is n-dimensional state variable, u is l-dimensional vector of control inputs, $f(x,u)$, $\sigma(x,u)$ are vector-functions of the appropriate dimension, $w_r(t)$ ($r = 1,\dots,m$) are independent standard Wiener processes. It is supposed that for $u = 0$ the system (3.1) has an invariant manifold M.

We shall select the stabilizing regulator from the class of admissible feedbacks $u = u(x)$ satisfying conditions:

(a) $u(x)$ is sufficiently smooth and $u|_M = 0$;

(b) for the deterministic system

$$dx = f(x, u(x))dt \tag{3.2}$$

the manifold M is exponentially stable in the neighbourhood U of M.

Without loss of generality, we can restrict our consideration by the regulator in the following form

$$u(x) = K(\gamma(x))\Delta(x). \tag{3.3}$$

Here $K(x)$ is a feedback matrix coefficient.

Consider a set $\mathbb{K}$ of $l \times n$-matrices $K(x)$ satisfying the following condition: the manifold M is exponentially stable for the closed-loop deterministic system (3.2), (3.3).

For the stabilization of the closed-loop stochastic system (3.1), (3.3) we will use a spectral criterion from Theorem 1.

Consider corresponding operators

$$\mathcal{A}_K[V] = \left(f_0, \frac{\partial V}{\partial x} \right) + (F + BK)^\top V + V(F + BK),$$

$$\mathcal{S}_K[V] = \sum_{r=1}^{m} (C_r + H_r K)^\top V (C_r + H_r K), \quad \mathcal{P}_K = -\mathcal{A}_K^{-1} \mathcal{S}_K,$$

where

$$f_0 = f(x,0), \quad F(x) = \frac{\partial f}{\partial x}(x,0), \quad B(x) = \frac{\partial f}{\partial u}(x,0),$$
$$C_r(x) = \frac{\partial \sigma_r}{\partial x}(x,0), \quad H_r(x) = \frac{\partial \sigma_r}{\partial u}(x,0).$$

The Theorem 1 implies the following.

Theorem 2. *The manifold M is EMS-stabilizable for the stochastic system* (3.1) *with the feedback* (3.3) *if and only if*

(a) $\mathbb{K} \neq \emptyset$,

(b) The inequality $\inf_{K\in\mathbb{K}} \rho(\mathcal{P}_K) < 1$ *holds.*

The feedback (3.3) *stabilizes the stochastic system* (3.1) *for any* $K \in \mathbb{K}$ *satisfying the inequality* $\rho(\mathcal{P}_K) < 1$.

This Theorem reduces a stabilization problem to the minimization of the spectral radius of the operator $\mathcal{P}_K$.

Example 2. Stabilization of the cycle for 2D-system. For the case of the cycle on a plane ($n = 2$), one can find for the spectral radius of the operator $\mathcal{P}_K$ the following explicit formula

$$\rho(\mathcal{P}_K) = -\frac{\langle \beta_K \rangle}{\langle \alpha_K \rangle}.$$

Here

$$\begin{aligned} \alpha_K &= p^\top \left[(F + BK)^\top + F + BK \right] p, \\ \beta_K &= \operatorname{tr} \left(\textstyle\sum_{r=1}^m (C_r + H_r K)(C_r + H_r K)^\top \right), \end{aligned} \tag{3.4}$$

$$F(t) = \frac{\partial f}{\partial x}(\xi(t),0), \quad B(t) = \frac{\partial f}{\partial u}(\xi(t),0),$$
$$C_r(t) = \frac{\partial \sigma_r}{\partial x}(\xi(t),0), \quad H_r(t) = \frac{\partial \sigma_r}{\partial u}(\xi(t),0),$$

$p(t)$ is a vector orthonormal to the limit cycle at the point $\xi(t)$.

The condition of the stabilizability $\inf_{K\in\mathbb{K}} \rho(\mathcal{P}_K) < 1$ is equivalent to the inequality

$$\inf_{K\in\mathbb{K}} I(K) < 0, \quad I(K) = \langle \alpha_K + \beta_K \rangle.$$

So, a solution of the stabilization problem is reduced to the minimizing of the quadratic functional $I(K)$.

4. NUMERICAL SIMULATIONS. STABILIZATION OF CYCLES FOR THE STOCHASTIC HOPF SYSTEM

Consider stochastically forced Hopf system with control

$$\begin{aligned} \dot{x} &= \mu x - y - (x^2 + y^2)x + u + \sigma_1 (x^2 + y^2 - \mu)\dot{w}_1(t) + \sigma_2 u \dot{w}_2(t) \\ \dot{y} &= x + \mu y - (x^2 + y^2) y. \end{aligned} \tag{4.1}$$

Here w_1, w_2 are standard Wiener processes, σ_1 is an intensity of state-dependent noise, and σ_2 is an intensity of control-dependent noise, u is a scalar control input. For $u = 0$, $\mu > 0$, $\sigma_1 = 0$, this system has a limit cycle $x^2 + y^2 = \mu$. For this cycle, we use the parametrization $x = \sqrt{\mu}\cos t, y = \sqrt{\mu}\sin t$. The aim of the control is to stabilize this cycle in the mean square sense.

Feedback matrix in the regulator (3.3) for Hopf system (4.1) is $K(t) = k(t)p(t)$, where $p(t) = (\cos t, \sin t)^\top$ and $k(t)$ is a scalar function. Functions α_k, β_k in (3.4) have an explicit representation

$$\alpha_k = -4\mu + 2k\cos t, \quad \beta_k = 4\sigma_1^2\mu + k^2\sigma_2^2.$$

So, the quadratic functional $I(k) = \langle \alpha_k + \beta_k \rangle$ is as follows

$$I(k) = 4\mu(\sigma_1^2 - 1) + \frac{1}{2\pi}\int_0^{2\pi} \left(2k(t)\cos t + k^2(t)\sigma_2^2\right) dt. \tag{4.2}$$

For $u = 0$, a necessary and sufficient condition of the stochastic stability of the cycle is $\sigma_1^2 < 1$.

For numerical simulations, fix $\mu = 1$, $\sigma_1 = 2$. Theoretically, for these parameters, the cycle $x^2+y^2 = 1$ of the uncontrolled $(u = 0)$ system (4.1) is stochastically unstable in the mean square sense.

In Fig. 1, by dashed line, we plot a function $M(t) = \mathrm{E}\left(\sqrt{x^2(t) + y^2(t)} - 1\right)^2$, where $x(t), y(t)$ is a solution of the Hopf system with $u = 0$ for initial conditions $x(0) = 1.01$, $y(0) = 0$. For numerical simulations, we use Euler-Maruyama scheme with time step $\Delta t = 10^{-5}$ and averaging of 5000 random trajectories. Here, an exponential growth of the quadratic deviation of solutions from the cycle is observed.

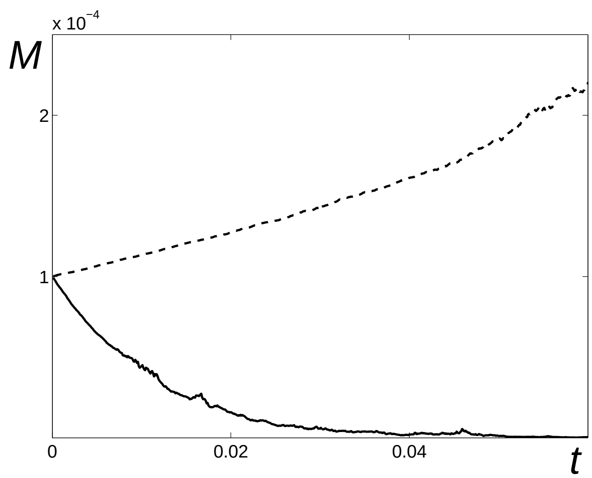

Fig. 1. Mean square deviation $M(t)$ for uncontrolled (dashed line) and controlled (solid line) stochastic Hopf system.

t	$M_{\text{uncontrolled}}$	$M_{\text{controlled}}$
0.000	$1.000000 \cdot 10^{-4}$	$1.000000 \cdot 10^{-4}$
0.005	$1.064833 \cdot 10^{-4}$	$6.468021 \cdot 10^{-5}$
0.010	$1.133401 \cdot 10^{-4}$	$4.113804 \cdot 10^{-5}$
0.015	$1.205908 \cdot 10^{-4}$	$2.815260 \cdot 10^{-5}$
0.020	$1.273460 \cdot 10^{-4}$	$1.771679 \cdot 10^{-5}$
0.025	$1.346127 \cdot 10^{-4}$	$7.500728 \cdot 10^{-6}$
0.030	$1.431145 \cdot 10^{-4}$	$5.176267 \cdot 10^{-6}$
0.035	$1.522734 \cdot 10^{-4}$	$2.452727 \cdot 10^{-6}$
0.040	$1.611460 \cdot 10^{-4}$	$1.974542 \cdot 10^{-6}$
0.045	$1.701024 \cdot 10^{-4}$	$1.581786 \cdot 10^{-6}$
0.050	$1.855306 \cdot 10^{-4}$	$5.316145 \cdot 10^{-7}$
0.055	$2.040819 \cdot 10^{-4}$	$2.395845 \cdot 10^{-7}$
0.060	$2.200374 \cdot 10^{-4}$	$5.251944 \cdot 10^{-8}$

TABLE 1. Mean square deviation for uncontrolled and controlled stochastic Hopf system.

Consider now abilities of the stabilization. The function $k_o(t) = -\frac{\cos(t)}{\sigma_2^2}$ minimizes the functional (4.2). The minimal value of this functional is

$$I(k_o) = 4\mu(\sigma_1^2 - 1) - \frac{1}{2\sigma_2^2}.$$

For $\sigma_1^2 > 1$, a necessary and sufficient condition of the stabilizability can be written in a parametrical form:

$$\sigma_2^2 < \frac{1}{8\mu(\sigma_1^2 - 1)}.$$

For the considered set of parameters $\mu = 1, \sigma_1 = 2$, the stabilizability condition is $\sigma_2^2 < 1/24$. In this case, the feedback regulator is the following:

$$u = -\frac{x}{\sigma_2^2\sqrt{x^2+y^2}}\left(\sqrt{x^2+y^2} - 1\right).$$

In Fig. 1, by solid line, we plot a function $M(t)$ for the system (4.1) with this regulator and $\sigma_2 = 0.1$. Corresponding numerical data are presented in the Table 1. An exponential decrease of the quadratic deviation of solutions from the cycle demonstrates a stabilization.

5. CONCLUSION

A problem of the mean square stabilization of the general invariant manifolds for nonlinear stochastic systems was reduced to the minimization of the spectral radius of the corresponding operator. Constructive abilities of this theory were demonstrated for the important problem of the stabilization of the stochastically forced limit cycle. The problem of the stabilization of the limit cycle was turned to the classical mathematical problem of the quadratic functional minimization. This theory was successfully applied to the stabilization of the cycles of Hopf system via numerical simulations.

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