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THE STABILITY, BULLWHIP EFFECT AND OPTIMIZATION OF A HORIZONTAL COLLABORATION SUPPLY CHAIN NETWORK

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ABSTRACT. In this paper, we study the stability, bullwhip effect and optimization of the order policy system for a horizontal collaboration supply chain network in which two or more retailers or distributors may mutually accept a small order. Two Z-transform expressions of the solutions to the order equations are obtained respectively for the horizontal collaboration and non-horizontal collaboration supply chain networks. It is shown that a stable (unstable) order policy system for the non-horizontal collaboration supply network can remain (become) stable in the horizontal collaboration supply network as long as suitable order policies are adopted. Conditions sufficient for the absence or presence of the bullwhip effect in the order policy system are obtained. A special relation between the horizontal collaboration and non-horizontal collaboration supply chain networks is proved. This relation shows that the horizontal collaboration supply chain network can either enhance or reduce the bullwhip effect in the non-horizontal collaboration supply chain network. Moreover, we present an approach to obtain the optimal order policy indirectly.

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1. Introduction

Supply chain networks often face three problems: intense competition, unpredictable customer demand and constrained supplies. In order to survive and develop in such complicated and changing environment, collaboration in supply chain networks has become one of the most effective ways to create business value.

Soosay, Hyland and Ferrer (2008) summarized various types of collaboration in supply chain networks, including vertical, horizontal, lateral and virtual collaboration. The so-called vertical collaboration takes place at different levels of a supply chain network. The most important type of vertical collaboration is information sharing — retailers and suppliers share demand information and action plans in order to align their forecasts for capacity and long-term planning, yet still order independently. Information sharing may reduce the bullwhip effect which refers to the phenomenon where the variance of the orders amplifies as one moves upstream, enhancing profitability (Lee et al. 2000, Dejonckheere et al. 2004, Ouyang 2007). Another type of

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vertical collaboration, which has been studied by Caridi et al. (2005), Chung and Leung (2005), Shirodkar and Kempf (2006), is the collaborative planning, forecasting and replenishment (CPFR) process, which enhances the cooperation of exchanging sales and order forecasts between trading partners. The horizontal collaboration proposed by Barratt (2004), Simatupang and Sridharan (2002, 2005) occurs when two or more unrelated or competing partners (e.g., suppliers, retailers, etc.) cooperate at the same level of the supply chain to share their private information or resources such as inventory levels or distribution centers. Such horizontal collaboration can overcome financial barriers to trade (Manning and Baines 2004) and reduce overall cost of the supply chain (Prakash and Deshmukh 2010).

From the review of literature, it can be seen that information sharing is key to vertical and horizontal collaboration in the supply chain network. Though information sharing can reduce the bullwhip effect and overall cost of the supply chain, and can help supply chain members to effectively match demand and supply to increase overall supply chain profitability, the idea is not easy to realize since it is based on mutual trust, openness, shared risk and shared rewards (Barratt 2004). It is even more difficult to realize when the benefit to supply chain members is not consistent with overall supply chain profitability. So we consider in the present paper an easily executed horizontal collaboration supply chain network without information sharing. Such a horizontal collaboration occurs when two or more partners (e.g., retailers, distributors, etc.) competing at the same level of the supply chain network mutually accept a relatively small order. Here, we say that it is an easily executed horizontal collaboration supply chain since the mutual order quantity is relative small. In fact, such a horizontal collaboration often occurs in real supply chain networks. If there are no mutual order goods between the distributors or retailers we call it a competition supply network since any two retailers or distributors can be regarded as the competitors. It will be shown in the paper that horizontal collaboration can improve greatly the competitive performance of supply networks. Note that the horizontal collaboration (accepting mutually an order for a relative small quantity) is similar to the model of *transshipment* or *inventory sharing* which refers to the lateral transfer of inventory among depots, dealers, or retailers. This model has been studied by many authors such as Krishnan and Rao (1965), Rudi, Kapur and Pyke (2001), Dong and Rudi (2005), Zhao, Deshpande and Ryan (2006), and Hanany, Tzur and Levran (2010).

Previous quantitative studies of supply chain (network) have mainly focused on three problems: stability of supply chain, the bullwhip effect in supply chains and optimization of order policies. In the present paper, we study these three problems in the horizontal collaboration and the competition supply chain networks. The paper is novel in that (1) it gives two Z-transform expressions of the solutions to the order equations respectively for the horizontal collaboration and the competition supply networks; (2) it shows that a stable (unstable) order policy system for the competition supply network can remain (become) stable in the horizontal collaboration supply network as long as we adopt suitable order policies; (3) it gives some sufficient conditions for avoiding or producing the bullwhip effect in the order policy systems; (4) it proves a special relation between the horizontal collaboration and competition supply chain networks, and by this relation we can show that the bullwhip effect in the competition supply network can either be reduced or enhanced in the corresponding horizontal collaboration supply network; and (5) it presents an approach to obtain the optimal order policy indirectly.

The paper is organized as follows. We survey relevant literature in Section 2. The horizontal collaboration and competition supply chain networks and their order equations for the order policies are described in Section 3. Section 4 gives the solutions to the order equations, and analyzes the stability in the order policy system for the horizontal collaboration and the competition supply chain networks. In Section 5, after presenting some sufficient conditions for avoiding or producing the bullwhip effect in the order policy system, we present a special relation between the horizontal collaboration and the competition supply chain networks. Section 6 shows an approach to obtain the optimal order policy indirectly. As an application of the main results, an example analysis is performed in Section 7. In Section 8, we conclude and discuss the limitations of our model as well as several extensions. The proofs of theorems are given in the Appendix.

2. **Literature Review**

Stability is a fundamental characteristic of order policy systems in supply chain networks. Riddalls and Bennett (2002) and Warburton et al. (2004) study stability properties and present a stability criterion for a continuous version of the supply chain. Disney and Towill (2002) develops a discrete transfer function model to determine the dynamic stability of a Vendor Managed Inventory supply chain. Hoberg et al. (2007) proves that a two-echelon supply chain with a stationary demand that operates under inventory-on-hand policies are not stable, but supply chains that operate under installation-stock and echelon-stock policies are. Disney (2008) investigates the discrete time order-up-to policy with two independent proportional controllers in the policy's feedback loop and identifies the conditions of stability using Jury's inners approach (Jury, 1974). Ostrovsky (2008) studies a set of bilateral contracts — another kind of stable supply chain network — no upstream-downstream sequence of agents can add a chain of contracts (or drop, if necessary, some other contracts) to make themselves better off, and shows that under same-side substitutability and cross-side complementarity, the chain-stable networks are guaranteed to exist. What we consider in the paper is different from those studied by the above authors, and our focus is on the relation between the horizontal collaboration and non-horizontal collaboration supply chain networks in terms of stability.

The bullwhip effect was originally observed and studied by Forrester (1961). Sterman (1989) reports evidence of the bullwhip effect in the Beer Distribution Game. Lee et al. (1997) give five important causes of the bullwhip effect: the use of demand signal processing, non-zero lead times, order batching, supply shortages and price fluctuations. They show that the information transferred in the form of orders tends to be distorted and the distortion tends to increase the bullwhip effect. Chen et al. (2000) quantifies the bullwhip effect in order-up-to policies based on exponential smoothing forecasts as well as moving average forecasts. Alwan (2003) reveals that there is actually no bullwhip effect in a negatively correlated process under an mean square error optimal forecasting scheme. Daganzo (2004) presents a policy-specific but demand-independent upper bound for the order variance amplification factor of any decentralized policy and shows that the bound is always tight for the suppliers at the end of a long chain so that a policy exhibits the bullwhip effect if and only if its bound is greater than 1. Hosoda and Disney (2006) reveals that the level of a three-echelon supply chain has no impact on the bullwhip effect, which instead is determined by combining the lead-time from the customer and the local replenishment lead-time. Strozzi et al. (2008) finds that stabilizing the dynamics of a single-product one-echelon supply chain can reduce the total costs and the bullwhip effect. By using the discrete control theory model and z-transform techniques, Dejonckheere et al. (2003), Disney and Towill (2003), Disney et al. (2004), Ouyang and Daganzo (2006) and Disney (2008) obtain analytical expressions for the bullwhip effect and some analytical conditions for its absence or presence. Lee et al. (2000), Chen et al. 2000, Dejonckheere et al. (2004), Ouyang (2007), Madlberger (2008) and Ren et al. (2010) investigate the value, impact and benefits of information sharing in supply chains. Sucky (2009) extends the analysis of Chen et al. (2000) to a supply chain with a network structure in which risk pooling can reduce the bullwhip effect at every individual stage and shows that the bullwhip effect may be overestimated if a simple supply chain is assumed. Ouyang and Li (2010) analyze the bullwhip effect in supply chains with a general network topology, general linear ordering policies (information sharing schemes), and various customer demand(s). They present robust formulas to test the existence of the bullwhip effect in the worst-case metric, which do not require knowledge of the demand process. Our model is essentially a simplification of Ouyang and Li's model. It is the simplification that allows us to obtain the Z-transform expressions of the solutions to the dynamics equations for order policy systems. Our work on the bullwhip effect in supply chain networks extends or complements the above works.

The optimization of supply chain networks is one of the most important tasks in supply chain management. Santoso et al. (2005) develops a practical methodology for large-scale supply chain network design problems under uncertainty and provide an efficient framework for identifying and statistically testing a variety of candidate design solutions. Altiparmak et al. (2006) proposes a new solution procedure based on genetic algorithms to find the set of Pareto-optimal solutions for multi-objective supply chain network design problems. Herty and Ringhofer (2007) develops a methodology based on a fluid dynamic model for a supply chain to investigate optimal dynamic policies. Kaplan et al. (2007) presents an optimization model for integrated multi-product and multi-echelon supply chain networks with price elasticity. Sheremetov and Rocha-Mier (2008) investigates the problem of dynamic optimization of the supply chain network within the framework of the collective intelligence theory. Shukla et al. (2010) proposes a hybrid approach incorporating simulation, Taguchi methods, robust multiple non-linear regression analysis and the Psychoclonal algorithm to identify the optimal operating conditions for supply chain networks. Gottlich et al. (2010) presents a model for a production network with order and distribution policies and money flow, and the distribution and order rates are determined by an optimization problem for maximizing the money flow where the discretized maximization problem is solved by mixed-integer programming. Different from the above-mentioned works on the optimization of supply chains, we derive the exact expression of the optimal vector solution to the optimization model for the horizontal collaboration supply chain network.

On the topic transshipment problem, Rudi et al. (2001) examines how the possibility of transshipment between two independent locations (sellers) affects the optimal inventory orders at each location, and find transshipment prices that induce the locations to choose inventory levels consistent with joint profit maximization. Dong and Rudi (2004) further shows that the impact of transshipment on the manufacturer and that on the retailers are very different depending on whether the manufacturer is a price taker or a price setter. Zhao et al. (2006) proves that the optimal inventory and transshipment decisions for an individual dealer are controlled by threshold rationing and requested levels. Hanany et al. (2010) proposes a new mechanism based on a contract between the retailers and a transshipment fund, and show that the mechanism strongly coordinates the system, i.e., achieves coordination as the unique equilibrium. These papers and other related literature (see the references in Hanany et al. (2010)) deal mainly with transshipment problems in a single-period supply chain network. We, on the other hand, are concerned with how horizontal collaboration can affect the stability, bullwhip effect and optimization in multiple-period (dynamic) supply chain networks.

3. Modeling Supply Chain Networks

3.1. **Competition and horizontal collaboration networks.** Consider a supply chain network with one manufacturer, several distributors and many retailers (or specialty stores). If there is no mutual order goods between the distributors or retailers (see Fig. 1) we call it a *competition or non-horizontal collaboration* supply network since any two retailers or distributors can be regarded as the competitors. If there is a small quantity of mutual ordering goods between the distributors or retailers (see Fig. 2), we call it an easily executed *horizontal collaboration* supply chain network. As can be seen, if one removes all collaborations in Fig. 2, what remains is the *competition* supply network. Note that there is no information sharing in both the competition and horizontal collaboration supply chain networks.

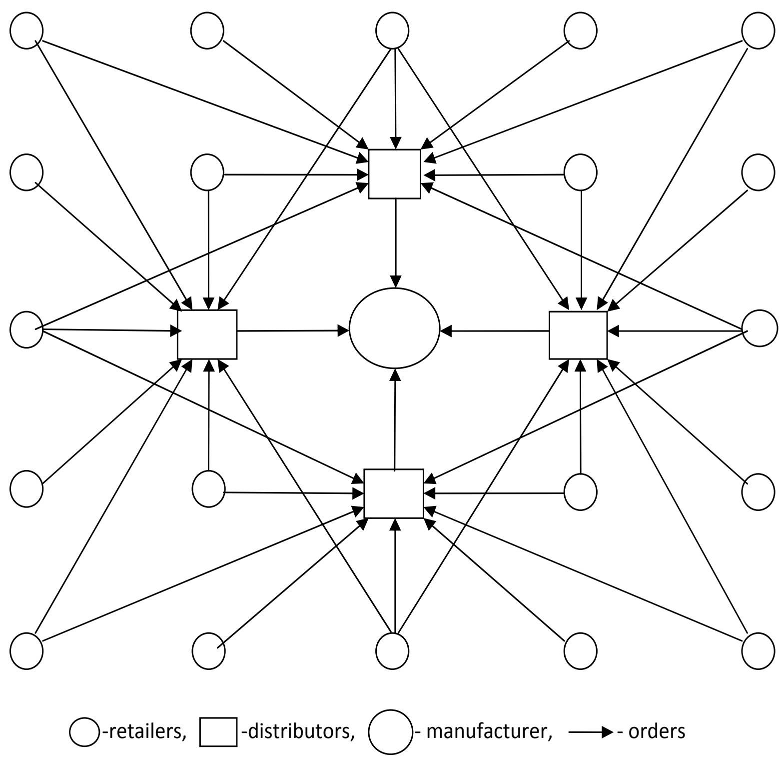

FIGURE 1. The competition supply network

Let $S = S_3 \cup S_2 \cup S_1 \cup S_0$, where $S_3 = \{m\}$ denotes the manufacturer, and $S_2 = \{d_1, \ldots, d_k\}$, $S_1 = \{r_1, \ldots, r_n\}$ and $S_0 = \{c_1, \ldots, c_n\}$ denote respectively the set of distributors, the set of retailers and the set of customers. Note that all customers ordering goods from the retailer r_i are denoted by c_i $(i = 1, \ldots, n)$. Let $x_i(t), y_i(t)$, $i \in S_2 \cup S_1$ and $z_{ab}(t)$, $(a, b \in S)$ denote respectively inventory position, in-hand inventory and the items ordered by a from b at discrete time $t = 0, 1, \ldots$. Since the manufacturer does not order the items from distributors, retailers and customers, distributors do not order from retailers or customers, and retailers do not order from customers, it follows that $z_{ab}(t) = 0$ for $a \in S_3$, $b \in S_2 \cup S_1 \cup S_0$, $a \in S_2$, $b \in S_1 \cup S_0$, and $a \in S_1, b \in S_0$.

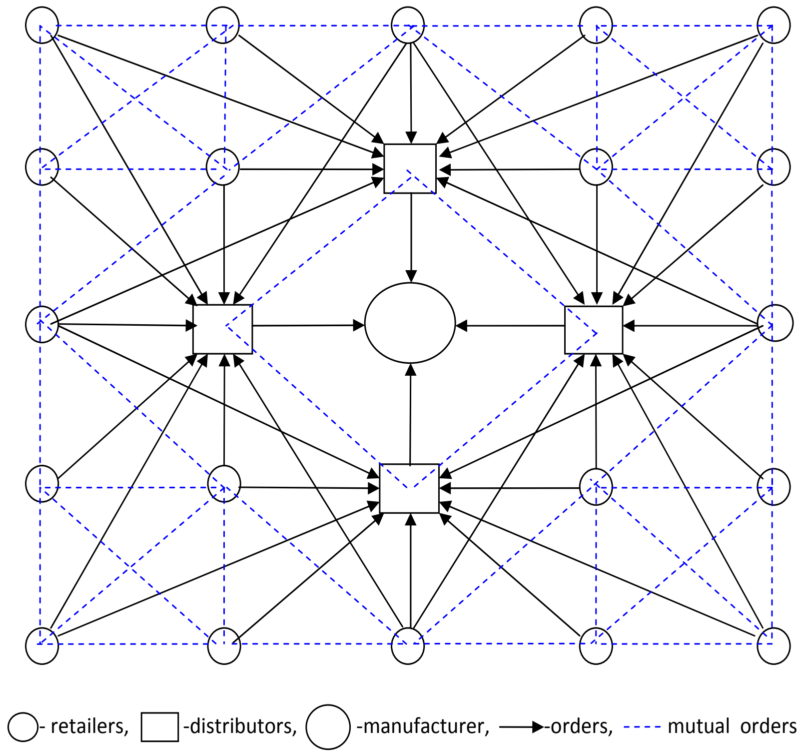

FIGURE 2. The horizontal collaboration network

Like Ouyang and Li (2010), we write the conservation equations for the inventory position $x_i(t)$ and the in-hand inventory $y_i(t)$ for $i \in S_j$, $j = 1, 2$, in the competition supply network as

$$x_i(t+1) = x_i(t) + \sum_{k \in S_{j+1}} z_{ik}(t) - \sum_{k \in S_{j-1}} z_{ki}(t) \tag{3.1}$$

and

$$y_i(t+1) = y_i(t) + \sum_{k \in S_{j+1}} z_{ik}(t - l_{ik}) - \sum_{k \in S_{j-1}} z_{ki}(t), \tag{3.2}$$

where $e_{ik} = 0, 1, 2, \ldots$ denotes the lead time by which i receives the ordered items from k. The conservation equations for the horizontal collaboration network can be similarly written as

$$x_i(t+1) = x_i(t) + \sum_{k \in S_{j+1}} z_{ik}(t) - \sum_{k \in S_{j-1}} z_{ki}(t) + \sum_{i \neq k \in S_j} z_{ik}(t) - \sum_{i \neq k \in S_j} z_{ki}(t) \tag{3.3}$$

and

$$y_i(t+1) = y_i(t) + \sum_{k \in S_{j+1}} z_{ik}(t - e_{ik}) - \sum_{k \in S_{j-1}} z_{ki}(t)$$

$$+\sum_{i\neq k\in S_j} z_{ik}(t-e_{ik})-\sum_{i\neq k\in S_j} z_{ki}(t) \tag{3.4}$$

3.2. **The dynamics equations for order policies.** Similar to Daganzo (2004) and Ouyang and Li (2010), we assume that the order $z_{ik}(t)$ can be described linearly by its past order and inventory history. Without loss generality we assume that $x_a(t)=y_a(t)=0$ for $t\leq 0$ and $z_{ab}(t)=0$ for $t<0$, $a,b\in S$. Let $A_i(p)$, $B_i(p)$, $C_{il}(p)$, $D_{li}(p)$, $U_{il}(p)$ and $V_{li}(p)$ be six polynomials with respect to the unit shift operator p for a time series.

(I) The set of order equations for the competition supply network is

$$\begin{aligned} z_{ik}(t) &= A_i(p)x_i(t)+B_i(p)y_i(t)+\sum_{l\in S_{j+1}} C_{il}(p)z_{il}(t-1) \\ &\quad +\sum_{l\in S_{j-1}} D_{li}(p)z_{li}(t-1)+\psi_{ik}(t) \end{aligned} \tag{3.5}$$

for $i\in S_j$, $k\in S_{j+1}$, $j=1,2$, where $\{\psi_{ik}(t),\, t\geq 0\}$ is a random variable series with $E(\psi_{ik}(t))=\mu_{ik}(t)$ and $Var(\psi_{ik}(t))=\sigma^2_{ik}(t)$, which can be considered as the random error series of the order policy system.

(II) The set of order equations for the horizontal collaboration network is

$$\begin{aligned} \hat{z}_{ik}(t) &= A_i(p)\hat{x}_i(t)+B_i(p)\hat{y}_i(t)+\sum_{l\in S_{j+1}} C_{il}(p)\hat{z}_{il}(t-1)+\sum_{l\in S_{j-1}} D_{li}(p)\hat{z}_{li}(t-1) \\ &\quad +\sum_{i\neq l\in S_j} U_{il}(p)\hat{z}_{il}(t-1)+\sum_{i\neq l\in S_j} V_{li}(p)\hat{z}_{li}(t-1)+\hat{\psi}_{ik}(t) \end{aligned} \tag{3.6}$$

for $i\in S_j$, $k\in S_{j+1}$, $j=1,2$, where $\{\hat{\psi}_{ik}(t), t\geq 0\}$ is a random variable series with $E(\hat{\psi}_{ik}(t))=\hat{\mu}_{ik}(t)$, $Var(\hat{\psi}_{ik}(t))=\hat{\sigma}^2_{ik}(t)$, which denotes the random error of the order policy system, and $\{\hat{x}_i(t)\}, \{\hat{y}_i(t)\}$ and $\{\hat{z}_{ik}(t)\}$ satisfy (3.3) and (3.4).

Since the mutual order quantities between retailers or distributors considered in the paper is relative small in the horizontal collaboration network, we may assume that the order by retailer (distribution) i from retailer (distribution) l at time t only dependents on the order by retailer (distribution) i from all distributions (manufacturer) and the order by all customers (retailers) from retailer (distribution) i at time $t-1$. That is, the mutual order between distributors or retailers is assumed to satisfy the following equations

$$\hat{z}_{il}(t)=\sum_{k\in S_{j+1}} \eta_{ik}(p)\hat{z}_{ik}(t-1)+\sum_{k\in S_{j-1}} \zeta_{ki}(p)\hat{z}_{ki}(t-1) \tag{3.7}$$

for $i\neq l$, $i,l\in S_j$, $j=1,2$, where both $\eta_{ik}(p)$ and $\zeta_{ki}(p)$ are two polynomials with respect to the unit shift operator p for a time series. To guarantee the order $\hat{z}_{il}(t)$ in

(3.7) is relative small, we assume that the following numbers

$$|\eta_{il}(p)| = \sum_{j=1}^{p} |c_{ilj}(\eta)|, \quad |\zeta_{li}(p)| = \sum_{j=1}^{p} |c_{lij}(\zeta)|$$

are small, where both $\{c_{ilj}(\eta), 1 \leq j \leq p\}$ and $\{c_{ilj}(\zeta), 1 \leq j \leq p\}$ are the coefficients of two polynomials $\eta_{il}(p)$ and $\zeta_{li}(p)$, respectively.

Note that it is assumed that the upstream has ample stock to satisfy downstream demand. Moreover, the above order policy allows the order $z_{ik}(t)$ (or $\hat{z}_{ik}(t)$) to be negative. In such a case, we assume, like Lee et al. (1997) and Chen et al. (2000), that excess inventory can be returned without a cost penalty.

4. **Stability of Order Equations**

The first problem we are concerned with is the stability of the order policy system. The stability means that the order $z_{ik}(t)$ is bounded for all time t as long as the market demand is bounded. Solving equations (3.5) and (3.6), we can obtain the necessary and sufficient conditions for the stability of the competition and horizontal collaboration supply chain networks, respectively.

Denote the Z-transforms (Graf 2004) of $\{x_i(t)\}$, $\{\hat{x}_i(t)\}$, $\{y_i(t)\}$, $\{\hat{y}_i(t)\}$, $\psi_{ik}(t)$, $\hat{\psi}_{ik}(t)$, $\{z_{ik}(t)\}$ and $\{\hat{z}_{ik}(t)\}$ by $X_i(z)$, $\hat{X}_i(z)$, $Y_i(z)$, $\hat{Y}_i(z)$, $\Psi_{ik}(z)$, $\hat{\Psi}_{ik}(z)$, $Z_{ik}(z)$ and $\hat{Z}_{ik}(z)$, respectively.

For the competition supply network, it follows from (3.1), (3.2) and (3.5) that

$$X_i(z) = \frac{1}{z-1}[\sum_{l\in S_{j+1}} Z_{il}(z) - \sum_{l\in S_{j-1}} Z_{li}(z)] \tag{4.1}$$

$$Y_i(z) = \frac{1}{z-1}[\sum_{l\in S_{j+1}} z^{e_{il}} Z_{il}(z) - \sum_{l\in S_{j-1}} Z_{li}(z)] \tag{4.2}$$

$$\begin{aligned} Z_{ik}(z) &= A_i(z^{-1})X_i(z) + B_i(z^{-1})Y_i(z) + \sum_{l\in S_{j+1}} z^{-1}C_{il}(z^{-1})Z_{il}(z) \\ &\quad + \sum_{l\in S_{j-1}} z^{-1}D_{li}(z^{-1})Z_{li}(z) + \Psi_{ik}(z) \end{aligned} \tag{4.3}$$

for $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$. By (4.1) and (4.2) we can rewrite (4.3) as

$$Z_{ik}(z) = \sum_{l\in S_{j+1}} \alpha_{il}(z)Z_{il}(z) + \sum_{l\in S_{j-1}} \beta_{li}(z)Z_{li}(z) + \Psi_{ik}(z) \tag{4.4}$$

for $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$, where

$$\alpha_{il}(z) = \frac{A_i(z^{-1})}{z-1} + \frac{z^{-e_{il}}B_i(z^{-1})}{z-1} + \frac{C_{il}(z^{-1})}{z} \tag{4.5}$$

$$\beta_{li}(z) = \frac{D_{li}(z^{-1})}{z} - \frac{A_i(z^{-1})}{z-1} - \frac{B_i(z^{-1})}{z-1}. \tag{4.6}$$

Without loss of generalization, we assume that $\alpha_{il}(z)$ and $\beta_{li}(z)$ are not both equal to zero.

By taking $l' = c_i \in S_0$, $l = r_j \in S_1$ and $i \neq j$, we have $z_{l'l}(t) = 0$ for all t, and therefore, $Z_{l'l}(z) = 0$. Since $z_{c_j r_j}(t)$ can be considered as the demand of customer c_j (the set of all customers who order or buy goods from retailer r_j) at time t, we denote $z_{c_j r_j}(t)$ and $Z_{c_j r_j}(z)$ simply by $z_{c_j}(t)$ and $Z_{c_j}(z)$ respectively.

Theorem 1. *The order policy for the competition supply network in* (4.4) *is stable in time if and only if all zeros of the following equations*

$$(1 - z^{-1})(1 - \sum_{l \in S_{j+1}} \alpha_{il}(z)) = 0, \quad i \in S_j,\ j = 1, 2, \tag{4.7}$$

are located inside the unit circle of the complex z-plane. Furthermore, if the order policy is stable, then the Z-transform, $Z_{d_i m}(z)$, of the order $\{z_{d_i m}(t)\}$ placed by distributor $d_i \in S_2$ with manufacturer m, has the following form

$$\begin{aligned} Z_{d_i m}(z) &= \sum_{l=1}^{n} \frac{\beta_{r_l d_i}(z)}{(1 - \alpha_{d_i m}(z))} \frac{\beta_{c_l r_l}(z) Z_{c_l}(z)}{(1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z))} \\ &\quad + \sum_{l=1}^{n} \frac{\beta_{r_l d_i}(z)}{(1 - \alpha_{d_i m}(z))} \frac{\sum_{j=1}^{k} \alpha_{r_l d_j}(z) \Psi_{r_l d_j}(z)}{(1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z))} \\ &\quad + \frac{\Psi_{d_i m}(z) + \sum_{l=1}^{n} \beta_{r_l d_i}(z) \Psi_{r_l d_i}(z)}{1 - \alpha_{d_i m}(z)} \end{aligned} \tag{4.8}$$

for $1 \leq i \leq k$.

Now we consider the stability of the order policy for the horizontal collaboration network. By (3.3), (3.4) and (3.6) we can similarly obtain

$$\begin{aligned} \hat{Z}_{ik}(z) &= \sum_{l \in S_{j+1}} \alpha_{il}(z) \hat{Z}_{il}(z) + \sum_{l \in S_{j-1}} \beta_{li}(z) \hat{Z}_{li}(z) \\ &\quad + \sum_{i \neq l \in S_j} \gamma_{il}(z) \hat{Z}_{il}(z) + \sum_{i \neq l \in S_j} \delta_{li}(z) \hat{Z}_{li}(z) + \hat{\Psi}_{ik}(z) \end{aligned} \tag{4.9}$$

for $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$, where

$$\gamma_{il}(z) = \frac{A_i(z^{-1})}{z - 1} + \frac{z^{-e_{il}} B_i(z^{-1})}{z - 1} + \frac{U_{il}(z^{-1})}{z} \tag{4.10}$$

$$\delta_{li}(z) = \frac{V_{li}(z^{-1})}{z} - \frac{A_i(z^{-1})}{z - 1} - \frac{B_i(z^{-1})}{z - 1}. \tag{4.11}$$

Plugging the Z-transforms of (3.7) into (4.9) we get

$$\begin{aligned} \hat{Z}_{ik}(z) &= \sum_{l \in S_{j+1}} \tilde{\alpha}_{il}(z)) \hat{Z}_{il}(z) + \sum_{l \in S_{j-1}} \tilde{\beta}_{li}(z) \hat{Z}_{li}(z) \\ &\quad + \sum_{l \in S_{j+1}} \sum_{i \neq l' \in S_j} \hat{\alpha}_{il'l}(z) \hat{Z}_{l'l}(z) + \sum_{l \in S_{j-1}} \sum_{i \neq l' \in S_j} \hat{\beta}_{ill'} \hat{Z}_{ll'}(z) + \hat{\Psi}_{ik}(z) \end{aligned} \tag{4.12}$$

for $i \in S_j$, $j = 1, 2$, where

$$\tilde{\alpha}_{il}(z) = \alpha_{il}(z) + z^{-1}\eta_{il}(z^{-1}) \sum_{i \neq l' \in S_j} \gamma_{il'}(z), \quad \hat{\alpha}_{il'l}(z) = \delta_{l'i}(z) z^{-1} \eta_{l'l}(z^{-1}) \tag{4.13}$$

$$\tilde{\beta}_{li}(z) = \beta_{li}(z) + z^{-1}\zeta_{li}(z^{-1}) \sum_{i \neq l' \in S_j} \gamma_{il'}(z), \quad \hat{\beta}_{ill'}(z) = \delta_{l'i}(z) z^{-1} \zeta_{ll'}(z^{-1}). \tag{4.14}$$

In order to obtain the closed form solution to equation (4.12), we assume that $\delta_{l'i}(z)$ does not depend on i, that is,

$$\delta_{l'i}(z) = \delta_{l'}(z). \tag{4.15}$$

for $i, l' \in S_j$, $j = 1, 2$. To guarantee (4.15) we assume in the following discussion that $V_{l'i}(z^{-1}) = V_{l'}(z^{-1})$,

$$A_{d_i}(z^{-1}) + B_{d_i}(z^{-1}) = A_{d_1}(z^{-1}) + B_{d_1}(z^{-1}) \tag{4.16}$$

for $1 \leq i \leq k$, and

$$A_{r_l}(z^{-1}) + B_{r_l}(z^{-1}) = A_{r_1}(z^{-1}) + B_{r_1}(z^{-1}) \tag{4.17}$$

for $1 \leq l \leq n$. Of course (4.15) can be true in other cases. For example, if $\frac{A_i(z^{-1})}{z-1} + \frac{B_i(z^{-1})}{z-1}$ is a polynomial and we take $V_{l'i}(z^{-1}) = z\frac{A_i(z^{-1})}{z-1} + z\frac{B_i(z^{-1})}{z-1} + V_{l'}(z^{-1})$, then (4.15) holds, where $V_{l'}(p)$ is a polynomial with respect to the unit shift operator p for a time series. Here we do not consider this case since the main results for this case are similar to those of the previous case.

Let $\hat{\alpha}_{l'l}(z) = \hat{\alpha}_{il'l}(z)$ and $\hat{\beta}_{ll'}(z) = \hat{\beta}_{ill'}(z)$ if (4.15) holds.

Theorem 2. *Let* (4.15) *be true. Then the order policy for the horizontal collaboration supply network in* (4.12) *is stable in time if and only if all zeros of the following equations*

$$(1 - z^{-1})\Big(1 - \sum_{l \in S_{j+1}} [\tilde{\alpha}_{il}(z) - \hat{\alpha}_{il}(z)]\Big)\Big(1 - \sum_{l' \in S_j} \frac{\sum_{l \in S_{j+1}} \hat{\alpha}_{l'l}(z)}{1 - \sum_{l \in S_{j+1}} [\tilde{\alpha}_{l'l}(z) - \hat{\alpha}_{l'l}(z)]}\Big) = 0 \tag{4.18}$$

for $i \in S_j$, $j = 1, 2$ are located inside the unit circle of the complex z-plane. If the order policy is stable, then the Z-transform, $\hat{Z}_{d_im}(z)$, of the order $\{\hat{z}_{d_im}(t)\}$ can be expressed as

$$\begin{aligned}\hat{Z}_{d_im}(z) &= \frac{\sum_{l=1}^{n} [\tilde{\beta}_{r_ld_i}(z)\hat{Z}_{r_ld_i}(z) + \sum_{j \neq i} \hat{\beta}_{r_ld_j}(z)\hat{Z}_{r_ld_j}(z)] + \hat{\Psi}_{d_im}(z)}{1 - \tilde{\alpha}_{d_im}(z) + \hat{\alpha}_{d_im}(z)} \\ &\quad + \frac{\sum_{j=1}^{k}[\sum_{l=1}^{n}(\tilde{\beta}_{r_ld_j}(z)\hat{Z}_{r_ld_j}(z) + \sum_{j' \neq j} \hat{\beta}_{r_ld'_j}(z)\hat{Z}_{r_ld'_j}(z)) + \hat{\Psi}_{d_jm}(z)]\kappa_j(z)}{(1 - \tilde{\alpha}_{d_im}(z) + \hat{\alpha}_{d_im}(z))(1 - \sum_{j=1}^{k} \kappa_j(z))}\end{aligned} \tag{4.19}$$

where

$$\hat{Z}_{r_l d_1}(z) = \frac{\tilde{\beta}_{c_l r_l}(z) Z_{c_l}(z) + \sum_{j\neq l} \hat{\beta}_{c_j r_j}(z) Z_{c_j}(z) + \hat{b}_l(z)}{1 - \sum_{j=1}^k (\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z))} + \frac{b(z) + \sum_{i=1}^n [\tilde{\beta}_{c_i r_i}(z) Z_{c_i}(z) + \sum_{j\neq i} \hat{\beta}_{c_j r_j}(z) Z_{c_j}(z) + \hat{b}_i(z)]\hat{\kappa}_i(z)}{(1 - \sum_{j=1}^k [\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z)])(1 - \sum_{j=1}^n \hat{\kappa}_j(z))},$$

$$\hat{Z}_{r_l d_j}(z) = \hat{Z}_{r_l d_1}(z) + \hat{\Psi}_{r_l d_j}(z) - \hat{\Psi}_{r_l d_1}(z), \quad 2 \le j \le k, \tag{4.20}$$

and $b(z) = \sum_{i=1}^n \sum_{j=1}^k (\hat{\Psi}_{r_i d_j}(z) - \hat{\Psi}_{r_i d_1}(z))\hat{\alpha}_{r_i d_j}(z)$,

$$\kappa_i(z) = \frac{\hat{\alpha}_{d_i m}(z)}{1 - \tilde{\alpha}_{d_i m}(z) + \hat{\alpha}_{d_i m}(z)}, \quad \hat{\kappa}_l(z) = \frac{\sum_{j=1}^k \hat{\alpha}_{r_l d_j}(z)}{1 - \sum_{j=1}^k (\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z))} \tag{4.21}$$

$$\hat{b}_l(z) = \hat{\Psi}_{r_l d_1}(z) + \sum_{j=1}^k (\hat{\Psi}_{r_l d_j}(z) - \hat{\Psi}_{r_l d_1}(z))(\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z))$$

for $1 \le i \le k$, $1 \le l \le n$.

As can be seen, the stability does not depend on the polynomial $\zeta_{il}(p)$, $i \in S_{j-1}$, $l \in S_j$, $j = 1, 2$. Next we discuss the stability of the two order policy systems. Let the order policy for the competition supply network in (4.4) be stable in time. It follows from (4.7) in Theorem 1 that

$$\begin{aligned} a_i(z) &\triangleq (1 - z^{-1})(1 - \sum_{l \in S_{j+1}} \alpha_{il}(z)) \\ &= 1 - z^{-1} - \sum_{l \in S_{j+1}} [z^{-1} A_i(z^{-1}) + z^{-e_{il}-1} B_i(z^{-1}) + (z-1) z^{-2} C_{il}(z^{-1})] \neq 0 \end{aligned}$$

for $i \in S_j$, $j = 1, 2$, and all $|z| \ge 1$. Let $a_i = \min_{|z|\ge 1} |a_i(z)|$. Note that A_i, B_i and C_{il} are three polynomial functions of z^{-1}, and $\lim_{z\to\infty} a_i(z) = 1$. Hence $a_i \neq 0$. It follows from (4.10), (4.11) and (4.13) that

$$\begin{aligned} b_i(z) &\triangleq \sum_{l \in S_{j+1}} (1 - z^{-1})[\tilde{\alpha}_{il}(z) - \alpha_{il}(z) - \hat{\alpha}_{il}(z)] \\ &= z^{-2} \sum_{l \in S_{j+1}} \eta_{il}(z^{-1}) \Big((1 - z^{-1})[\sum_{i \neq l' \in S_j} U_{il'}(z^{-1}) - V_i(z^{-1})] + A_i(z^{-1}) + B_i(z^{-1}) \\ &\quad \sum_{i \neq l' \in S_j} (A_i(z^{-1}) + z^{-e_{il'}} B_i(z^{-1})) \Big) \end{aligned}$$

for $i \in S_j$, $j = 1, 2$. Thus, we can take small values of $|\eta_{il}(p)|$ for $i \in S_j$, $l \in S_{j+1}$ $j = 1, 2$, such that $a_i(z) - b_i(z) \neq 0$ for all $|z| \ge 1$ and $i \in S_j$, $j = 1, 2$. Similarly, we can take small values of $|\eta_{il}(p)|$ for $i \in S_j, l \in S_{j+1}$ $j = 1, 2$ such that

$$c_j(z) \triangleq 1 - \sum_{l' \in S_j} \frac{\sum_{l \in S_{j+1}} \hat{\alpha}_{l'l}(z)}{1 - \sum_{l \in S_{j+1}} [\tilde{\alpha}_{l'l}(z) - \hat{\alpha}_{l'l}(z)]} \neq 0$$

for all $|z| \geq 1$. That is, the equations in (4.18) satisfy $(a_i(z) - b_i(z))c_j(z) \neq 0$ for all $|z| \geq 1$ and $i \in S_j$, $j = 1, 2$. Thus, if the order policy system for the competition supply network is stable, then the order policy system for the horizontal collaboration supply network is also stable as long as we take suitably small values of $|\eta_{il}(p)|$ for $i \in S_j$, $l \in S_{j+1}$ $j = 1, 2$.

Let the order policy for the competition supply network be unstable over time. For example, there is a z_0 ($|z_0| \geq 1$) such that $a_i(z_0) = 0$ for some $i \in S_j$. Assume that

$$A_{d_1}(1) + B_{d_1}(1) \neq 0, \quad A_{r_1}(1) + B_{r_1}(1) \neq 0. \tag{4.22}$$

Hence, $z_0 \neq 1$ since $a_i(1) = |S_{j+1}|(A_i(1) + B_i(1)) \neq 0$ for $i \in S_j$, $j = 1, 2$, where $|S_{j+1}|$ is the number of elements of set S_{j+1}. Take the polynomials $\eta_{il}(p)$ such that $\sum_{l \in S_{j+1}} \eta_{il}(z^{-1}) \neq 0$ for $i \in S_j$, $j = 1, 2$ and all $|z| \geq 1$. If $b_i(z_0) \neq 0$, we can take small values of $|\eta_{il}(p)|$ for $i \in S_j$, $l \in S_{j+1}$ $j = 1, 2$, such that $(a_i(z_0) - b_i(z_0))c_j(z_0) = -b_i(z_0)c_j(z_0) \neq 0$. If $b_i(z_0) = 0$, we can take suitable values of $U_{il'}(p)$ and $V_i(p)$, and small values of $|\eta_{i'l}(p)|$ for $i \neq i' \in S_j$, $l \in S_{j+1}$ $j = 1, 2$ such that $a_{i'}(z_0) - b_{i'}(z_0) \neq 0$ for $i' \neq i$, and therefore

$$\begin{aligned}(a_i(z_0) - b_i(z_0))c_j(z_0) &= (1 - z_0^{-1}) \sum_{l \in S_{j+1}} \hat{\alpha}_{il}(z_0) \\ &= z_0^{-2}(z_0 - 1)\delta_i(z_0) z_0^{-2} \sum_{l \in S_{j+1}} \eta_{il}(z_0^{-1}) \neq 0\end{aligned}$$

as long as $\delta_i(z_0) \neq 0$. In fact, we can take the polynomials $V_i(p)$ such that $\delta_i(z) \neq 0$ for all $|z| \geq 1$ and $i \in S_j$, $j = 1, 2$. That is, an unstable order policy for the competition supply network can be made stable in the horizontal collaboration supply network. We sum up the discussion above in the following corollary.

Corollary 1. *Let (4.22) be true. A stable (unstable) order policy system for the competition supply network can remain (become) stable in the horizontal collaboration supply network as long as we adopt suitable order policies $\{\eta_{il}(p)\}$, $\{U_{il}(p)\}$ and $\{V_i(p)\}$.*

5. **The Bullwhip Effect**

Let the customer demand $z_{c_l}(t)$ be a random variable. The sum $\sum_{l=1}^{n} z_{c_l}(t)$ can be considered as the market demand at time t. The sum $\sum_{i=1}^{k} z_{d_i m}(t)$ is the total order of distributors, d_i, $1 \leq i \leq k$ at time t, which can be considered as the output of the manufacturer m at time t. According to the definition of the bullwhip effect (Lee et al. 1997), let

$$BW(T) = \frac{\sum_{t=0}^{T} Var(\sum_{i=1}^{k} z_{d_i m}(t))}{\sum_{t=0}^{T} Var(\sum_{l=1}^{n} z_{c_l}(t))},$$

where $Var(X)$ denotes the variance of random variable X. We say that there is no bullwhip effect in the supply chain network after time T_0 if $BW(T) \leq 1$ for $T \geq T_0$.

By (4.8) and (4.19)-(4.21), both $\sum_{i=1}^k Z_{d_i m}(z)$ and $\sum_{i=1}^k \hat{Z}_{d_i m}(z)$ of solutions (4.8) and (4.19) can be written as

$$\begin{aligned}(5.1)\qquad &\sum_{i=1}^k Z_{d_i m}(z)\\ &= \sum_{l=1}^n \lambda_l(z) Z_{c_l}(z) + D(z), \;\; \sum_{i=1}^k \hat{Z}_{d_i m}(z) = \sum_{l=1}^n \hat{\lambda}_l(z) Z_{c_l}(z) + \hat{D}(z),\end{aligned}$$

where

$$(5.2)\qquad \lambda_l(z) = \frac{\beta_{c_l r_l}(z)}{1-\sum_{j=1}^k \alpha_{r_l d_j}(z)} \sum_{i=1}^k \frac{\beta_{r_l d_i}(z)}{1-\alpha_{d_i m}(z)}$$

$$\begin{aligned}(5.3)\qquad \hat{\lambda}_l(z) &= \rho_l(z)(\tilde{\beta}_{c_l r_l}(z) - \hat{\beta}_{c_l r_l}(z))\\ &\quad + \frac{\sum_{l'=1}^n \rho_{l'}(z)}{1-\sum_{j=1}^k \hat{\kappa}_j(z)}[(\tilde{\beta}_{c_l r_l}(z) - \hat{\beta}_{c_l r_l}(z))\hat{\kappa}_l(z) + \hat{\beta}_{c_l r_l}(z)]\end{aligned}$$

$$\begin{aligned}\rho_l(z) &= \frac{1}{1-\sum_{j=1}^k(\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z))} \sum_{i=1}^k \frac{\tilde{\beta}_{r_l d_i}(z) + \sum_{j\neq i} \hat{\beta}_{r_l d_j}(z)}{1-\tilde{\alpha}_{d_i m}(z) + \hat{\alpha}_{d_i m}(z)}\\ &\quad + \frac{\sum_{j=1}^k (1-\tilde{\alpha}_{d_j m}(z) + \hat{\alpha}_{d_j m}(z))^{-1} (\sum_{j=1}^k (\tilde{\beta}_{r_l d_j}(z) + \sum_{j'\neq j} \hat{\beta}_{r_l d_{j'}}(z))\kappa_j(z))}{(1-\sum_{j=1}^k \kappa_j(z))(1-\sum_{j=1}^k [\tilde{\alpha}_{r_l d_j}(z) - \hat{\alpha}_{r_l d_j}(z)])}\end{aligned}$$

for $1 \leq l \leq n$, and both $D(z)$ and $\hat{D}(z)$ are the linear functions of $\{\Psi_{ik}(z)\}$ and $\{\hat{\Psi}_{ik}(z)\}$, respectively. This means that the variance of the output of the manufacturer, $\sum_{i=1}^k Z_{d_i m}(z)$, consists of two parts: one is that of the market demand, the other comes from the error of the order policy system. In this section we only consider the bullwhip effect produced by the variance of the market demands $\sum_{l=1}^n \lambda_l(z) Z_{c_l}(z)$ and $\sum_{l=1}^n \hat{\lambda}_l(z) Z_{c_l}(z)$, so that we assume that $\sigma_{ik}^2(t) = 0$ and $\hat{\sigma}_{ik}^2(t) = 0$ for $t \geq 0$, $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$. This means that both $\{\psi_{ik}(t)\}$ and $\{\hat{\psi}_{ik}(t)\}$ for $t \geq 0$, $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$, are deterministic number series, and therefore, $D(z)$ and $\hat{D}(z)$ are two deterministic functions, that is, $Var(D(z)) = Var(\hat{D}(z)) = 0$.

Lemma 1. *Let*

$$(5.4)\qquad \lambda_l(z) = \sum_{j=0}^\infty f_{lj} z^{-j}, \quad \hat{\lambda}_l(z) = \sum_{j=0}^\infty \hat{f}_{lj} z^{-j}.$$

Then

$$(5.5)\qquad \sum_{j=0}^\infty f_{lj} = \lim_{z\to 1} \lambda_l(z) = 1, \quad \sum_{j=0}^\infty \hat{f}_{lj} = \lim_{z\to 1} \hat{\lambda}_l(z) = 1 + Q_l$$

for $1 \le l \le n$*, where* Q_l *is a function which only depends on* $\{\eta_{r_l d_i}\}$, $\{\eta_{d_i m}\}$, $\{\zeta_{r_l d_i}\}$ *and* $\{\zeta_{c_l r_l}\}$*. In particular, when* $\eta_{r_l d_i}(1) = 0$, $\eta_{d_i m}(1) = 0$, $\zeta_{r_l d_i}(1) = 0$ *and* $\zeta_{c_l r_l}(1) = 0$*, we have* $Q_l = 0$*, and therefore* $\sum_{j=0}^{\infty} \hat{f}_{lj} = 1$*.*

Theorem 3. *Suppose that*

$$\sum_{t \ge \max\{i,j\}}^{T} Cov(z_{c_l}(t-i), z_{c_{l'}}(t-j)) \le \sum_{t=0}^{T} Cov(z_{c_l}(t), z_{c_{l'}}(t)) \tag{5.6}$$

for $1 \le l,\ l' \le n$, $0 \le i,\ j \le T$ *and* $T \ge T_0$*, and the coefficient series* $\{f_{lj}\}$ *in* (5.4) *satisfies* $f_{lj} \ge 0$ *for* $1 \le l \le n$, $j \ge 0$*, and*

$$\sum_{l=1}^{n} \sum_{l'=1}^{n} \left[1 - \left(\sum_{j=0}^{T} f_{lj} \right) \left(\sum_{j=0}^{T} f_{l'j} \right) \right] \left(\sum_{t=0}^{T} Cov(z_{c_l}(t), z_{c_{l'}}(t)) \right) \ge 0 \tag{5.7}$$

for $T \ge T_0$*. Then there is no bullwhip effect in the stable order policy for the competition supply network after time* T_0*.*

Remark 1. By (5.5) we know that $\left(\sum_{j=0}^{T} f_{lj}\right)\left(\sum_{j=0}^{T} f_{l'j}\right) \le 1$ for $1 \le l,\ l' \le n$. It follows that both (5.6) and (5.7) are true if $\sum_{t=0}^{T} Cov(z_{c_l}(t), z_{c_{l'}}(t)) \ge 0$ for $1 \le l \ne l' \le n$ and $\sum_{t \ge \max\{i,j\}}^{T} Cov(z_{c_l}(t-i), z_{c_{l'}}(t-j)) \le 0$ for $1 \le l,\ l' \le n$ and $0 \le i \ne j \le T$. This means that there is no bullwhip effect in the stable order policy for the competition supply network for all time as long as $f_{lj} \ge 0$ for $1 \le l \le n$, $j \ge 0$, and all covariances of customer demands are nonnegative at the same time and negative at different times. This result is similar to that of Alwan (2003).

We can obtain the same result as Theorem 3 for the horizontal collaboration supply network as long as we add the condition, $\sum_{j=0}^{\infty} \hat{f}_{lj} \le 1$.

Theorem 4. *Suppose that the condition* (5.6) *holds, and the coefficient series* $\{\hat{f}_{lj}\}$ *in* (5.4) *satisfies* $\hat{f}_{lj} \ge 0$ *for* $1 \le l \le n$, $j \ge 0$, $\sum_{j=0}^{\infty} \hat{f}_{lj} \le 1$ *for* $1 \le l \le n$*, and*

$$\sum_{l=1}^{n} \sum_{l'=1}^{n} \left[1 - \left(\sum_{j=0}^{T} \hat{f}_{lj} \right) \left(\sum_{j=0}^{T} \hat{f}_{l'j} \right) \right] \left(\sum_{t=0}^{T} Cov(z_{c_l}(t), z_{c_{l'}}(t)) \right) \ge 0 \tag{5.8}$$

for $T \ge T_0$*. Then there is no bullwhip effect in the stable order policy for the horizontal collaboration supply network after time* T_0*.*

In the following theorem we give other sufficient conditions for the bullwhip effect to exist.

Theorem 5. *Let two customer demands be unrelated and the demands at different times also be unrelated. If* $\sum_{j=0}^{\infty} f_{lj}^2 \le 1$ $\left(\sum_{j=0}^{\infty} \hat{f}_{lj}^2 \le 1\right)$ *for all* $1 \le l \le n$*, then there is no bullwhip effect in the stable order policy for the the competition (horizontal collaboration) supply network. If* $\sum_{j=0}^{\infty} f_{lj}^2 > 1$ $\left(\sum_{j=0}^{\infty} \hat{f}_{lj}^2 > 1\right)$ *for all* $1 \le l \le n$ *and the variances* $Var(z_{c_l}(t))$ *are bounded for* $t \ge 0$ *and* $1 \le l \le n$*, then the bullwhip* *effect exists in the stable order policy for the competition (horizontal collaboration) supply network.*

In other words, the result of Theorem 5 implies that if the bullwhip effect exists, then there is at least one l ($1 \le l \le n$) such that $\sum_{j=0}^{\infty} f_{lj}^2 > 1$ (or $\sum_{j=0}^{\infty} \hat{f}_{lj}^2 > 1$).

Remark 2. According to Disney and Towill (2003), $\sum_{j=0}^{\infty} f_{lj}^2$ and $\sum_{j=0}^{\infty} \hat{f}_{lj}^2$ can be expressed as

$$\sum_{j=0}^{\infty} f_{lj}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda_l(e^{i\omega})\lambda_l(e^{-i\omega})d\omega, \quad \sum_{j=0}^{\infty} \hat{f}_{lj}^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\lambda}_l(e^{i\omega})\hat{\lambda}_l(e^{-i\omega})d\omega,$$

where $i = \sqrt{-1}$. Thus, it can be said that the bullwhip effect in the horizontal collaboration supply network is smaller than that in the competition supply network as long as

$$\int_{-\pi}^{\pi}[\lambda_l(e^{i\omega})\lambda_l(e^{-i\omega}) - \hat{\lambda}_l(e^{i\omega})\hat{\lambda}_l(e^{-i\omega})]d\omega > 0.$$

We next present a close relation between $\{\hat{f}_{lj}\}$ and $\{f_{lj}\}$. To this end we first give a lemma which describes a property of $\{f_{lj}\}$. Let

$$[(1 - z^{-1})(1 - \alpha_{d_i m}(z))]^{-1} = \sum_{j=0}^{\infty} g_{ij} z^{-j},$$

$$\left[(1 - z^{-1})\left(1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z)\right)\right]^{-1} = \sum_{j=0}^{\infty} h_{lj} z^{-j}$$

and

$$\frac{\beta_{r_l d_i}(z)}{1 - \alpha_{d_i m}(z)} = \sum_{j=0}^{\infty} g_{lij} z^{-j}, \quad \frac{\beta_{c_l r_l}(z)}{1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z)} = \sum_{j=0}^{\infty} h'_{lj} z^{-j}$$

for $1 \le i \le k$, $1 \le l \le n$. Note that g_{ij}, g_{lij}, h_{lj} and h'_{lj} can be written as $g_{ij} = a_{ij}(g_i)^j$, $g_{lij} = a_{lij}(g_i)^j$, $h_{lj} = b_{lj}(h_l)^j$, $h'_{lj} = b'_{lj}(h_l)^j$, where g_i and h_l are the largest roots in absolute value respectively of the equations $(1 - z^{-1})(1 - \alpha_{d_i m}(z)) = 0$ and $(1 - z^{-1})(1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z)) = 0$.

Lemma 2. *Let* $g^* = \max_{1 \le i \le k} |g_i|$ *and* $h^* = \max_{1 \le l \le n} |h_i|$*. If* $g_{i_0 j}$ *satisfies* $a_{i_0 j} \neq 0$, $\lim_{j\to\infty} a_{i_0, j-k}/a_{i_0 j} = 1$ *for each fixed* k*, and*

$$|g_{i_0}| = g^* > h^*, \quad |g_{i_0}| > |g_{i'}|, \quad a_{i_0 j} \neq 0 \tag{5.9}$$

for $1 \le i' \neq i_0 \le k$*, then* f_{lj} *can be written as*

$$f_{lj} = c_{lj} g_{i_0 j} = c_{lj} a_{i_0 j} (g_{i_0})^j \tag{5.10}$$

for $j \ge 0$*, where* $\{c_{lj}\}$ *satisfies*

$$c_l \triangleq \lim_{j\to\infty} c_{lj} = (1 - 1/g_{i_0})\beta_{r_l d_{i_0}}(g_{i_0}) \frac{\beta_{c_l r_l}(g_{i_0})}{1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(g_{i_0})} \tag{5.11}$$

for $1 \leq l \leq n$.

Lemma 2 holds also for $\{\hat{f}_{lj}\}$ as long as g_{i_0} is replaced by $\hat{g}_{i_1}$ which is the largest root in absolute value among all roots of the equations $(1-z^{-1})(1-\tilde{\alpha}_{d_im}(z)+\hat{\alpha}_{d_im}(z))$ and $(1-z^{-1})(1-\sum_{j=1}^{k}[\tilde{\alpha}_{r_ld_j}(z)-\hat{\alpha}_{r_ld_j}(z)])$.

By the lemma we can now establish a special relation between $\{\hat{f}_{lj}\}$ and $\{f_{lj}\}$.

Theorem 6. *Let* $f_{lj} \neq 0, c_l \neq 0$ *for some* l $(1 \leq l \leq n)$ *and* $j \geq 0$, $\{g_{i_0j},\ j \geq 0\}$ *satisfy* (5.9). *Suppose that* $\hat{g}_{i_1} = g_{i_0}$. *Then, for any small positive number* $\epsilon < 1$ *we can choose the order policy of horizontal collaboration* $\{\eta_{r_ld_i},\ \eta_{d_im},\ \zeta_{r_ld_i},\ \zeta_{c_lr_l}\}$ *with small values of* $|\eta_{r_ld_i}(p)|$, $|\eta_{d_im}(p)|$, $|\zeta_{r_ld_i}(p)|$ *and* $|\zeta_{c_lr_l}(p)|$ *such that there exists a number series* ε_{lj} *for* $j \geq 0$ *which satisfies* $|\varepsilon_{lj}| \leq \epsilon$ *for* $j \geq 0$, $\lim_{j\to\infty}\varepsilon_{lj}$,

$$\hat{f}_{lj} = (1+\varepsilon_{lj})f_{lj}, \tag{5.12}$$

and

$$\sum_{j=0}^{\infty}\varepsilon_{lj}f_{lj} = Q_l. \tag{5.13}$$

As an application of Theorem 6, we discuss in the following the bullwhip effect in the competition and horizontal collaboration supply networks. By (5.12) we can choose two kinds of order policy for horizontal collaboration $\{\eta^{(1)}_{r_ld_i},\ \eta^{(1)}_{d_im},\ \zeta^{(1)}_{r_ld_i},\ \zeta^{(1)}_{c_lr_l}\}$ and $\{\eta^{(2)}_{r_ld_i},\ \eta^{(2)}_{d_im},\ \zeta^{(2)}_{r_ld_i},\ \zeta^{(2)}_{c_lr_l}\}$ such that the corresponding number series $\{\varepsilon^{(1)}_{lj}\}$ and $\{\varepsilon^{(2)}_{lj}\}$ satisfy

$$\sum_{j=0}^{\infty}(\hat{f}^{(s)}_{lj})^2 \begin{cases} = \sum_{j=0}^{\infty}(1+\varepsilon^{(1)}_{lj})^2 f_{lj}^2 > \sum_{j=0}^{\infty} f_{lj}^2 \\ = \sum_{j=0}^{\infty}(1+\varepsilon^{(2)}_{lj})^2 f_{lj}^2 < \sum_{j=0}^{\infty} f_{lj}^2, \end{cases} \tag{5.14}$$

where the coefficient series $\{\hat{f}^{(s)}_{lj}\}$ in (5.14) corresponds to the order policy of horizontal collaboration $\{\eta^{(s)}_{r_ld_i},\ \eta^{(s)}_{d_im},\ \zeta^{(s)}_{r_ld_i},\ \zeta^{(s)}_{c_lr_l}\}$ for $s = 1, 2$. Thus, we have the following corollary.

Corollary 2. *Let two customer demands be unrelated and the demands at different times also be unrelated. We can choose two kinds of stable order policy for horizontal collaboration such that the bullwhip effect in the stable order policy for the competition supply network can either be reduced or enhanced in the corresponding horizontal collaboration supply network.*

6. **Optimal Order Policy**

A good order policy should not only satisfy market demand, but also produce no bullwhip effect. Additionally, it should be stable. So the optimal order policy considered in the paper should not only satisfy the market demand best, but also keep the variance of the order to a minimum.

In this section we assume that $\psi_{ik}(t) = 0$ and $\hat{\psi}_{ik}(t) = 0$ for $t \geq 0$, $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$, that is, the order policy system doest not err. This implies that $D(z) = 0$ and $\hat{D}(z) = 0$. Let $C(t) = \sum_{l=1}^{n} z_{c_l}(t)$ and $M(t) = \sum_{i=1}^{k} z_{d_i m}(t)$, which can be considered as the market demand and output of the manufacturer at time t, respectively. It follows from (5.1)–(5.4) that

$$\begin{aligned}\sum_{i=1}^{k} Z_{d_i m}(z) &= \sum_{l=1}^{n} \lambda_l(z) Z_{c_l}(z) = \sum_{l=1}^{n}\left(\sum_{j=0}^{\infty} f_{lj} z^{-j}\right) Z_{c_l}(z)\\ \sum_{i=1}^{k} \hat{Z}_{d_i m}(z) &= \sum_{l=1}^{n} \hat{\lambda}_l(z) Z_{c_l}(z) = \sum_{l=1}^{n}\left(\sum_{j=0}^{\infty} \hat{f}_{lj} z^{-j}\right) Z_{c_l}(z).\end{aligned}$$

Hence

$$\sum_{i=1}^{k} z_{d_i m}(t) = \sum_{l=1}^{n}\sum_{j=0}^{t} f_{lj} z_{c_l}(t-j), \quad \sum_{i=1}^{k} \hat{z}_{d_i m}(t) = \sum_{l=1}^{n}\sum_{j=0}^{t} \hat{f}_{lj} z_{c_l}(t-j) \tag{6.1}$$

for $t \geq 0$. Assume that

$$0 < \sum_{t=0}^{T} C(t) \to C \leq \infty \quad (T \to \infty), \qquad f_{lj} \geq 0 \tag{6.2}$$

for $j \geq 0$ and $1 \leq l \leq n$. Note that $\sum_{j=0}^{\infty} f_{lj} = 1$. Thus

$$\sum_{t=0}^{T} M(t) = \sum_{l=1}^{n}\sum_{j=0}^{T} f_{lj} \sum_{t=0}^{T-j} z_{c_l}(t) \tag{6.3}$$

and therefore

$$\frac{\sum_{t=0}^{T} M(t)}{\sum_{t=0}^{T} C(t)} = \frac{\sum_{l=1}^{n}[(\sum_{j=0}^{T} f_{lj}) \sum_{t=0}^{T} z_{c_l}(t) - \sum_{j=0}^{T} f_{lj} \sum_{t=T-j+1}^{T} z_{c_l}(t)]}{\sum_{l=1}^{n}\sum_{t=0}^{T} z_{c_l}(t)} < 1$$

for $T > 0$, and

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T} M(t)}{\sum_{t=0}^{T} C(t)} = 1.$$

This means that the total output of the manufacturer can never satisfy the total market demand. That is, the competition supply network cannot satisfy the market demand within any finite time under condition (6.2). However, by (5.12) we know that the horizontal collaboration supply network may be able to satisfy the market demand within a finite time under condition (6.2). We first discuss the following optimal order policy for the horizontal collaboration supply network,

$$Var_{\{optimal\ policy\}}\left(\sum_{t=0}^{T} M(t)\right) = \min_{\{order\ policies\}} Var\left(\sum_{t=0}^{T} M(t)\right) \tag{6.4}$$

which is subject to

$$E_{\{order\ policies\}}\left(\sum_{t=0}^{T} M(t)\right) = E\left(\sum_{t=0}^{T} C(t)\right), \tag{6.5}$$

where $E(X)$ is the expectation of the random variable X. That is, the optimal order policy will not only enable the total output of the manufacturer to satisfy the total market demand within time T but also keep the variance of the total output of the manufacturer to a minimum value. Similar to (6.3) we have

$$\sum_{t=0}^{T} M(t) = \sum_{l=1}^{n}\sum_{j=0}^{T} \hat{f}_{lj} \sum_{t=0}^{T-j} z_{c_l}(t). \tag{6.6}$$

Let $f_l = (\hat{f}_{l0}, \ldots, \hat{f}_{lT})$, $F_T = (f_1, \ldots, f_n)$, $z_{lj} = \sum_{t=0}^{T-j} z_{c_l}(t)$, $\mu_{lj} = E(z_{lj})$, $\mu_l = (\mu_{l0}, \ldots, \mu_{lT})$, and $V_{ll'} = (Cov(z_{li}, z_{l'j}))$ be the covariance matrix for $1 \le l,\ l' \le n$. Let the notation A' denote the transpose of matrix or vector A. Then, (6.4) and (6.5) can be written in the following vector form

$$Var_{F_T^*}\left(\sum_{t=0}^{T} M(t)\right) = \min_{F_T} Var\left(\sum_{t=0}^{T} M(t)\right) = \min_{F_T} \sum_{l=1}^{n}\sum_{l'=1}^{n} f_l V_{ll'} f'_{l'} \tag{6.7}$$

which is subject to

$$\sum_{l=1}^{n} f_l \mu'_l = c_T, \quad \sum_{l=1}^{n} f_l \mathbf{1}' = n, \tag{6.8}$$

where $c_T = \sum_{t=0}^{T} E(C(t))$ and $\mathbf{1} = (1, \ldots, 1)$.

An order policy that makes the n vectors $F_T^* = (f_1^*, \ldots, f_n^*)$ satisfy (6.7) is called the optimal order policy or the optimal vector solution. The following theorem gives the expression of the optimal vector solution.

Theorem 7. *Let the covariance matrix V_{ll} be nonsingular for $1 \le l \le n$. Then the optimal vector solution can be expressed as*

$$f_l^* = \left[\lambda_1^*(\mu_l + \sum_{i=l+1}^{n} \alpha_i V_{il}) + \lambda_2^*\left(\mathbf{1} + \sum_{i=l+1}^{n} \beta_i V_{il}\right)\right] V_{ll}^{-1} \tag{6.9}$$

for $1 \le l \le n$ when $a_2 b_1 - a_1 b_2 \neq 0$, where

$$\begin{aligned}
(6.10)\quad a_1 &= \sum_{l=1}^{n}\left(\mu_l + \sum_{i=l+1}^{n} \alpha_i V_{il}\right) V_{ll}^{-1}\mathbf{1}', \quad a_2 = \sum_{l=1}^{n}\left(\mu_l + \sum_{i=l+1}^{n} \alpha_i V_{il}\right) V_{ll}^{-1}\mathbf{1}' \\
(6.11)\quad b_1 &= \sum_{l=1}^{n}\left(\mathbf{1} + \sum_{i=l+1}^{n} \beta_i V_{il}\right) V_{ll}^{-1}\mathbf{1}', \quad b_2 = \sum_{l=1}^{n}\left(\mathbf{1} + \sum_{i=l+1}^{n} \beta_i V_{il}\right) V_{ll}^{-1}\mu'_l \\
\lambda_1^* &= \frac{b_1 c_T - n b_2}{a_2 b_1 - a_1 b_2}, \quad \lambda_2^* = \frac{n a_2 - a_1 c_T}{a_2 b_1 - a_1 b_2}
\end{aligned}$$

and α_i and β_i are two vector functions which depend only on μ_j, V_{jj}^{-1} and $V_{j,j+1}, \ldots V_{j,n}$ for $i \le j \le n$. In particular, $a_2 b_1 - a_1 b_2 > 0$ when $V_{ll'} = 0$ for $l \neq l'$ and $\mu_l \neq c\mathbf{1}$ for $1 \le l \le n$, where c is a constant.

As can be seen from the proof of Theorem 7, if the restriction $\sum_{l=1}^{n} f_l \mathbf{1}' = n$ is removed, the optimal vector solution becomes

$$f_l^* = \frac{c_T}{a_2}\left(\mu_l + \sum_{i=l+1}^{n} \alpha_i V_{il}\right) V_{ll}^{-1}$$

for $1 \le l \le n$ when $a_2 \ne 0$.

Remark 3. Taking $\hat{f}_{l0} = 1$ for $1 \le l \le n$ and $\hat{f}_{lj} = 0$ for $1 \le j \le T$, $1 \le l \le n$, we have

$$\sum_{t=0}^{T} M(t) = \sum_{l=1}^{n}\sum_{j=0}^{T} \hat{f}_{lj} \sum_{t=0}^{T-j} z_{c_l}(t) = \sum_{t=0}^{T} C(t).$$

This means that

$$\min_{F_T} Var\left(\sum_{t=0}^{T} M(t)\right) \le Var\left(\sum_{t=0}^{T} C(t)\right).$$

That is, the optimal order policy avoids the bullwhip effect.

Note that the order policy corresponding to the optimal vector solution F_T^* in (6.9) is optimal only at time T, it may not be optimal at time t, $0 \le t < T$. On the other hand, the two conditions in (6.8) are too restrict to be realistic. Thus we consider the following model.

$$\min_{F_T}\left[\sum_{t=0}^{T} w_t \left[\sum_{s=0}^{t} E(M(s)) - c_t\right]^2 + \sum_{t=0}^{T} v_t Var\left(\sum_{s=0}^{t} M(s)\right)\right], \tag{6.12}$$

where, $c_t = \sum_{s=0}^{t} E(C(s))$ and both $\{w_t\}$ and $\{v_t\}$ are positive weight coefficients satisfying $\sum_{t=0}^{T}(w_t + v_t) = 1$. Model (6.12) means that the order policy that corresponds to the optimal solution F_T^* satisfies the market demand c_t best; it also has the minimum variance for all time $0 \le t \le T$. When $\sum_{t=0}^{T} w_t > \sum_{t=0}^{T} v_t$, it implies that satisfying the market demand is more important than reducing the variance of order quantity of all distributors.

Let $z_{lj}^{(t)} = \sum_{s=0}^{t-j} z_{c_l}(s)$ for $0 \le j \le t$, $z_{lj}^{(t)} = 0$ for $t+1 \le j \le T$, $\mu_{lj}^{(t)} = E(z_{lj}^{(t)})$, $\mu_l(t) = (\mu_{l0}^{(t)}, \ldots, \mu_{lt}^{(t)}, \ldots, \mu_{lT}^{(t)})$, and $V_{ll'}(t) = \left(Cov(z_{li}^{(t)}, z_{l'j}^{(t)})\right)$ be the covariance matrix for $1 \le l, l' \le n$. Then, (6.12) can also be written in the following form

$$\min_{F_T}\left[\sum_{t=0}^{T} w_t \left[\sum_{l=1}^{n} f_l \mu_l'(t) - c_t\right]^2 + \sum_{l=1}^{n}\sum_{l'=1}^{n}\sum_{t=0}^{T} v_t f_l V_{ll'}(t) f_{l'}'\right]. \tag{6.13}$$

To get the optimal solution, let $V = (V_{ij}(v))$, $W = (W_1, \ldots W_n)$ and $C = (C_1, \ldots, C_n)$, where $V_{ij}(v) = \sum_{t=0}^{T} v_t V_{ij}(t)$ for $j \le i$, $V_{ij}(v) = 0$ for $j > i$, and

$$W_l' = \sum_{t=0}^{T} w_t \mu_l'(t)(\mu_1(t), \ldots, \mu_n(t))$$

and $C_l = \sum_{t=0}^{T} w_t c_t \mu_l(t)$ for $1 \le l \le n$.

Theorem 8. *Let the matrix $V + W$ be nonsingular. Then the optimal vector solution F_T^* for the model* (6.13) *can be expressed as*

$$F_T^* = C(V+W)^{-1}. \tag{6.14}$$

Note that in order to obtain the optimal solution, the numbers $\hat{f}_{lj}$, $0 \le j \le T$, $1 \le l \le n$, in Theorems 7 and 8 are regarded as the independent variables. In fact, $\hat{f}_{lj}$ can be written as

$$\hat{f}_{lj} = \hat{f}_{lj}(x_1, \dots, x_d, y_1, \dots y_q) = a_{lj}(x_1, \dots, x_d, y_1, \dots y_q)[b_l(y_1, \dots, y_q)]^j \tag{6.15}$$

for $j \ge 0$, $1 \le l \le n$, where $|b_l(y_1, \dots, y_q)| < 1$, $a_{lj}(x_1, \dots, x_d, y_1, \dots y_q)$ and $b_l(y_1, \dots, y_q)$ are the functions of independent variables $x_1, \dots, x_d, y_1, \dots y_q$, that is, $\hat{f}_{lj}$, $1 \le l \le n$, $j \ge 0$, are the functions of independent variables $x_1, \dots, x_d, y_1, \dots y_q$. An optimal solution $\hat{f}_{lj}^* \triangleq \hat{f}_{lj}(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*)$ can be obtained using the following corollary.

Corollary 3. *Let $\{c_{lj}^*, 0 \le j \le T, 1 \le l \le n\}$, be the optimal solution to Theorems* 7 *or* 8 *and let*

$$L(x_1, \dots, x_d, y_1, \dots y_q) = \sum_{l=1}^{n} \sum_{j=0}^{T} [\hat{f}_{lj}(x_1, \dots, x_d, y_1, \dots y_q) - c_{lj}^*]^2.$$

If D is a closed and bounded space of real numbers of $d + p$ dimension and $\hat{f}_{lj}$, $1 \le l \le n$, $0 \le j \le T$, are continuous functions, then there exists a vector $(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*)$ such that

$$L(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*) = \min_{(x_1, \dots, x_d, y_1, \dots y_q) \in D} L(x_1, \dots, x_d, y_1, \dots y_q). \tag{6.16}$$

The results of Theorem 8 and Corollary 3 hold also for the competition supply chain network if $\hat{f}_{lk}$ is replaced by f_{lk}.

From the above discussion we see that to overcome the difficulty of getting the optimal vector for model (6.7) or (6.13) under condition (6.15), we first obtain the optimal solution $\{c_{lj}^*\}$ in (6.9) or (6.14) under the assumption that $\{\hat{f}_{lj}\}$ are all independent variables, then find the optimal vector $(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*)$ in (6.16) such that the difference of $\{\hat{f}_{lj}(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*)\}$ and $\{c_{lj}^*\}$ arrives at the minimum value. Thus, the order policy corresponding to the optimal vector $(x_1^*, \dots, x_d^*, y_1^*, \dots y_q^*)$ is the optimal order policy we seek. That is, the optimal order policy is obtained indirectly.

7. An Example

Example. Let $A_{d_i}(p) = A_d \neq 0$, $A_{r_i}(p) = A_r \neq 0$ for $i = 1, 2$, and $B_i(p) = 0$ for $i \in S_j$, $j = 1, 2$. Let $C_{d_i m}(p) = C_d$, $C_{r_i d_j}(p) = C_r$, $D_{r_i d_j}(p) = D_d$, $D_{c_j r_j} = D_r$ for

$i, j = 1, 2$, and $U_{il}(p) = V_{il}(p) = U \neq 0$ for $i \neq l$, $i, l \in S_j$, $j = 1, 2$. Let $\eta_{d_i m}(p) = \eta_d$, $\eta_{r_i d_j}(p) = \eta_r$, $\zeta_{r_i d_j}(p) = \zeta_d$ and $\zeta_{c_i r_i}(p) = \zeta_r$ for $i, j = 1, 2$. (see Fig. 3).

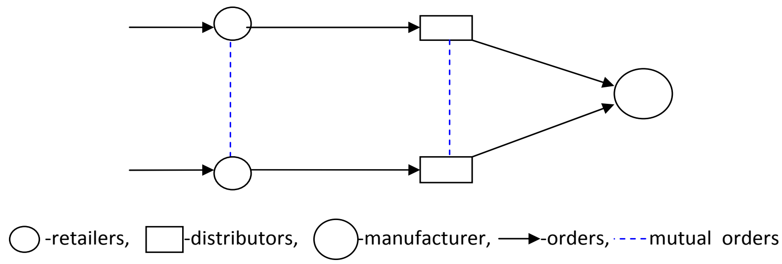

FIGURE 3. A horizontal collaboration network with two retailers

Assume that $\psi_{ik}(t) = 0$ and $\hat{\psi}_{ik}(t) = 0$ for $t \geq 0$, $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$. The order equations for the competition and horizontal collaboration supply networks are given by the following.

(I) Order equations for the competition supply network

$$\begin{aligned} z_{d_i}(t) &= A_d x_{d_i}(t) + C_d z_{d_i}(t-1) + D_d(z_{r_1}(t-1) + z_{r_2}(t-1)) \\ z_{r_i}(t) &= A_r x_{r_i}(t) + C_r z_{r_i}(t-1) + D_r(z_{c_1}(t-1) + z_{c_2}(t-1)) \end{aligned}$$

for $i = 1, 2$.

(II) Order equations for the horizontal collaboration network

$$\begin{aligned} \hat{z}_{d_i}(t) &= A_d x_{d_i}(t) + C_d \hat{z}_{d_i}(t-1) + D_d(\hat{z}_{r_1}(t-1) + \hat{z}_{r_2}(t-1)) \\ &\quad + U(\hat{z}_{d_1 d_2}(t-1) + \hat{z}_{d_2 d_1}(t-1)) \\ \hat{z}_{r_i}(t) &= A_r x_{r_i}(t) + C_r \hat{z}_{r_i}(t-1) + D_r(z_{c_1}(t-1) + z_{c_2}(t-1)) \\ &\quad + U(\hat{z}_{r_1 r_2}(t-1) + \hat{z}_{r_2 r_1}(t-1)) \\ \hat{z}_{d_i d_j}(t-1) &= \eta_d \hat{z}_{d_i}(t-2) + \zeta_d(\hat{z}_{r_1}(t-2) + \hat{z}_{r_2}(t-2)) \\ \hat{z}_{r_i r_j}(t-1) &= 2\eta_r \hat{z}_{r_i}(t-2) + \zeta_r z_{c_i}(t-2) \end{aligned}$$

for $i, j = 1, 2$, and $i \neq j$.

Next we shall discuss respectively the stability, the relation between $\{f_{lk}, k \geq 0\}$ and $\{\hat{f}_{lk}, k \geq 0\}$, the bullwhip effect and the optimal order policy.

(1) **Stability**. Since

$$\begin{aligned} a_d(z) &\triangleq (1 - z^{-1})(1 - \alpha_{d_i m}(z)) = z^{-2}[z^2 - (1 + A_d + C_d)z + C_d], \\ a_r(z) &\triangleq (1 - z^{-1})\left(1 - \sum_{j=1}^{2} \alpha_{r_i d_j}(z)\right) = z^{-2}[z^2 - (1 + 2A_r + 2C_r)z + 2C_r] \end{aligned}$$

it follows from (4.7) that the order policy for the competition supply network (I) is stable if and only if the roots, r_1, r_2, of $a_d(z) = 0$ and r_3, r_4, of $a_r(z) = 0$ satisfies

$$|r_{1,2}| = \left|1 + A_d + C_d \pm \sqrt{(1 + A_d + C_d)^2 - 4C_d}\right| / 2 < 1, \tag{7.1}$$

$$(7.2)\qquad |r_{3,4}| \;=\; \left|1+2A_r+2C_r \pm \sqrt{(1+2A_r+2C_r)^2-8C_r}\right|/2<1.$$

As an application of Corollary 1, we next discuss the stability of the horizontal collaboration network (II). Note that

$$\begin{aligned} b_d(z) &\triangleq (1-z^{-1})(\tilde{\alpha}_{d_im}(z)-\alpha_{d_im}(z)-\hat{\alpha}_{d_im}(z)) = \frac{2\eta_d A_d}{z^2} \\ b_r(z) &\triangleq \sum_{j=1}^{2}(1-z^{-1})(\tilde{\alpha}_{r_id_j}(z)-\alpha_{r_id_j}(z)-\hat{\alpha}_{r_id_j}(z)) = \frac{4\eta_r A_r}{z^2} \end{aligned}$$

for $i=1,2$, and

$$c_r(z) \;\triangleq\; 1-\frac{4\eta_r[(1-z^{-1})U-A_r]}{z^2(a_r(z)-b_r(z))}, \quad c_d(z)\triangleq 1-\frac{2\eta_d[(1-z^{-1})U-A_d]}{z^2(a_d(z)-b_d(z))}.$$

Hence, (4.18) can be rewritten as

$$(7.3)\quad a_d(z)-b_d(z))c_d(z) = z^{-3}[z^3-(1+A_d+C_d)z^2+(C_d-2\eta_dU)z+2\eta_dU]$$

$$(7.4)\quad (a_r(z)-b_r(z))c_r(z) = z^{-3}[z^3-(1+2A_r+2C_r)z^2+2(C_r-2\eta_rU)z+4\eta_rU].$$

Clearly, we can take small values of η_d and η_r such that $(a_d(z)-b_d(z))c_d(z)\neq 0$ and $(a_r(z)-b_r(z))c_r(z)\neq 0$ for all $|z|\geq 1$ as long as (7.1) and (7.2) hold. That is, we can take small values of η_d and η_r such that the order policy system for the horizontal collaboration supply network still remains stable.

Let $a_d(z_0)=0$ and $a_r(z_0)=0$ for some $|z_0|\geq 1$. Here, $z_0\neq 1$ since $a_d(1)=-A_d\neq 0$ and $a_r(1)=-2A_r\neq 0$. Then

$$\begin{aligned} (a_d(z_0)-b_d(z_0))c_d(z_0) &= z_0^{-3}2\eta_dU(1-z_0)\neq 0 \\ (a_r(z_0)-b_r(z_0))c_r(z_0) &= z_0^{-3}4\eta_rU(1-z_0)\neq 0 \end{aligned}$$

as long as $\eta_d\neq 0$ and $\eta_r\neq 0$. This means that we can take small values of $\eta_d(\neq 0)$ and $\eta_r(\neq 0)$ such that $(a_d(z)-b_d(z))c_d(z)\neq 0$ and $(a_r(z)-b_r(z))c_r(z)\neq 0$ for all $|z|\geq 1$. That is, an unstable order policy system for the competition supply network can be made stable in the horizontal collaboration supply network as long as we take suitable values of η_d, η_r and U.

(2) **Relation between $\{f_{lj}\}$ and $\{\hat{f}_{lj}\}$.** It follows from (5.2) and (5.3) that

$$\begin{aligned} \lambda_l(z) &= \frac{2z^{-2}[D_d(1-z^{-1})-A_d][D_r(1-z^{-1})-A_r]}{a_d(z)a_r(z)} \\ \hat{\lambda}_l(z) &= \frac{2z^{-2}[(D_d+2z^{-1}\zeta_dU)(1-z^{-1})-A_d][(D_r+2z^{-1}\zeta_rU)(1-z^{-1})-A_r]}{(a_r(z)-b_r(z))c_r(z)(a_d(z)-b_d(z))c_d(z)} \end{aligned}$$

for $l=1,2$. Let the numbers $\hat{r}_i$, $i=1,2,3$ and $\hat{r}_i$, $i=4,5,6$ be the roots of equations $(a_d(z)-b_d(z))c_d(z)=0$ and $(a_r(z)-b_r(z))c_r(z)$ respectively, which satisfy $|\hat{r}_i|<1, 1\leq i\leq 6$. Let $r_i\neq r_j$ and $\hat{r}_i\neq\hat{r}_j$ for $i\neq j$. Then, $\lambda_l(z)$ and $\hat{\lambda}_l(z)$ can be rewritten as

$$\lambda_l(z)=2F(z^{-1})G(z^{-1}), \quad \hat{\lambda}_l(z)=2\hat{F}(z^{-1})\hat{G}(z^{-1}),$$

where

$$G(z^{-1}) = \sum_{i=1}^{4} \left[\prod_{j\neq i}(r_i - r_j)^{-1}\right] \frac{r_i}{1 - r_i z^{-1}}, \quad \hat{G}(z^{-1}) = \sum_{i=1}^{6} \left[\prod_{j\neq i}(\hat{r}_i - \hat{r}_j)^{-1}\right] \frac{\hat{r}_i^3}{1 - \hat{r}_i z^{-1}}$$

$$F(z^{-1}) = D_d D_r z^{-2} - (2D_d D_r - D_d A_r - D_r A_d) z^{-1} + (D_d - A_d)(D_r - A_r)$$

and

$$\begin{aligned}
\hat{F}(z^{-1}) &= F(z^{-1}) + 2U[\zeta_d(D_r - A_r) + \zeta_r(D_d - A_d)]z^{-1} \\
&\quad + 2U[2\zeta_d\zeta_r U - \zeta_r A_d - \zeta_d A_r]z^{-2} \\
&\quad - 2U[\zeta_d(2\zeta_r U - D_r) + \zeta_r(2\zeta_d U - D_d)]z^{-3} + 4\zeta_r\zeta_d U^2 z^{-4}.
\end{aligned}$$

Note that $\lambda_1(z) = \lambda_2(z)$ and $\hat{\lambda}_1(z) = \hat{\lambda}_2(z)$. Let $f_{lk} = f_k$ and $\hat{f}_{lk} = \hat{f}_k$ for $l = 1, 2$. Let $s = z^{-1}$. Thus

$$\begin{aligned}
f_k &= 2(k!)^{-1} \frac{d^k(F(s)G(s))}{d^k s}\Big|_{s=0} \\
&= 2(k!)^{-1}\left[F(0)G^{(k)}(0) + kF'(0)G^{(k-1)}(0) + \frac{k(k-1)}{2}F''(0)G^{(k-2)}(0)\right] \\
&= 2\left[F(0)R_k + F'(0)R_{k-1} + \frac{1}{2}F''(0)R_{k-2}\right]
\end{aligned}$$

for $k \geq 2$, where $f_1 = f_2 = 0$, $R_0 = R_1 = 0$ and $R_k = \sum_{i=1}^{4}\left[\prod_{j\neq i}(r_i - r_j)^{-1}\right](r_i)^{1+k}$ for $k \geq 2$. Similarly, we have

$$\hat{f}_k = 2\sum_{j=0}^{4} \frac{\hat{F}^{(j)}(0)\hat{R}_{k-j}}{j!}$$

for $k \geq 2$, where $\hat{f}_0 = \hat{f}_1 = 0$, $\hat{R}_k = \sum_{i=1}^{6}\left[\prod_{j\neq i}(\hat{r}_i - \hat{r}_j)^{-1}\right](\hat{r}_i)^{3+k}$ for $k \geq 2$, $\hat{R}_k = 0$ for $k = -2, -1, 0, 1$, and

$$\begin{aligned}
\hat{F}^{(0)}(0) &= F(0) \\
\hat{F}^{(1)}(0) &= F^{(1)}(0) + 2U[\zeta_d(D_r - A_r) + \zeta_r(D_d - A_d)] \\
\hat{F}^{(2)}(0) &= F^{(2)}(0) + 4U(2\zeta_d\zeta_r U - \zeta_r A_d - \zeta_d A_r) \\
\hat{F}^{(3)}(0) &= 12U[\zeta_d(D_r - 2\zeta_r U) + \zeta_r(D_d - 2\zeta_d U)] \\
\hat{F}^{(4)}(0) &= 4!4U^2\zeta_d\zeta_r.
\end{aligned}$$

As can be seen

$$\sum_{k=0}^{\infty} f_k = \lambda_l(1) = \frac{2A_d A_r}{2A_d A_r} = 1 = \hat{\lambda}_l(1) = \sum_{k=0}^{\infty} \hat{f}_k.$$

Let $|r_{i_0}| = \max_{1\leq i\leq 4}\{|r_i|\}$, $|\hat{r}_{i_0}| = \max_{1\leq i\leq 6}\{|\hat{r}_i|\}$ and $|r_{i_0}| > |r_j|$, $|\hat{r}_{i_0}| > |\hat{r}_j|$ for $j \neq i_0$. Let r_{i_0} and z_1 be two roots of the equation $a_d(z) = 0$. In order to get the root $\hat{r}_{i_0}$ of $(a_d(z) - b_d(z))c_d(z) = 0$ such that $\hat{r}_{i_0} = r_{i_0}$, let

$$(a_d(z) - b_d(z))c_d(z) = (1 - r_{i_0}/z)(1 - \hat{z}_1/z)(1 - \hat{z}_2/z) = 0.$$

By (7.3) we have

$$\hat{z}_{1,2} = \frac{1}{2}\left[1 + A_d + C_d - r_{i_0} \pm \sqrt{(1 + A_d + C_d - r_{i_0})^2 + 8\eta_d U}\right].$$

Clearly, $\hat{z}_1 \to z_1$ and $\hat{z}_2 \to 0$ as $\eta_d \to 0$. Note that

$$\lim_{k\to\infty} \frac{R_k}{r_{i_0}^k} = R(r_{i_0}) = r_{i_0} \prod_{j\neq i_0} (r_{i_0} - r_j)^{-1},$$

$$\lim_{k\to\infty} \frac{\hat{R}_k}{r_{i_0}^k} = \hat{R}(r_{i_0}) = r_{i_0}^3 \prod_{j\neq i_0} (r_{i_0} - \hat{r}_j)^{-1},$$

it follows that

$$\lim_{k\to\infty} \frac{f_k}{r_{i_0}^k} = R(r_{i_0})F(r_{i_0}^{-1}), \quad \lim_{k\to\infty} \frac{\hat{f}_k}{r_{i_0}^k} = \hat{R}(r_{i_0})\hat{F}(r_{i_0}^{-1}).$$

Moreover, the numbers $\{\varepsilon_{lk}\}$ in Theorem 6 can be written as

$$\begin{aligned}
\varepsilon_{lk} &= f_k^{-1}(\hat{f}_k - f_k) \\
&= 2f_k^{-1}\Big(F(0)(\hat{R}_k - R_k) + F'(0)(\hat{R}_{k-1} - R_{k-1}) \\
&\quad + \frac{F''(0)}{2}(\hat{R}_{k-2} - R_{k-2}) + (\hat{F}'(0) - F'(0))\hat{R}_{k-1} + \frac{1}{2}(\hat{F}''(0) - F''(0))\hat{R}_{k-2} \\
&\quad + \frac{1}{3!}(\hat{F}^{(3)}(0) - F^{(3)}(0))\hat{R}_{k-3} + \frac{1}{4!}(\hat{F}^{(4)}(0) - F^{(4)}(0))\hat{R}_{k-4}\Big)
\end{aligned}$$

for $k \geq 2$, and

$$\lim_{k\to\infty} \varepsilon_{lk} = \frac{\hat{F}(r_{i_0}^{-1})(\hat{R}(r_{i_0}) - R(r_{i_0})) + (\hat{F}(r_{i_0}^{-1}) - F(r_{i_0}^{-1}))R(r_{i_0})}{R(r_{i_0})F(r_{i_0}^{-1})}.$$

Since $\hat{R}(r_{i_0}) - R(r_{i_0}) \to 0$ as $\eta_d \to 0$, $\eta_r \to 0$, and $\hat{F}(r_{i_0}^{-1}) - F(r_{i_0}^{-1}) \to 0$ as $\zeta_d \to 0$, $\zeta_r \to 0$, it follows that ε_{lk} can be uniformly small when η_d, η_r, ζ_d and ζ_r are all small.

(3) **The bullwhip effect.** Taking $A_d = -1/6$, $C_d = 1/4$, $A_r = -5/48$, $C_r = 1/8$, it follows from (7.1) and (7.2) that $r_1 = 3/4$, $r_2 = 1/3$, $r_3 = 2/3$ and $r_4 = 3/8$. Similarly, taking $\eta_d = 1/36$, $\eta_r = 1/128$ and $U = 1/2$ we can obtain the following six roots of the equations $(a_d(z) - b_d(z))c_d(z) = 0$ and $(a_r(z) - b_r(z))c_r(z) = 0$

$$\begin{aligned}
\hat{r}_1 &= 0.7226, \quad \hat{r}_2 = 0.4468, \quad \hat{r}_3 = -0.086 \\
\hat{r}_4 &= 0.631, \quad \hat{r}_5 = 0.464, \quad \hat{r}_6 = -0.0534.
\end{aligned}$$

It can be checked that $R_k > 0$ and $\hat{R}_k > 0$ for $k \geq 2$ since $G^{(k)}(0) > 0$ and $\hat{G}^{(k)}(0) > 0$ for $k \geq 2$. In fact, we have

$$\begin{aligned}
G(z^{-1}) &= \frac{1}{z^2 a_d(z) a_r(z)} = \frac{1}{z^2(1 - r_1/z)(1 - r_2/z)(1 - r_1/z)(1 - r_2/z)} \\
&= \left(\frac{1}{r_1 - r_2}\left[\frac{1}{1 - r_1/z} - \frac{1}{1 - r_1/z}\right]\right)\left(\frac{1}{r_3 - r_4}\left[\frac{1}{1 - r_3/z} - \frac{1}{1 - r_4/z}\right]\right)
\end{aligned}$$

$$= \left(\sum_{k=1}^{\infty}\left[\frac{r_1^k - r_2^k}{r_1 - r_2}\right] z^{-k}\right)\left(\sum_{k=1}^{\infty}\left[\frac{r_3^k - r_4^k}{r_3 - r_4}\right] z^{-k}\right).$$

This implies that $G^{(k)}(s)|_{s=0} > 0$ for $k \geq 2$ since $r_1 = 3/4 > r_2 = 1/3$ and $r_3 = 2/3 > r_4 = 3/8$. Similarly, we can check that $\hat{G}^{(k)}(s)|_{s=0} > 0$ for $k \geq 2$.

Let $0 \geq D_d \geq A_d = -1/6$ and $0 \geq D_r \geq A_r = -5/48$. Then $-D_d(D_r - A_r) \geq 0$ and $-D_r(D_d - A_d) \geq 0$. Thus

$$\begin{aligned} f_k/2 &= (D_d - A_d)(D_r - A_r)R_k \\ &\quad -[D_d(D_r - A_r) + D_r(D_d - A_d)]R_{k-1} + D_d D_r R_{k-2} \geq 0 \end{aligned}$$

for $k \geq 2$. By Theorem 3 we see that the bullwhip effect does not exist in the competition supply network when (5.6) and (5.7) hold for $\{f_k, k \geq 0\}$. Taking ζ_d and ζ_r such that $D_d \leq \zeta_d \leq 0$ and $D_r \leq \zeta_r \leq 0$, we have

$$\begin{aligned} \hat{f}_k/2 &= (D_d - A_d)(D_r - A_r)\hat{R}_k - [(D_d - \zeta_d)(D_r - A_r) + (D_r - \zeta_r)(D_d - A_d)]\hat{R}_{k-1} \\ &\quad +[(D_d - \zeta_d)(D_r - \zeta_r) + \zeta_d(D_r - A_r) + \zeta_r(D_d - A_d)]\hat{R}_{k-2} \\ &\quad +[\zeta_d(D_r - \zeta_r) + \zeta_r(D_d - \zeta_d)]\hat{R}_{k-3} + \zeta_d\zeta_r\hat{R}_{k-4} \geq 0 \end{aligned}$$

for $k \geq 2$ when ζ_d and ζ_r are small. Thus, by Theorem 4, the bullwhip effect does not exist in the horizontal collaboration supply network when (5.6) and (5.8) hold for $\{\hat{f}_k, k \geq 0\}$. Note that $\hat{R}_2 = 1$, $\hat{R}_3 = 2.125$ and $\hat{R}_4 = 2.9306$. If we take $\zeta_d = 1$, $\zeta_r = 1$ $D_d = -1/12$ and $D_r = -1/48$, then

$$\hat{f}_2 = 0.0139, \quad \hat{f}_3 = 0.3802, \quad \hat{f}_4 = 2.6644.$$

By Theorem 5, this means that the the bullwhip effect exists in the horizontal collaboration supply network when two customer demands are unrelated and the demands at different time is also irrelative since $\sum_{j=0}^{4} \hat{f}_j^2 > 1$.

(4) **The optimal order policy.** We first calculate the optimal solution to Theorem 8. Consider $T = 3$ and $w_t = v_t = 1/8$ for $0 \leq t \leq 3$. Let $z_{c_1}(t)$, $z_{c_2}(t)$, $0 \leq t \leq 3$, be mutually independent with the expectations $E(z_{c_i}(0)) = 1/2$, $E(z_{c_i}(1)) = 1/2$, $E(z_{c_i}(2)) = 1$ and $E(z_{c_i}(3)) = 1$ for $i = 1, 2$ and the variances $Var(z_{c_i}(t)) = 1$ for $i = 1, 2$ and $0 \leq t \leq 3$. Then, $C = \frac{1}{2}(13\ 7\ 3\ 1)$ and

$$V + W = \frac{1}{2}\begin{pmatrix} 39 & 23 & 11 & 4 \\ 23 & 33/2 & 8 & 3 \\ 11 & 8 & 11/2 & 2 \\ 4 & 3 & 2 & 3/2 \end{pmatrix}$$

Thus, the optimal solution in (6.14) can be written as $c_{l0}^* = 0.0084$, $c_{l1}^* = 0.5334$, $c_{l2}^* = -0.148$ and $c_{l3}^* = -0.2252$ for $l = 1, 2$. Now, we consider the optimal solution f_0^*, f_1^*, f_2^* and f_3^* which satisfy (6.16) for the competition supply network. Here, we only consider a special case: let $D_d - A_d = D_r - A_r = x$, that is, $D_d = -1/6 + x$ and

$D_r = -5/48 + x$, where x is a variable. Since $f_0 = f_1 = 0$, $R_2 = 1$ and $R_3 = 2.125$, it follows that

$$f_2 = 2x^2, \quad f_3 = 2[x^2 R_3 - (-1/6 + x)x - (-5/24 + x)x] = \frac{x^2}{4} + \frac{x}{144}.$$

Hence

$$L(x) = 2\left[(c_{l0}^*)^2 + (c_{l1}^*)^2 + (2x^2 - c_{l2}^*)^2 + \left(\frac{x^2}{4} + \frac{x}{144} - c_{l3}^*\right)^2\right].$$

Let $L'(x) = 0$, then

$$x^3 + 0.0083x^2 + 0.0877x + 0.0025 = 0.$$

It can be checked that there is only one real root, $x^* = -0.0284$, for this equation. This means that $L(x)$ is at the minimum at $x = x^*$. Thus, we obtain the optimal solution: $f_0^* = f_1^* = 0$, $f_2^* = 0.0016$ and $f_3^* = 0.0004$ when $D_d = -0.1951$ and $D_r = -0.1326$.

The above optimal solution is restricted because the variables A_d, A_r, C_d and C_r were previously fixed. Since $f_k, k \geq 2$, are the functions of the variables A_d, A_r, C_d, C_r, D_d and D_r, it is better to seek the vector $(A_d^*, A_r^*, C_d^*, C_r^*, D_d^*, D_r^*)$ that satisfies (6.16) in Corollary 3, that is

$$L(A_d^*, A_r^*, C_d^*, C_r^*, D_d^*, D_r^*) = 2 \min_{\{A_d, A_r, C_d, C_r, D_d, D_r\}} \sum_{k=0}^{T} [f_k(A_d, A_r, C_d, C_r, D_d, D_r) - c_{lk}^*]^2.$$

Thus, the vector $(A_d^*, A_r^*, C_d^*, C_r^*, D_d^*, D_r^*)$ is the best order policy for the competition supply chain network.

8. **Conclusion and Discussion**

In the paper, we compared the horizontal collaboration supply network with the competition (non-horizontal collaboration) supply network in terms of the stability and the bullwhip effect in the respective order policy systems. By solving the order equations, we can get the necessary and sufficient conditions for judging the stability of the respective order policy systems for the horizontal collaboration and competition supply networks. It is shown that a stable (unstable) order policy system for the competition supply network can remain (become) stable in the horizontal collaboration supply network as long as we adopt suitable order policies $\{\eta_{il}(p)\}$, $\{U_{il}(p)\}$ and $\{V_i(p)\}$. By using the expressions of the solutions we obtain two number series $\{f_{lj}\}$ and $\{\hat{f}_{lj}\}$, which are the coefficients of the linear combination of market demands $\{z_{c_l}(t)\}$. Making use of the coefficients we get the sufficient conditions that lead to the absence or presence of the bullwhip effect in the two supply chain networks. The special relation between the two supply networks in Theorem 6 shows that the horizontal collaboration supply network can either reduce or enhance the bullwhip effect in the competition supply network. It is difficult to get the optimal solution $\{\hat{f}^*_{lj}\}$ for model (6.7) or (6.13) since $\{\hat{f}_{lj}\}$ is a series of functions with variables, $x_1, \ldots, x_d, y_1, \ldots y_q$. Hence, we first obtain the optimal solution $\{c^*_{lj}\}$, then we get the optimal solution $\{\hat{f}^*_{lj}\}$ by choosing vector $(x^*_1, \ldots, x^*_d, y^*_1, \ldots y^*_q)$ such that the difference between $\{\hat{f}_{lj}(x^*_1, \ldots, x^*_d, y^*_1, \ldots y^*_q)\}$ and $\{c^*_{lj}\}$ is at the minimum value. That is, we can obtain indirectly the best order policy corresponding to $x^*_1, \ldots, x^*_d, y^*_1, \ldots y^*_q$ by using the optimal solution $\{c^*_{lj}\}$. Finally, the example analysis shows that all results obtained in this paper are applicable.

Our model has several limitations. First, the order $\{z_{ik}(t)\}$ is assumed to be determined by the linear combination of its past order and inventory levels, that is, the order satisfies the linear equations with the random errors. Secondly, we assume in Section 5 that $\sigma^2_{ik}(t) = 0$ and $\hat{\sigma}^2_{ik}(t) = 0$, and in Sections 6 and 7, that $\psi_{ik}(t) = 0$ and $\hat{\psi}_{ik}(t) = 0$ for $i \in S_j$, $k \in S_{j+1}$, $j = 1, 2$. These assumptions restrict our model's practicalness. Third, there is no information sharing in the order policy system. Fourth, the prices of the manufacturer, distributor and retailer, were not considered when seeking the optimal order policy. There are a few possible extensions. It would be interesting to generalize our model to one in which the orders satisfy some nonlinear and stochastic equations with information sharing. It would be worthwhile to study the bullwhip effect in the optimal order policy without neglecting its random errors. The following model would also be of interest: Let p_m, p_r, α and β denote the manufacturer's price, retail price, inventory cost (per unit) and loss (per unit) out of stock, respectively. Assume that $p_m < p_r$. In order to satisfy the market demand, $\sum_{t=0}^{T} C(t)$ where $C(t) = \sum_{l=1}^{n} z_{c_l}(t)$, with the best economic benefit under the condition that there is no bullwhip effect, we may take an optimal order $\{\hat{f}^*_{lj}\}$ such that the following total average profit (TAP)

$$TAP(\{\hat{f}_{lj}\}) = E\left(\sum_{t=0}^{T} H(t, \{\hat{f}_{lj}\})\right)$$

arrives at the maximum value, that is,

$$TAP(\{\hat{f}^*_{lj}\}) = \max_{\{\hat{f}_{lj}\}} TAP(\{\hat{f}_{lj}\})$$

subject to $BW(T) \leq 1$, where

$$\begin{aligned} H(t, \{\hat{f}_{lj}\}) &= (p_r - p_m) \min\{C(t), M(t)\} - (p_m + \alpha)(M(t) - C(t))^+ \\ &\quad -(p_r - p_m + \beta)(C(t) - M(t))^+ + \lambda |M(t) - C(t)|^\gamma, \end{aligned}$$

$M(t) = \sum_{l=1}^{n} \sum_{j=0}^{t} \hat{f}_{lj} z_{c_l}(t - j)$, and both nonnegative numbers λ and γ can be considered as reward parameters.

APPENDIX A: Proofs of Theorems

Proof of Theorem 1. Let $i \in S_2$. Then for $k \in S_3 = \{m\}$, i.e., $k = m$, it follows from (4.4) that

$$\begin{aligned} Z_{im}(z) &= \alpha_{im}(z)Z_{im}(z) + \sum_{l\in S_1} \beta_{li}(z)Z_{li}(z) + \Psi_{im}(z) \\ &= \frac{1}{1-\alpha_{im}(z)}\left(\sum_{l\in S_1} \beta_{li}(z)Z_{li}(z) + \Psi_{im}(z)\right) \end{aligned} \tag{A.1}$$

and

$$Z_{li}(z) = \sum_{l'\in S_2} \alpha_{ll'}(z)Z_{ll'}(z) + \sum_{l'\in S_0} \beta_{l'l}(z)Z_{l'l}(z) + \Psi_{li}(z) \tag{A.2}$$

for $l \in S_1$. In order to solve equation (A.2), for fixed $r_l \in S_1$ we let $d_i \in S_2$, $z_i = Z_{r_l d_i}(z)$, $a_i = \alpha_{r_l d_i}(z)$, $b_i = \Psi_{r_l d_i}(z)$ for $1 \le i \le k$, and $c = \beta_{c_l r_l}(z)Z_{c_l}(z)$. Thus, (A.2) can be rewritten as

$$\begin{aligned} (1-a_1)z_1 - a_2z_2 - a_3z_3 - \cdots - a_kz_k &= b_1 + c \\ -a_1z_1 + (1-a_2)z_2 - a_3z_3 - \cdots - a_kz_k &= b_2 + c && \text{(A.3)} \\ \cdots & && \text{(A.4)} \\ \cdots & && \text{(A.5)} \\ \cdots & \\ -a_1z_1 - \cdots - a_{k-1}z_{k-1} + (1-a_k)z_k &= b_k + c. \end{aligned}$$

Now we solve (A.3). The first equation multiplied by -1 is added to the second, through to the kth equation, then the jth equation multiplied by a_j, $1 \le j \le k$, is added to the first equation. Thus we can obtain the following solution to (A.3)

$$z_i = \frac{c + \sum_{j=1}^k a_j b_j}{1 - \sum_{j=1}^k a_j} + b_i$$

for $1 \le i \le k$. That is, the solutions to (A.2) or (A.3) can be written as

$$Z_{r_l d_i}(z) = \frac{\beta_{c_l r_l}(z)Z_{c_l}(z) + \sum_{j=1}^k \alpha_{r_l d_j}(z)\Psi_{r_l d_j}(z)}{1 - \sum_{j=1}^k \alpha_{r_l d_j}(z)} + \Psi_{r_l d_i}(z) \tag{A.6}$$

for $1 \le l \le n$ and $1 \le i \le k$. Plugging (A.4) into (A.1) we have (15).

It follows from (A.1) and (A.4) that $\{Z_{d_i m}(z), 1 \le i \le k\}$ and $\{Z_{r_l d_j}(z), 1 \le l \le n, 1 \le j \le k\}$ are stable (Graf 2004, P. 109) if and only if $\{\alpha_{d_i m}(z), 1 \le i \le k\}$ and $\{\alpha_{r_l d_j}(z), 1 \le l \le n, 1 \le j \le k\}$ satisfy (4.7). This completes the proof.

Proof of Theorem 2. Let $i \in S_2$. It follows from (4.12) that

$$\hat{Z}_{im}(z) = \tilde{\alpha}_{im}(z)\hat{Z}_{im}(z) + \sum_{l\in S_1} \tilde{\beta}_{li}(z)\hat{Z}_{li}(z)$$

$$+\sum_{i\neq l'\in S_2}\hat{\alpha}_{l'm}(z)\hat{Z}_{l'm}(z)+\sum_{l\in S_1}\sum_{i\neq l'\in S_2}\hat{\beta}_{ll'}\hat{Z}_{ll'}(z)+\hat{\Psi}_{im}(z) \tag{A.7}$$

and

$$\begin{aligned}\hat{Z}_{li}(z) \;&=\; \sum_{l'\in S_2}\tilde{\alpha}_{ll'}(z)\hat{Z}_{ll'}(z)+\sum_{l'\in S_0}\tilde{\beta}_{l'l}(z)\hat{Z}_{l'l}(z)\\ &\quad+\sum_{l'\in S_2}\sum_{l\neq j\in S_1}\hat{\alpha}_{jl'}(z)\hat{Z}_{jl'}(z)+\sum_{l'\in S_0}\sum_{l\neq j\in S_1}\hat{\beta}_{l'j}\hat{Z}_{l'j}(z)+\hat{\Psi}_{li}(z) \qquad \text{(A.8)}\end{aligned}$$

for $l\in S_1$, where $\hat{\alpha}_{kk'}(z)=\delta_k(z)z^{-1}\eta_{kk'}(z^{-1})$ and $\hat{\beta}_{kk'}=\delta_{k'}(z)z^{-1}\zeta_{kk'}(z^{-1})$, $k\in S_j$, $k'\in S_{j+1}$, $j=1,2$. Let $d_i\in S_2$, $z_i=\hat{Z}_{d_im}(z)$, $a_i=\tilde{\alpha}_{d_im}(z)$, $a_i'=\hat{\alpha}_{d_im}(z)$, $b_i=\sum_{l\in S_1}\tilde{\beta}_{ld_i}(z)\hat{Z}_{ld_i}(z)$ and $c_i=\sum_{l\in S_1}\sum_{i\neq l'\in S_2}\hat{\beta}_{ll'}\hat{Z}_{ll'}(z)+\hat{\Psi}_{im}(z)$. Then, (A.5) can be rewritten simply as

$$\begin{aligned}(1-a_1)z_1-a_2'z_2-a_3'z_3-\cdots-a_k'z_k \;&=\; b_1+c_1\\ -a_1'z_1+(1-a_2)z_2-a_3'z_3-\cdots-a_k'z_k \;&=\; b_2+c_2 \qquad \text{(A.9)}\\ \cdots \qquad \cdots \qquad \cdots&\\ -a_1'z_1-\cdots-a_{k-1}'z_{k-1}+(1-a_k)z_k \;&=\; b_k+c_k.\end{aligned}$$

Now we solve the equations. The first equation above multiplied by -1 is added to the second, through to the kth equation, then the jth equation multiplied by $a_j'/(1-a_j+a_j')$, $1\le j\le k$, is added to the first equation. Thus we can obtain the following solutions to (A.5) or (A.7).

$$z_i=\frac{1}{1-a_i+a_i'}\left(b_i+c_i+\frac{\sum_{j=1}^k\frac{(b_j+c_j)a_j'}{1-a_j+a_j'}}{1-\sum_{j=1}^k\frac{a_j'}{1-a_j+a_j'}}\right) \tag{A.10}$$

for $1\le i\le k$. (A.8) can be written as (4.19) in Theorem 2.

To solve (A.6), let $r_l\in S_1, d_j\in S_2$ and let $z_{lj}=\hat{Z}_{r_ld_j}(z)$, $a_{lj}=\tilde{\alpha}_{r_ld_j}(z))$, $a_{lj}'=\hat{\alpha}_{r_ld_j}(z)$, $b_{lj}=\hat{\Psi}_{r_ld_j}(z)$, $c_l=\tilde{\beta}_{c_lr_l}(z)\hat{Z}_{c_l}(z)+\sum_{j\neq l}\hat{\beta}_{c_jr_j(z)}\hat{Z}_{c_j}(z)$ for $1\le l\le n, 1\le j\le k$. Thus, (A.6) can be written as

$$\begin{aligned}(1-a_{1j})z_{1j}-\sum_{l\neq j}a_{1l}z_{1l}-\sum_{i\neq 1}^{n}\sum_{l=1}^{k}a_{il}'z_{il} \;&=\; c_1+b_{1j}, quad1\le j\le k,\\ (1-a_{2j})z_{2j}-\sum_{l\neq j}a_{2l}z_{2l}-\sum_{i\neq 2}^{n}\sum_{l=1}^{k}a_{il}'z_{il} \;&=\; c_2+b_{2j}, \quad 1\le j\le k,\\ \cdots \qquad \cdots \qquad \cdots&\\ (1-a_{nj})z_{nj}-\sum_{l\neq j}a_{nl}z_{nl}-\sum_{i\neq n}^{n}\sum_{l=1}^{k}a_{il}'z_{il} \;&=\; c_n+b_{nj}, \quad 1\le j\le k.\end{aligned}$$

There are $n\times k$ equations. For each i $(1\le i\le n)$, the $((i-1)k+1)$th equation above multiplied by -1 is added to the $((i-1)k+j)$th equation for $2\le j\le k$, then the first equation multiplied by -1 is added to the $((i-1)k+1)$th equation for $2 \le i \le k$. Thus, we have

$$
\begin{aligned}
(1-a_{11})z_{11} - \sum_{l\neq 1} a_{1l}z_{1l} - \sum_{i\neq 1}^{n}\sum_{l=1}^{k} a'_{il}z_{il} &= c_1 + b_{11},\\
-z_{11} + z_{1j} &= b_{1j} - b_{11}, \quad 2 \le j \le k,\\
-(1-a_{11}+a'_{11})z_{11} + \sum_{l\neq 1}(a_{1l}-a'_{1l})z_{1l} &\\
+(1-a_{21}+a'_{21})z_{21} - \sum_{l\neq 1}(a_{2l}-a'_{2l})z_{2l} &= c_2 - c_1 + b_{21} - b_{11},\\
-z_{21} + z_{2j} &= b_{2j} - b_{21}, \quad 2 \le j \le k,\\
\cdots \qquad & \qquad \cdots \qquad \cdots\\
-(1-a_{11}+a'_{11})z_{11} + \sum_{l\neq 1}(a_{1l}-a'_{1l})z_{1l} &\\
+(1-a_{n1}+a'_{n1})z_{n1} - \sum_{l\neq 1}(a_{nl}-a'_{nl})z_{nl} &= c_n - c_1 + b_{n1} - b_{11}\\
-z_{n1} + z_{nj} &= b_{nj} - b_{n1}, \quad 2 \le j \le k.
\end{aligned}
$$

Furthermore, for each i $(2 \le i \le n)$, the $((i-1)k+1)$th equation is added to the $((i-1)k+j)$th equation multiplied by $a_{ij} - a'_{ij}$, $2 \le j \le k$, and the first equation is added to the $((i-1)k+j)$th equation multiplied by a'_{ij}, $2 \le i \le n$, $2 \le j \le k$. Then, for each i $(2 \le i \le n)$, the $((i-1)k+1)$th equation is added to the jth equation multiplied by $a'_{1j} - a_{1j}$, $2 \le j \le k$, and the first equation is added to the jth equation multiplied by a_{1j}, $2 \le j \le k$. Hence

$$
\begin{aligned}
(1-\alpha_1)z_{11} - \sum_{i=2}^{n} \alpha'_i z_{i1} &= c_1 + b + \hat{b}_1,\\
-z_{11} + z_{1j} &= b_{1j} - b_{11}, \quad 2 \le j \le k,\\
-(1-\alpha_1+\alpha'_1)z_{11} + (1-\alpha_2+\alpha'_2)z_{21} &= c_2 - c_1 + \hat{b}_2 - \hat{b}_1,\\
-z_{21} + z_{2j} &= b_{2j} - b_{21}, \quad 2 \le j \le k,\\
\cdots \qquad & \qquad \cdots \qquad \cdots\\
-(1-\alpha_1+\alpha'_1)z_{11} + (1-\alpha_n+\alpha'_n)z_{n1} &= c_n - c_1 + \hat{b}_n - \hat{b}_1\\
-z_{n1} + z_{nj} &= b_{nj} - b_{n1}, \quad 2 \le j \le k.
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha_i &= \sum_{l=1}^{k} a_{il}, \quad \alpha'_i = \sum_{l=1}^{k} a'_{il}, \quad b = \sum_{i=1}^{n}\sum_{j=1}^{k}(b_{ij}-b_{i1})a'_{ij}\\
\hat{b}_i &= b_{i1} + \sum_{j=1}^{k}(b_{ij}-b_{i1})(a_{ij}-a'_{ij}), \qquad 1 \le i \le n.
\end{aligned}
$$

Finally, we can obtain the solutions to (A.6) in the following as long as the first equation above is added to the $((i-1)k+1)th$ equation multiplied by $\alpha'_i/(1-\alpha_i+\alpha'_i)$, $2 \le i \le k$.

$$\begin{aligned} z_{i1} &= \frac{1}{1-\alpha_i+\alpha'_i}\left(\hat{b}_i + c_i + \frac{b+\sum_{l=1}^{n}\frac{(\hat{b}_l+c_l)\alpha'_l}{1-\alpha_l+\alpha'_l}}{1-\sum_{l=1}^{n}\frac{\alpha'_l}{1-\alpha_l+\alpha'_l}}\right) \\ z_{ij} &= z_{i1} + b_{ij} - b_{i1} \end{aligned} \tag{A.11}$$

for $1 \le i \le n$, $2 \le j \le k$. Thus, (4.20) in Theorem 2 can be obtained from (A.9).

From (A.8) and (A.9) we see that the order policy for the horizontal collaboration supply network in (2.12) is stable in time if and only if $\{\tilde{\alpha}_{d_im}(z)\}$, $\{\hat{\alpha}_{d_im}(z)\}$, $\{\tilde{\alpha}_{r_ld_j}(z)\}$ and $\{\hat{\alpha}_{r_ld_j}(z)\}$ satisfy (4.18). This completes the proof Theorem 2.

Proof of Lemma 1. Using (4.5) and (4.6) it can be checked that

$$\begin{aligned} \sum_{j=0}^{\infty} f_{lj} &= \lim_{z\to 1}\frac{\beta_{c_lr_l}(z)}{1-\sum_{j=1}^{k}\alpha_{r_ld_j}(z)}\sum_{i=1}^{k}\frac{\beta_{r_ld_i}(z)}{1-\alpha_{d_im}(z)} \\ &= \frac{A_{r_l}(1)+B_{r_l}(1)}{\sum_{j=1}^{k}(A_{r_l}(1)+B_{r_l}(1))}\sum_{i=1}^{k}\frac{A_{d_i}(1)+B_{d_i}(1)}{A_{d_i}(1)+B_{d_i}(1)} = 1. \end{aligned}$$

It follows from (4.5), (4.6), (4.10), (4.11), (4.13) and (4.14) that

$$\begin{aligned} \lim_{z\to 1}(z-1)(\tilde{\beta}_{c_lr_l}(z)-\hat{\beta}_{c_lr_l}(z)) &= -(A_{r_l}(1)+B_{r_l}(1))(1-n\zeta_{c_lr_l}(1)) \\ \lim_{z\to 1}(z-1)(\tilde{\beta}_{r_ld_j}(z)-\hat{\beta}_{r_ld_j}(z)) &= -(A_{d_j}(1)+B_{d_j}(1))(1-k\zeta_{r_ld_j}(1)), \\ \lim_{z\to 1}(z-1)(1-\tilde{\alpha}_{d_im}(z)+\hat{\alpha}_{d_im}(z)) &= -(A_{d_i}(1)+B_{d_i}(1))(1+k\eta_{d_im}(1)), \\ \lim_{z\to 1}(z-1)(1-\sum_{j=1}^{k}[\tilde{\alpha}_{r_ld_j}(z)-\hat{\alpha}_{r_ld_j}(z)]) &= -(A_{r_l}(1)+B_{r_l}(1))\sum_{j=1}^{k}(1+n\zeta_{r_ld_j}(1)) \end{aligned}$$

and

$$\kappa_j(1) = \lim_{z\to 1}\kappa_j(z) = \frac{\eta_{d_jm}(1)}{1+k\eta_{d_jm}(1)}, \quad \hat{k}_j(1) = \lim_{z\to 1}\hat{\kappa}_j(z) = \frac{\sum_{i=1}^{k}\eta_{r_jd_i}(1)}{\sum_{i=1}^{n}(1+n\eta_{r_jd_i}(1))}.$$

Thus

$$\sum_{j=0}^{\infty}\hat{f}_{lj} = \lim_{z\to 1}\hat{\lambda}_l(z) = 1+Q_l$$

for $1 \le l \le n$, where

$$\begin{aligned} Q_l = \; & -\frac{\zeta_{c_lr_l}(1)+\frac{1}{k}\sum_{i=1}^{k}\eta_{r_ld_i}(1)}{\frac{1}{n}+\sum_{i=1}^{k}\eta_{r_ld_i}(1)} - \sum_{i=1}^{k}\frac{\zeta_{r_ld_i}(1)+\eta_{d_im}(1)}{1+k\eta_{d_im}(1)} \\ & + \frac{\zeta_{c_lr_l}(1)+\frac{1}{k}\sum_{i=1}^{k}\eta_{r_ld_i}(1)}{\frac{1}{n}+\sum_{i=1}^{k}\eta_{r_ld_i}(1)}\sum_{i=1}^{k}\frac{\zeta_{r_ld_i}(1)+\eta_{d_im}(1)}{1+k\eta_{d_im}(1)} + R_l, \end{aligned}$$

$$
\begin{aligned}
R_l &= \frac{(n\zeta_{c_l r_l}(1)-1)\sum_{i=1}^k (1+k\eta_{d_i m}(1))^{-1}\sum_{j=1}^k[\zeta_{r_l d_j}(1)-(1-k\zeta_{r_l d_j}(1))\kappa_j(1)]}{(1-\sum_{j=1}^k \kappa_j(1))\sum_{i=1}^k(1+n\eta_{r_l d_i}(1))} \\
&\quad + \frac{\sum_{l'=1}^n \bar{\rho}_{l'}(1)}{1-\sum_{j=1}^k \hat{\kappa}_j(1)}[\zeta_{c_l r_l}(1)-(1-n\zeta_{c_l r_l}(1))\hat{\kappa}_l(1)]
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\rho}_l(1) = \lim_{z\to 1}\frac{\rho_l(z)}{z-1} &= -\frac{\sum_{i=1}^k (1+k\eta_{d_i m}(1))^{-1}\sum_{j=1}^k[\zeta_{r_l d_j}(1)-(1-k\zeta_{r_l d_j}(1))\kappa_j(1)]}{(1-\sum_{j=1}^k \kappa_j(1))\sum_{i=1}^k(1+n\eta_{r_l d_i}(1))} \\
&\quad + \frac{\sum_{i=1}^k \frac{1-k\zeta_{r_l d_i}(1)}{1+k\eta_{d_i m}(1)}}{\sum_{i=1}^k (1+n\eta_{r_l d_i}(1))}.
\end{aligned}
$$

As can be seen, the expression of Q_l is very complex, and it only depends on $\{\eta_{r_l d_i}(1)\}$, $\{\eta_{d_i m}(1)\}$, $\{\zeta_{r_l d_i}(1)\}$ and $\{\zeta_{c_l r_l}(1)\}$. In particular, when $\eta_{r_l d_i}(1) = 0$, $\eta_{d_i m}(1) = 0$, $\zeta_{r_l d_i}(1) = 0$ and $\zeta_{c_l r_l}(1) = 0$, we have $Q_l = 0$, and therefore $\sum_{j=0}^{\infty} \hat{f}_{lj} = 1$.

Proof of Theorem 3. Denote $z_{c_l}(t)$ by $z_l(t)$ briefly in the following. Let $D(z) = \sum_{j=0}^{\infty} d_j z^{-j}$. It follows from (5.1) and (6.1) that $\sum_{i=1}^k z_{d_i m}(t) = \sum_{l=1}^n \sum_{j=0}^t f_{lj} z_{c_l}(t-j) + d_t$, and therefore

$$
\begin{aligned}
&\sum_{t=0}^T Var(\sum_{i=1}^k z_{d_i m}(t)) \\
&= \sum_{l=1}^n \sum_{t=0}^T \sum_{j=0}^t f_{lj}^2 Var(z_l(t-j)) + \sum_{l=1}^n \sum_{t=1}^T \sum_{0\le i\ne j\le t} f_{li} f_{lj} Cov(z_l(t-i), z_l(t-j)) \\
&\quad + \sum_{1\le l\ne l'\le n} \sum_{t=0}^T \sum_{j=0}^t f_{lj} f_{l'j} Cov(z_l(t-j), z_{l'}(t-j)) \\
&\quad + \sum_{1\le l\ne l'\le n} \sum_{t=0}^T \sum_{0\le i\ne j\le t} f_{li} f_{l'j} Cov(z_l(t-i), z_{l'}(t-j)).
\end{aligned}
$$

Note that

$$
\sum_{t=0}^T \sum_{j=0}^t f_{lj}^2 Var(z_l(t-j)) = \sum_{j=0}^T f_{lj}^2 \sum_{t=0}^{T-j} Var(z_l(t)) \le \left(\sum_{j=0}^T f_{lj}^2\right)\left(\sum_{t=0}^T Var(z_l(t))\right).
$$

From (5.6) it follows that

$$
\begin{aligned}
\sum_{t=1}^T \sum_{0\le i\ne j\le t} f_{li} f_{lj} Cov(z_l(t-i), z_l(t-j)) &\le \left(\sum_{0\le i\ne j\le T} f_{li} f_{lj}\right)\left(\sum_{t=0}^T Var(z_{c_l}(t))\right), \\
\sum_{t=0}^T \sum_{j=0}^t f_{lj} f_{l'j} Cov(z_l(t-j), z_{l'}(t-j)) &\le (\sum_{j=0}^T f_{lj} f_{l'j})\left(\sum_{t=0}^T Cov(z_l(t), z_{l'}(t))\right)
\end{aligned}
$$

and

$$\sum_{t=0}^{T} \sum_{0\le i\ne j\le t} f_{li} f_{l'j} Cov(z_l(t-i), z_{l'}(t-j)) \le \left(\sum_{0\le i\ne j\le T} f_{li} f_{l'j} \right) \left(\sum_{t=0}^{T} Cov(z_l(t), z_{l'}(t)) \right).$$

Thus, by (5.7) we have

$$\begin{aligned}
\sum_{t=0}^{T} Var\left(\sum_{i=1}^{k} z_{d_i m}(t)\right) &\le \sum_{l=1}^{n} \left(\sum_{j=0}^{T} f_{lj}\right)^2 \left(\sum_{t=0}^{T} Var(z_l(t))\right) \\
&\quad + \sum_{1\le l\ne l'\le n} \left(\sum_{j=0}^{T} f_{lj}\right) \left(\sum_{j=0}^{T} f_{l'j}\right) \left(\sum_{t=0}^{T} Cov(z_l(t), z_{l'}(t))\right) \\
&= \sum_{l=1}^{n} \left[\left(\sum_{j=0}^{T} f_{lj}\right)^2 - 1\right] \left(\sum_{t=0}^{T} Var(z_l(t))\right) \\
&\quad + \sum_{1\le l\ne l'\le n} \left[\left(\sum_{j=0}^{T} f_{lj}\right) \left(\sum_{j=0}^{T} f_{l'j}\right) - 1\right] \left(\sum_{t=0}^{T} Cov(z_l(t), z_{l'}(t))\right) \\
&\quad + \sum_{l=1}^{n} \sum_{t=0}^{T} Var(z_l(t)) + \sum_{1\le l\ne l'\le n} \sum_{t=0}^{T} Cov(z_l(t), z_{l'}(t)) \\
&= -\sum_{l=1}^{n} \sum_{l'=1}^{n} \left[1 - \left(\sum_{j=0}^{T} f_{lj}\right) \left(\sum_{j=0}^{T} f_{l'j}\right)\right] \left(\sum_{t=0}^{T} Cov(z_l(t), z_{l'}(t))\right) \\
&\quad + \sum_{t=0}^{T} Var\left(\sum_{l=1}^{n} z_l(t)\right) \\
&\le \sum_{t=0}^{T} Var\left(\sum_{l=1}^{n} z_l(t)\right)
\end{aligned}$$

and therefore

$$BW(T) = \frac{\sum_{t=0}^{T} Var(\sum_{i=1}^{k} z_{d_i m}(t))}{\sum_{t=0}^{T} Var(\sum_{l=1}^{n} z_l(t))} \le 1$$

for $T \ge T_0$. This completes the proof of Theorem 3.

Proof of Theorem 4. We omit the proof of Theorem 4 since it is the same as that of Theorem 3.

Proof of Theorem 5. Since any two customer demands are unrelated and the demands at different times are also unrelated, that is, $Cov(z_{c_l}(t-i),\, z_{c_{l'}}(t-j)) = 0$ for $i \ne j$, $1 \le l,\, l' \le n$, it follows that

$$\sum_{t=0}^{T} Var\left(\sum_{i=1}^{k} z_{d_i m}(t)\right) = \sum_{l=1}^{n} \sum_{t=0}^{T} \sum_{j=0}^{t} f_{lj}^2 Var(z_{c_l}(t-j)) = \sum_{l=1}^{n} \sum_{j=0}^{T} f_{lj}^2 \sum_{t=0}^{T-j} Var(z_{c_l}(t)).$$

This implies that there is no bullwhip effect in the stable order policy for the competition (or horizontal collaboration) supply network when $\sum_{j=0}^{\infty} f_{lj}^2 \leq 1$ (or $\sum_{j=0}^{\infty} \hat{f}_{lj}^2 \leq 1$) for $1 \leq l \leq n$, since

$$\begin{aligned}\sum_{t=0}^{T} Var\left(\sum_{i=1}^{k} z_{d_i m}(t)\right) &= \sum_{l=1}^{n}\sum_{j=0}^{T} f_{lj}^2 \sum_{t=0}^{T-j} Var(z_{c_l}(t)) \\ &\leq \sum_{l=1}^{n}\sum_{t=0}^{T} Var(z_{c_l}(t)) = \sum_{t=0}^{T} Var\left(\sum_{l=1}^{n} z_{c_l}(t)\right),\end{aligned}$$

when $\sum_{j=0}^{\infty} f_{lj}^2 \leq 1$ (or $\sum_{j=0}^{\infty} \hat{f}_{lj}^2 \leq 1$) for all $1 \leq l \leq n$.

Let $\sum_{j=0}^{\infty} f_{lj}^2 > 1$ (or $\sum_{j=0}^{\infty} \hat{f}_{lj}^2 > 1$) for $1 \leq l \leq n$. Then there is a positive number $a > 1$ such that $\sum_{j=0}^{T} f_{lj}^2 \geq a$ for $T \geq T_0$ and $1 \leq l \leq n$. Note that

$$\begin{aligned}\sum_{t=0}^{T} Var\left(\sum_{i=1}^{k} z_{d_i m}(t)\right) &= \sum_{l=1}^{n}\left(\sum_{j=0}^{T} f_{lj}^2\right)\sum_{t=0}^{T} Var(z_{c_l}(t)) \\ &\quad - \sum_{l=1}^{n}\sum_{j=1}^{T} f_{lj}^2 \sum_{t=0}^{j-1} Var(z_{c_l}(T-t)).\end{aligned}$$

Since $\sum_{j=0}^{\infty} f_{lj}^2 < \infty$ and $Var(z_{c_l}(t))$ are bounded for $1 \leq l \leq n, t \geq 0$, it follows that

$$\lim_{T\to\infty} \frac{\sum_{l=1}^{n}\sum_{j=1}^{T} f_{lj}^2 \sum_{t=0}^{j-1} Var(z_{c_l}(T-t))}{\sum_{l=1}^{n}\sum_{t=0}^{T} Var(z_{c_l}(t))} = 0$$

for both $\sum_{l=1}^{n}\sum_{t=0}^{\infty} Var(z_{c_l}(t)) < \infty$ and $\sum_{l=1}^{n}\sum_{t=0}^{\infty} Var(z_{c_l}(t)) = \infty$. Thus,

$$\lim_{T\to\infty} \frac{\sum_{t=0}^{T} Var\left(\sum_{i=1}^{k} z_{d_i m}(t)\right)}{\sum_{t=0}^{T} Var\left(\sum_{l=1}^{n} z_{c_l}(t)\right)} \geq a.$$

That is $\sum_{t=0}^{T} Var(\sum_{i=1}^{k} z_{d_i m}(t)) > \sum_{t=0}^{T} Var(\sum_{l=1}^{n} z_{c_l}(t))$ for $T \geq T_1$. This means that the bullwhip effect exists in the the competition (or horizontal collaboration) supply network. This completes the proof of Theorem 5.

Proof of Lemma 2. Let $c_{lj} = f_{lj}/g_{i_0 j}$ and $(z-1)\beta_{r_l d_{i_0}}(z) = \sum_{s=0}^{p} \beta_{ls} z^{-s}$. Note that $a_{i_0,j-1}/a_{i_0 j} = 1$ as $j \to \infty$ and $(g_i)/(g_{i_0})^j \to 0$ as $j \to \infty$ for $i \neq i_0$. Hence

$$\lim_{j\to\infty} \frac{g_{l i_0 j}}{g_{i_0 j}} = \lim_{j\to\infty} \frac{\sum_{s=0}^{p} \beta_{ls} g_{i_0,j-s}}{g_{i_0 j}} = (g_{i_0} - 1)\beta_{r_l d_{i_0}}(g_{i_0})$$

and $g_{lis}/g_{i_0 j} \to 0$ as $j \to \infty$ for $i \neq i_0, 0 \leq s \leq j$. Since

$$\sum_{j=0}^{\infty} f_{lj} z^{-j} = \frac{\beta_{c_l r_l}(z)}{1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(z)} \sum_{i=1}^{k} \frac{\beta_{r_l d_i}(z)}{1 - \alpha_{d_i m}(z)} = \sum_{j=0}^{\infty}\left[\sum_{i=1}^{k}\sum_{s=0}^{j} g_{lis} h'_{l,j-s}\right] z^{-j},$$

and

$$\sum_{j=0}^{\infty} h'_{lj}/(g_{i_0})^j = \frac{\beta_{c_l r_l}(g_{i_0})}{1 - \sum_{j=1}^{k} \alpha_{r_l d_j}(g_{i_0})},$$

it follows that

$$\begin{aligned}\lim_{j\to\infty} c_{lj} &= \lim_{j\to\infty}\frac{\sum_{i=1}^{k}\sum_{s=0}^{j} g_{lis}h'_{l,j-s}}{g_{i_0 j}} = \lim_{j\to\infty}\sum_{s=0}^{j}\frac{g_{li_0 s}}{g_{i_0 s}}\frac{a_{i_0 s}}{a_{i_0 j}}\frac{h'_{l,j-s}}{(g_{i_0})^{j-s}}\\ &= (1-1/g_{i_0})\beta_{r_l d_{i_0}}(g_{i_0})\frac{\beta_{c_l r_l}(g_{i_0})}{1-\sum_{j=1}^{k}\alpha_{r_l d_j}(g_{i_0})}.\end{aligned}$$

Proof of Theorem 6. Let

$$\begin{aligned}\alpha_{d_i}(z) &= 1-\alpha_{d_i m}(z),\quad \alpha_{r_l}(z) = 1-\sum_{j=1}^{k}\alpha_{r_l d_j}(z)\\ \bar\beta_l(z) &= \tilde\beta_{c_l r_l}(z)-\beta_{c_l r_l}(z)-\hat\beta_{c_l r_l}(z)\\ \bar\beta_{li}(z) &= \tilde\beta_{r_l d_i}(z)-\beta_{r_l d_i}(z)+\sum_{j\neq i}\hat\beta_{r_l d_j}(z)\\ \eta_i(z) &= \sum_{l\in S_{j+1}}[\tilde\alpha_{il}(z)-\alpha_{il}(z)-\hat\alpha_{il}(z)],\quad i\in S_j, j=1,2,\end{aligned}$$

and

$$\frac{\tilde\beta_{c_l r_l}(z)-\hat\beta_{c_l r_l}(z)}{1-\sum_{j=1}^{k}[\tilde\alpha_{r_l d_j}(z)-\tilde\alpha_{r_l d_j}(z)]}\sum_{i=1}^{k}\frac{\tilde\beta_{r_l d_i}(z)+\sum_{j\neq i}\hat\beta_{r_l d_j}(z)}{1-\tilde\alpha_{d_i m}(z)+\hat\alpha_{d_i m}(z)} = \sum_{j=0}^{\infty}\tilde f_{lj}z^{-j}.$$

It follows that

$$\begin{aligned}\sum_{j=0}^{\infty}[\tilde f_{lj}-f_{lj}]z^{-j} &= \frac{\beta_{c_l r_l}(z)+\bar\beta_l(z)}{\alpha_{r_l}(z)-\eta_{r_l}(z)}\sum_{i=1}^{k}\frac{\beta_{r_l d_i}(z)+\bar\beta_{li}(z)}{\alpha_{d_i}(z)-\eta_{d_i}(z)}-\frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)}\sum_{i=1}^{k}\frac{\beta_{r_l d_i}(z)}{\alpha_{d_i}(z)}\\ &= \frac{[\bar\beta_l(z)\alpha_{r_l}(z)/\beta_{c_l r_l}(z)+\eta_{r_l}(z)]}{\alpha_{r_l}(z)-\eta_{r_l}(z)}\frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)}\sum_{i=1}^{k}\frac{\beta_{r_l d_i}(z)}{\alpha_{d_i}(z)}\\ &\quad+\frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)}\sum_{i=1}^{k}\frac{[\bar\beta_{li}(z)\alpha_{d_i}(z)/\beta_{r_l d_i}(z)+\eta_{d_i}(z)]}{\alpha_{d_i}(z)-\eta_{d_i}(z)}\frac{\beta_{r_l d_i}(z)}{\alpha_{d_i}(z)}\\ &\quad+\frac{\bar\beta_l(z)\alpha_{r_l}(z)+\beta_{c_l r_l}(z)\eta_{r_l}(z)}{(\alpha_{r_l}(z)-\eta_{r_l}(z))\alpha_{r_l}(z)}\sum_{i=1}^{k}\frac{\bar\beta_{li}(z)\alpha_{d_i}(z)+\beta_{r_l d_i}(z)\eta_{d_i}(z)}{(\alpha_{d_i}(z)-\eta_{d_i}(z))\alpha_{d_i}(z)}.\end{aligned}$$

Without loss of generality, we assume that $i_1 = i_0$, that is $\hat g_{i_0} = g_{i_0}$. This means that g_{i_0} is a root of the equation $(1-z^{-1})(\alpha_{d_{i_0}}(z)-\eta_{d_{i_0}}(z)) = 0$, and g_{i_0} is the largest root in absolute value of the equations $(1-z^{-1})(\alpha_{d_i}(z)-\eta_{d_i}(z)) = 0$ and $(1-z^{-1})(\alpha_{r_l}(z)-\eta_{r_l}(z)) = 0$ for $i\neq i_0$, $1\le i\le k$ and $1\le l\le n$. Let

$$\begin{aligned}q_l(z) &= \frac{\bar\beta_l(z)\alpha_{r_l}(z)/\beta_{c_l r_l}(z)+\eta_{r_l}(z)}{\alpha_{r_l}(z)-\eta_{r_l}(z)} = \sum_{j=0}^{\infty}q_{lj}z^{-j}\\ q_{li}(z) &= \frac{\bar\beta_{li}(z)\alpha_{d_i}(z)/\beta_{r_l d_i}(z)+\eta_{d_i}(z)}{\alpha_{d_i}(z)-\eta_{d_i}(z)} = \sum_{j=0}^{\infty}q_{lij}z^{-j}\end{aligned}$$

and

$$\frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)}(1-1/z)\beta_{r_l d_i}(z) = \sum_{j=0}^{\infty} f_{lij} z^{-j}.$$

Since $(1-z^{-1})\alpha_{d_{i_0}}(z)$ and $(1-z^{-1})(\alpha_{d_{i_0}}(z) - \eta_{d_{i_0}}(z))$ can be rewritten as

$$\begin{aligned}
(1-z^{-1})\alpha_{d_{i_0}}(z) &= (1-g_{i_0}/z)\alpha'_{d_{i_0}}(z),\\
(1-z^{-1})(\alpha_{d_{i_0}}(z) - \eta_{d_{i_0}}(z)) &= (1-g_{i_0}/z)\alpha''_{d_{i_0}}(z),
\end{aligned}$$

where all roots of the equations $\alpha'_{d_{i_0}}(z) = 0$ and $\alpha''_{d_{i_0}}(z) = 0$ in absolute value are less than $|g_{i_0|}$, and $c_l \neq 0$, that is, $\beta_{c_l r_l}(g_{i_0}) \neq 0$ and $\beta_{r_l d_{i_0}}(g_{i_0})$, it follows that $q_l(g_{i_0}) < \infty$, and

$$q_{li_0}(g_{i_0}) = \frac{[\bar{\beta}_{li}(g_{i_0})/\beta_{r_l d_i}(g_{i_0}) + 1]\alpha'_{d_{i_0}}(g_{i_0}) - \alpha''_{d_{i_0}}(g_{i_0})}{\alpha''_{d_{i_0}}(g_{i_0})} < \infty.$$

Moreover, $q_l(g_{i_0}) \to 0$ and $q_{li_0}(g_{i_0}) \to 0$ as $|\eta_{d_i m}(p)| \to 0$, $|\eta_{r_l d_j}(p)| \to 0$, $|\zeta_{c_l r_l}(p)| \to 0$ and $|\zeta_{r_l d_i}(p)| \to 0$.

Note that

$$q_l(z) \times \frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)} \sum_{i=1}^{k} \frac{\beta_{r_l d_i}(z)}{\alpha_{d_i}(z)} = \sum_{j=0}^{\infty}[\sum_{k=0}^{j} q_{l,j-k} f_{lk}] z^{-j}$$

$$\frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)} \frac{(1-1/z)\beta_{r_l d_i}(z)}{(1-1/z)\alpha_{d_i}(z)} = \sum_{j=0}^{\infty}[\sum_{k=0}^{j} f_{li,j-k} g_{ik}] z^{-j}$$

and

$$q_{li}(z) \times \frac{\beta_{c_l r_l}(z)}{\alpha_{r_l}(z)} \frac{\beta_{r_l d_i}(z)}{\alpha_{d_i}(z)} = \sum_{j=0}^{\infty}[\sum_{k=0}^{j} q_{li,j-k}(\sum_{k'=0}^{k} f_{li,k-k'} g_{ik'})] z^{-j}.$$

By Lemma 2 we have

$$\begin{aligned}
\lim_{j\to\infty} \frac{\sum_{k=0}^{j} q_{l,j-k} f_{lk}}{f_{lj}} &= \lim_{j\to\infty} \sum_{k=0}^{j} \frac{q_{l,j-k}}{(g_{i_0})^{j-k}} \frac{c_{lk}}{c_{lj}} \frac{a_{i_0 k}}{a_{i_0 j}} = q_l(g_{i_0})\\
\lim_{j\to\infty} \frac{\sum_{k=0}^{j} f_{li,j-k} g_{i_0 k}}{f_{lj}} &= \lim_{j\to\infty} \frac{1}{c_{lj}} \sum_{k=0}^{j} \frac{f_{li,j-k}}{(g_{i_0})^{j-k}} \frac{a_{i_0 k}}{a_{i_0 j}}\\
&= \frac{\beta_{c_l r_l}(g_{i_0})}{c_l \alpha_{r_l}(g_{i_0})}(1-1/g_{i_0})\beta_{r_l d_{i_0}}(g_{i_0}) = 1
\end{aligned}$$

and $\lim_{j\to\infty}(\sum_{k=0}^{j} f_{li,j-k} g_{ik})/f_{lj} = 0$ for $i \neq i_0$. Therefore

$$\begin{aligned}
\lim_{j\to\infty} \frac{\sum_{k=0}^{j} q_{li_0,j-k}(\sum_{k'=0}^{k} f_{li_0,k-k'} g_{i_0 k'})}{f_{lj}} &= \lim_{j\to\infty} \sum_{k=0}^{j} \frac{q_{li,j-k}}{(g_{i_0})^{j-k}} \frac{c_{lk} a_{i_0 k}}{c_{lj} a_{i_0 j}} \frac{(\sum_{k'=0}^{k} f_{li_0,k-k'} g_{i_0 k'})}{f_{lk}}\\
&= q_{li_0}(g_{i_0})\\
\lim_{j\to\infty} \frac{\sum_{k=0}^{j} q_{li,j-k}(\sum_{k'=0}^{k} f_{li,k-k'} g_{ik'})}{f_{lj}} &= 0
\end{aligned}$$

for $i \neq i_0$. Thus, $\tilde{f}_{lj} - f_{lj}$ can be written as

$$\tilde{f}_{lj} - f_{lj} = \vartheta_{lj} f_{lj}, \tag{A.12}$$

where the number series $\{\vartheta_{lj}, j \geq 0\}$ satisfies

$$\vartheta_l \triangleq \lim_{j\to\infty} \vartheta_{lj} = q_l(g_{i_0}) + q_{li_0}(g_{i_0}) + q_l(g_{i_0}) q_{li_0}(g_{i_0}).$$

Furthermore, for any fixed small positive number $\epsilon < 1/3$, by the definitions of $q_l(z)$ and $q_{li}(z)$ we can choose small values of $|\zeta_{c_l r_l}(p)|$, $|\zeta_{r_l d_i}(p)|$, $|\eta_{d_i m}(p)|$ and $|\eta_{r_l d_j}(p)|$ such that $|\vartheta_{lj}| \leq \epsilon$ for $j \geq 0$. That is, we can set the order policy for horizontal collaboration $\{\eta_{r_l d_i}, \eta_{d_i m}, \zeta_{r_l d_i}, \zeta_{c_l r_l}\}$ using small values of $|\eta_{r_l d_i}(p)|$, $|\eta_{d_i m}(p)|$, $|\zeta_{r_l d_i}(p)|$ and $|\zeta_{c_l r_l}(p)|$ such that $|\vartheta_{lj}| \leq \epsilon$ for $j \geq 0$.

Next we consider the relation between $\hat{f}_{lj}$ and f_{lj}. Let

$$\mu_l(z) \triangleq \hat{\lambda}_l(z) - \frac{\beta_{c_l r_l}(z) + \bar{\beta}_l(z)}{\alpha_{r_l}(z) - \eta_{r_l}(z)} \sum_{i=1}^{k} \frac{\beta_{r_l d_i}(z) + \bar{\beta}_{li}(z)}{\alpha_{d_i}(z) - \eta_{d_i}(z)}$$

$$[(1 - z^{-1})(\alpha_{d_i}(z) - \eta_{d_i}(z))]^{-1} = \sum_{j=0}^{\infty} \tilde{g}_{ij} z^{-j},$$

$$[(1 - z^{-1})(\alpha_{r_l}(z) - \eta_{r_l}(z))]^{-1} = \sum_{j=0}^{\infty} \tilde{h}_{lj} z^{-j}$$

and $\mu_l(z) = \sum_{j=0}^{\infty} \mu_{lj} z^{-j}$. Since $\tilde{\alpha}_{d_i m}(z) \to \alpha_{d_i m}(z)$, $\tilde{\alpha}_{r_l d_j}(z) \to \alpha_{r_l d_j}(z)$, and so $\tilde{g}_{ij} \to g_{ij}$, $\tilde{h}_{lj} \to h_{lj}$ as $|\eta_{d_i m}(p)| \to 0$ and $|\eta_{r_l d_j}(p)| \to 0$, so that we can choose small values of $|\eta_{d_i m}(p)|$ and $|\eta_{r_l d_j}(p)|$ such that $\tilde{g}_{ij} = \tilde{a}_{ij}(\tilde{g}_i)^j$, $j \geq 0$, satisfy the condition (5.9), and therefore, $\tilde{f}_{ij} = \tilde{c}_{lj}\tilde{g}_{ij}$. Note that $\tilde{g}_{i_0} = g_{i_0}$. Similar to (A.10), the coefficient series $\mu_{lj}, j \geq 0$, can be written as

$$\mu_{lj} = \omega_{lj} \tilde{f}_{ij}, \tag{A.13}$$

for $j \geq 0$, where the number series $\{\omega_{lj}, j \geq 0\}$ satisfies

$$\begin{aligned} \omega_l &= \lim_{j\to\infty} \omega_{lj} \\ &= \tau_l + \frac{\hat{\beta}_{c_l r_l}(g_{i_0})(1 - \hat{\kappa}_l(g_{i_0})) + \tilde{\beta}_{c_l r_l}(g_{i_0})\hat{\kappa}_l(g_{i_0})}{1 - \sum_{j=1}^{k} \hat{\kappa}_j(g_{i_0})} \sum_{j=1}^{n} \frac{1 + \tau_j}{\tilde{\beta}_{c_j r_j}(g_{i_0}) - \hat{\beta}_{c_j r_j}(g_{i_0})}, \end{aligned}$$

where

$$\tau_l = \frac{\sum_{j=1}^{k} [\tilde{\beta}_{r_l d_j}(g_{i_0}) + \sum_{i \neq j} \hat{\beta}_{r_l d_i}(g_{i_0})] \kappa_j(g_{i_0})}{(\tilde{\beta}_{r_l d_{i_0}}(g_{i_0}) + \sum_{j \neq i_0} \hat{\beta}_{r_l d_j}(g_{i_0}))(1 - \sum_{j=1}^{k} \kappa_j(g_{i_0}))}.$$

Moreover, $\hat{\beta}_{c_l r_l}(g_{i_0}) \to 0$, $\kappa_j(g_{i_0}) \to 0$ and $\hat{\kappa}_l(g_{i_0}) \to 0$ as $|\eta_{d_i m}(p)| \to 0$, $|\eta_{r_l d_j}(p)| \to 0$, $|\zeta_{c_l r_l}(p)| \to 0$ and $|\zeta_{r_l d_i}(p)| \to 0$, so $\omega_l \to 0$. Thus, we can take small values of $|\zeta_{c_l r_l}(p)|$, $|\zeta_{r_l d_i}(p)|$, $|\eta_{d_i m}(p)|$ and $|\eta_{r_l d_j}(p)|$ such that $|\omega_{lj}| \leq \epsilon$ for $j \geq 0$.

Since

$$\sum_{j=0}^{\infty} \hat{f}_{lj} z^{-j} = \hat{\lambda}_l(z) = \lambda_l(z) + \hat{\lambda}_l(z) - \lambda_l(z)$$
$$= \sum_{j=0}^{\infty} f_{lj} z^{-j} + \sum_{j=0}^{\infty} [\tilde{f}_{lj} - f_{lj}] z^{-j} + \sum_{j=0}^{\infty} \mu_{lj} z^{-j},$$

it follows from (A.10) and (A.11) that

$$\hat{f}_{lj} = f_{lj} + \vartheta_{lj} f_{lj} + \omega_{lj} \tilde{f}_{lj} = (1 + \varepsilon_{lj}) f_{lj},$$

where $\varepsilon_{lj} = \vartheta_{lj} + \omega_{lj} + \vartheta_{lj}\omega_{lj}$ and $|\varepsilon_{lj}| \leq 2\epsilon + \epsilon^2 < 1$. This is (5.12). Furthermore, by (5.5) and (5.12) we have

$$\sum_{j=0}^{\infty} \hat{f}_{lj} = 1 + Q_l = 1 + \sum_{j=0}^{\infty} \varepsilon_{lj} f_{lj}.$$

This implies (5.13). Finally, we have $\lim_{j\to\infty} \varepsilon_{lj} = \vartheta_l + \omega_l + \vartheta_l \omega_l$. This completes the proof of Theorem 6.

Proof of Theorem 7. By the Lagrange multiplier method, the equations (6.7) and (6.8) can be written as

$$L(f_1, \dots, f_n, \lambda_1, \lambda_2) = \frac{1}{2} \sum_{l=1}^{n} \sum_{l'=1}^{n} f_l V_{ll'} f'_{l'} + \lambda_1 \left(C_T - \sum_{l=1}^{n} f_l \mu'_l \right) + \lambda_2 \left(n - \sum_{l=1}^{n} f_l \mathbf{1}' \right).$$

Let

$$\frac{\partial L}{\partial f_l} = 2 \left[V_{ll} f'_l + \sum_{i=l+1}^{n} V_{li} f'_i - \lambda_1 \mu'_l - \lambda_2 \mathbf{1}' \right] = 0$$

for $1 \leq l \leq n$. Hence,

$$f'_n = \lambda_1 V_{ll}^{-1} \mu'_l + \lambda_2 V_{ll}^{-1} \mathbf{1}' \tag{A.14}$$

$$f'_l = \lambda_1 V_{ll}^{-1} \mu'_l + \lambda_2 V_{ll}^{-1} \mathbf{1}' - \sum_{i=l+1}^{n} V_{ll}^{-1} V_{li} f'_i \tag{A.15}$$

for $1 \leq l \leq n-1$. By (A.12) and (A.13) we see that f'_l can be expressed as $f'_l = \lambda_1 \alpha'_l + \lambda_2 \beta'_l$, where both α'_l and β'_l are vector functions that depend only on μ_j, V_{jj}^{-1} and $V_{j,j+1}, \dots V_{j,n}$ for $l \leq j \leq n$. Thus

$$f'_l = \lambda_1 V_{ll}^{-1} \mu'_l + \lambda_2 V_{ll}^{-1} \mathbf{1}' - \sum_{i=l+1}^{n} V_{ll}^{-1} V_{li} [\lambda_1 \alpha'_i + \lambda_2 \beta'_i]$$

$$= \lambda_1 \left(V_{ll}^{-1} \mu'_l - \sum_{i=l+1}^{n} V_{ll}^{-1} V_{li} \alpha'_i \right) + \lambda_2 \left(V_{ll}^{-1} \mathbf{1}' - \sum_{i=l+1}^{n} V_{ll}^{-1} V_{li} \beta'_i \right) \tag{A.16}$$

for $l \leq j \leq n$. Multiplying the left-hand side of the f'_l above by the vectors $\mathbf{1}$ and μ_l, and by (6.8), we have

$$n = a_1 \lambda_1 + b_1 \lambda_2$$

$$c_T \;=\; a_2\lambda_1 + b_2\lambda_2,$$

where a_1, a_2, b_1 and b_2 are defined in (6.10) and (6.11). Solving the above two equations, λ_1 and λ_2 become

$$\lambda_1^* \;=\; \frac{b_1c_T - nb_2}{a_2b_1 - a_1b_2}, \quad \lambda_2^* = \frac{na_2 - a_1c_T}{a_2b_1 - a_1b_2},$$

Plugging λ_1^* and λ_2^* into (A.14), we obtain the optimal vector solutions f_l^*, $1 \le l \le n$ in (6.8).

When $V_{ll'} = 0$ for $l \neq l'$, we have

$$a_2b_1 - a_1b_2 = \left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mu_l'\right)\left(\sum_{l=1}^n \mathbf{1}V_{ll}^{-1}\mathbf{1}'\right) - \left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mathbf{1}'\right)^2,$$

where $a_1 = b_2$. Since V_{ll}^{-1} is positive definite and $\mu_l \neq c\mathbf{1}$ for $1 \le l \le n$, it follows that

$$\sum_{l=1}^n (x\mu_l + \mathbf{1})V_{ll}^{-1}(x\mu_l + \mathbf{1})' > 0$$

for any real number x. Note that

$$\left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mu_l'\right)x^2 + 2\left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mathbf{1}'\right)x + \sum_{l=1}^n \mathbf{1}V_{ll}^{-1}\mathbf{1}'$$
$$= \sum_{l=1}^n (x\mu_l + \mathbf{1})V_{ll}^{-1}(x\mu_l + \mathbf{1})' > 0.$$

This implies that the quadratic discriminant satisfies

$$4\left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mathbf{1}'\right)^2 - 4\left(\sum_{l=1}^n \mu_l V_{ll}^{-1}\mu_l'\right)\left(\sum_{l=1}^n \mathbf{1}V_{ll}^{-1}\mathbf{1}'\right) < 0.$$

That is, $a_2b_1 - a_1b_2 > 0$. This completes the proof of Theorem 7.

Proof of Theorem 8. Let

$$\frac{\partial}{\partial f_l}\left[\sum_{t=0}^T w_t\left[\sum_{l=1}^n f_l\mu_l(t)' - c_t\right]^2 + \sum_{l=1}^n\sum_{l'=1}^n\sum_{t=0}^T v_t f_l V_{ll'}(t) f_{l'}'\right] = 0$$

for $1 \le l \le n$. Hence

$$V_{ll}(v)f_l' + \sum_{j=l+1} V_{lj}(v)f_j' + \sum_{t=0}^T\left[\sum_{j=1}^n w_t\mu_j(t)f_j'\right]\mu_l'(t) - \sum_{t=0}^T w_tc_t\mu_l'(t) = 0.$$

for $1 \le l \le n$, and so

$$V'F_T' + W'F_T' = C'.$$

That is, $F_T^* = C(V+W)^{-1}$.

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