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PRACTICAL DETERMINISTIC APPROACHES TO STOCHASTIC PROGRAMMING PROBLEMS

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ABSTRACT. We consider various formulations of stochastic programming problems in a unified way. The objective is to show how one can actually look at the problems in the same way as deterministic mathematical programming problems and gain practical insights to the problems as well as computational schemes. We also provide first and second order conditions, and infinite dimensional analysis.
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1. Statement of Problem

We start by considering the following stochastic programming problem. Later on we will consider various problems to give more comprehensive and practical views.

$$\min\{g(x) + Q_\mu(x) : x\epsilon C\}$$

subject to

$$\begin{aligned} Q_\mu(x) &= \int_{\mathcal{R}^l} \widetilde{Q}(h(x,z))d\mu(z) \\ \tilde{Q}(t) &= \min\{q^T y : Wy = t, y \geq 0\} \end{aligned} \tag{1}$$

Assumptions

- **(1):** The function g is real valued and continuously differentiable on $\mathcal{R}^n$.
- **(2):** The set C is a nonempty convex and closed subset of $\mathcal{R}^n$.
- **(3):** μ is a Borel probability measure on $\mathcal{R}^m$.
- **(4):** W is a linear transformation from $\mathcal{R}^{n_1}$ to $\mathcal{R}^k$.
- **(5):** There exists functions $\Lambda \in L_2(\mathcal{R}^m)$ and Θ such that

$$|h_i(x_2,z) - h_i(x_1,z)| \leq \Lambda(z)|x_2 - x_1|$$

$$|\partial^\alpha h_i(x_2,z) - \partial^\alpha h_i(x_1,z)| \leq \Lambda(z)|x_2 - x_1|, \quad |\alpha| \leq 2$$

$$h_i(x,z) \geq \Theta(x)$$

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(6): Later we will add inequality constraint of the form $f(x) \leq 0$. We assume that f and g in the objective functional above are twice continuously differentiable. We can relax this assumption. However, we can do more, as the reader will see, if we impose this assumption. We also assume that there exists $M > 0$ such that $f(x) > 0$ if $x \geq M$, and $h_i \in L_2$.

(7): $\int_{\mathcal{R}^m} \|z\|^2 d\mu(z) < \infty$.

Virtually the same problem was studied by Romish and Schultz [5] and others.

To handle the constraint $x \in C$ we may consider introducing the following function

$$I_C(x) = \begin{cases} 0 & : \quad x \in C \\ \infty & : \quad x \notin C \end{cases}$$

For $K > 0$ let

$$P_K(x) = \inf \left\{ \frac{K}{2} \|x - \zeta\| + I_C(\zeta) : \zeta \in \mathcal{R}^n \right\}.$$

Instead of problem (1) we consider

$$(\mathcal{P}_0) \quad \min\{g(x) + Q_\mu(x) : x \in C\}$$

subject to

$$\begin{aligned} x &\in C \\ Q_\mu(x) &= \int_{\mathcal{R}^l} \widetilde{Q}(h(x,z)) d\mu(z) \\ \tilde{Q}(t) &= \min\{q^T y : Wy = t, y \geq 0\} \\ f(x) &\leq 0 \end{aligned} \tag{2}$$

Let $\mathcal{P}_2(\mathcal{R}^m)$ be the set of all probability measures on $\mathcal{R}^m$ with finite quadratic moments. We can endow $\mathcal{P}_2(\mathcal{R}^m)$ with the Wasserstein distance defined as follows:

$$W_2^2(\mu^1, \mu^2) = \min \left\{ \int_{\mathcal{R}^m \times \mathcal{R}^m} \|z_1 - z_2\|^2 d\mu(z_1, z_2) : \mu \in \Gamma(\mu^1, \mu^2) \right\}, \tag{3}$$

where $\Gamma(\mu^1, \mu^2)$ is the set of all probability measures for which μ^1, μ^2 are the marginals. With the topology induced by the Wasserstein metric $\mathcal{P}_2(\mathcal{R}^m)$ is a complete separable metric space.

We may rewrite problem $(\mathcal{P}_0)$ as follows.

$$\min\{g(x) + Q_\mu(x)\}$$

subject to

$$x \in C, \quad \mu \in \mathcal{P}_2(\mathcal{R}^l), \quad f(x) \leq 0 \tag{4}$$

In (4) x and μ are decision variables.

We first consider the equation

$$Wy = h(x,z) \tag{5}$$

Without loss of generality assume that y_i, $i = k+1, \dots, n_1$ in (5) are free variables and

$$y_i = \bar{h}_i(x, z) + A_i \cdot \bar{y}, \quad i = 1, \dots, k \tag{6}$$

where

$$A_i, \quad \bar{y} \in \mathcal{R}^{n_1 - k}, \quad \bar{y} = (y_{k+1}, \dots, y_{n_1})^T$$

and $A_i \cdot \bar{y}$ denotes the inner product of A_i and $\bar{y}$. We will sometimes write $\langle A_i, \bar{y} \rangle$. We remark that, in (6), the functions $\bar{h}_i(x, z)$ are linear combinations of the coordinates of $h(x, z)$ in (5).

Then,

$$\begin{aligned} \tilde{Q}(h(x,z)) = & \sum_{l=1}^{k} q_l \bar{h}_l(x,z) \\ & + \min \left\{ \sum_{l=1}^{k} q_l \langle A_l, \bar{y} \rangle + \sum_{l=k+1}^{n_1} q_l y_l \mid y_{k+1} \geq 0, \dots, y_{n_1} \geq 0; \right. \\ & \left. \bar{h}_i(x,z) + A_i \cdot \bar{y} \geq 0, i = 1, \dots, k \right\} \end{aligned}$$

Instead of the constraint $x \in C$ we consider $\{x : f(x) \leq 0\}$. There is no problem keeping the constraint $x \in C$. However, the constraint $\{x : f(x) \leq 0\}$ occurs more in applications.

Let ω be a smooth real-valued function such that $\omega(t) = 0$ if $t \leq 0$ and $\omega(t) > 0$ if $t > 0$. Next, let $\tilde{\omega}$ be such that $\tilde{\omega}(t) = \omega(-t)$, and $\omega(t) \sim t^2$. Let

$$\bar{q} = (q_{k+1}, \dots, q_{n_1})^T \tag{7}$$

Assumption. We assume that

$$q_1 A_1 + \cdots + q_k A_k + \bar{q} \geq 0.$$

This assumption insures that the y_i values in $\bar{y}$ cannot be arbitrarily large. Next, for $K > 0$

$$\begin{aligned} & \min_{x,\bar{y}} \left\{ g(x) + \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x,z) d\mu(z) + \sum_{l}^{k} q_l \langle A_l, \bar{y} \rangle + \langle \bar{q}, \bar{y} \rangle \right. \\ & \quad \left. + K\omega(f(x)) + K \sum_{i=1}^{k} \int_{\mathcal{R}^m} \tilde{\omega}(\bar{h}_i(x,z) + \langle A_i, \bar{y} \rangle) \exp(-|z|^2) dz \right\} \\ & = \min_{x} \left\{ g(x) + \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x,z) d\mu(z) + K\omega(f(x)) \right. \\ & \quad \left. \min_{\bar{y}} \left\{ \sum_{l}^{k} q_l \langle A_l, \bar{y} \rangle + \langle \bar{q}, \bar{y} \rangle + K \sum_{i=1}^{k} \int_{\mathcal{R}^m} \tilde{\omega}(\bar{h}_i(x,z) + \langle A_i, \bar{y} \rangle) \exp(-|z|^2) dz \right\} \right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \min_{x,\bar{y}} \Bigg\{ g(x) + \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x,z) d\mu(z) \\
&\quad + \sum_{l=1}^{k} q_l \langle A_l, \bar{y}\rangle + \langle \bar{q}, \bar{y}\rangle \mid f(x) \leq 0, \bar{h}_i(x,z) + \langle A_i, \bar{y}\rangle \geq 0, i = 1, \dots, k. \Bigg\} \\
&= \min_{x} \{ g(x) + Q_\mu(x) \mid f(x) \leq 0 \}
\end{aligned}
$$

The last "equality" allows us to consider our entire problem as a nonlinear programming problem in the variables with the additional bonus to include the measure μ as part of the set of decision variables.

Let

$$
F(x,\bar{y},\mu) = g(x) + \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x,z) d\mu(z) + \sum_{l=1}^{k} q_l \langle A_l, \bar{y}\rangle + \langle \bar{q}, \bar{y}\rangle \tag{8}
$$

Let $(x^*, \bar{y}^*, \mu^*)$, where μ^* is a Gaussian measure, be a solution to the problem

$$
(\mathcal{P}_{00}) \quad \min_{x,\bar{y},\mu} F(x,\bar{y},\mu)
$$

subject to

$$
f(x) \leq 0, \quad \bar{y} \geq 0, \quad \bar{h}_i(x,z) + A_i \cdot \bar{y} \geq 0, \quad i = 1, \dots, k.
$$

Note that we are now allowing μ to be part of the decision variables.

Let

$$
\begin{aligned}
F_K(x,\bar{y},\mu) = F(x,\bar{y},\mu) &+ K\omega(f(x)) \\
&+ K \sum_{i=1}^{k} \int_{\mathcal{R}^m} \tilde{\omega}(\bar{h}_i(x,z) + \langle A_i, \bar{y}\rangle) \exp(-|z|^2) dz \\
&+ \|x - x^*\|^2 + W^2(\mu, \mu^*) + \epsilon \|\bar{y} - \bar{y}^*\|
\end{aligned} \tag{9}
$$

where μ is a Gaussian measure. Then,

$$
F_K(x^*, \bar{y}^*, \mu^*) \leq F(x^*, \bar{y}^*, \mu^*).
$$

Lemma 1.1. *For every $0 < \epsilon < 1$ there exists K_ϵ such that $F_K(x,\bar{y},\mu) > 0$ if any of the following inequalities is an equality*

$$
\|x - x^*\|^2 \leq \epsilon^2, \quad W^2(\mu, \mu^*) \leq \epsilon^2, \quad \|\bar{y} - \bar{y}^*\| \leq \epsilon.
$$

Proof. If the lemma were false there would exist $1 > \epsilon_1 > \epsilon_1 > \cdots > \epsilon_n > \cdots \longrightarrow 0$, $K(\epsilon_1), K(\epsilon_1), \dots, K(\epsilon_n) \cdots \longrightarrow \infty$, $\mu_1, \mu_2, \dots, \mu_n, \dots$; $x_1, x_2, \dots, x_n, \dots$; $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n, \dots$; such that one of the inequalities is an equality. There exists a subsequence $n_1 < n_2 < \cdots$; $x_\epsilon{}^*, \mu_\epsilon{}^*, \bar{y}_\epsilon^*$ such that $x_{n_i} \longrightarrow x_\epsilon{}^*$, $\bar{y}_{n_i} \longrightarrow \bar{y}_\epsilon^*$, $\mu_{n_i} \longrightarrow \mu_\epsilon{}^*$ weakly, $W(\mu_{n_i}, \mu_\epsilon{}^*) \longrightarrow 0$. That is,

$$
F(x_\epsilon^*, \bar{y}_\epsilon^*, \mu_\epsilon{}^*) = g(x_\epsilon^*) + \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x_\epsilon^*, z) d\mu_\epsilon{}^*(z) + \sum_{l=1}^{k} q_l \langle A_l, \bar{y}^*\rangle + \langle \bar{q}, \bar{y}^*\rangle,
$$

$$f(x_\epsilon^*) \le 0, \quad \bar{h}_i(x_\epsilon^*, z) + A_i \cdot \bar{y}_\epsilon^* \ge 0, \quad i = 1, \dots, k$$

giving $F(x_\epsilon^*, \bar{y}_\epsilon^*, \mu_\epsilon{}^*) < F(x^*, \bar{y}^*, \mu^*)$. This is a contradiction. □

Corollary 1.1. *Let $(x_\epsilon, \bar{y}_\epsilon, \mu_\epsilon)$ be such that*

$$F_{K_\epsilon}(x_\epsilon, \bar{y}_\epsilon, \mu_\epsilon) = \min\{F_K(x, \bar{y}, \mu) \mid \|x - x^*\| \le \epsilon, \|\bar{y} - \bar{y}^*\| \le \epsilon, W^2(\mu, \mu^*) \le \epsilon^2\}$$

Then,

$$\|x - x^*\| < \epsilon, \quad \|\bar{y} - \bar{y}^*\| < \epsilon, \quad W^2(\mu, \mu^*) < \epsilon^2$$

Next we have

Corollary 1.2. *From Corollary* 1.1 *we obtain that*

$$\begin{aligned} \partial_x F_{K_\epsilon}(x_\epsilon, \bar{y}_\epsilon, \mu_\epsilon) &= 0, \\ \partial_{\bar{y}} F_{K_\epsilon}(x_\epsilon, \bar{y}_\epsilon, \mu_\epsilon) &\ge 0. \end{aligned} \tag{10}$$

If μ and ν are probability measures on Euclidean space and m_μ, m_ν are their expectations, and C_μ, C_ν are their covariance matrices then we have [2]

$$W^2(\mu, \nu) \ge \|m_\mu - m_\nu\|^2 + trace(C_\mu + C_\nu - 2(C_\mu^{1/2} C_\nu C_\mu^{1/2})^{1/2}) \tag{11}$$

In (11), equality holds if μ and ν are Gaussian. We have

$$\sum_{i=1}^k \int_{\mathcal{R}^m} q_i \bar{h}_i(x, z) d\mu(z) = \sum_{i=1}^k \int_{\mathcal{R}^m} q_i \bar{h}_i(x, z) \frac{1}{(2\pi)^{m/2} |C^{-1}|^{1/2}} \exp^{\frac{-1}{2}(z-r)^T C (z-r)} dz \tag{12}$$

where r is the mean and C is the covariance of μ. In particular, in the case of μ_ϵ we write r_ϵ for the expectation and C_ϵ for the covariance. Perturbing the mean of μ_ϵ in $F_{K_\epsilon}(x_\epsilon, \bar{y}_\epsilon, \mu_\epsilon)$ we get

$$\begin{aligned} \frac{\partial}{\partial r} F_{K_\epsilon}(x_\epsilon,\ \bar{y}_\epsilon,\ \mu_\epsilon)|_{r=r_\epsilon} &= -\sum_{i=1}^k \int_{\mathcal{R}^m} q_i \bar{h}_i(x_\epsilon, z) C_\epsilon (z - r_\epsilon) d\mu_\epsilon(z) \\ &\quad + \partial_r W^2(\mu_\epsilon, \mu^*)|_{r=r_\epsilon} \\ &= 0 \end{aligned} \tag{13}$$

Next we make a variation in the covariance of μ_ϵ. Let D be a positive semidefinite matrix. Let $\mu_{\epsilon\theta}$, $0 < \theta < 1$, be such that

$$mean(\mu_{\epsilon\theta}) = r_\epsilon \quad \text{and} \quad cov(\mu_{\epsilon\theta}) = C_\epsilon + \theta(D - C_\epsilon)$$

Then,

$$\frac{d}{d\theta} F_{K_\epsilon}(x_\epsilon, \bar{y}_\epsilon, \mu_{\epsilon\theta})|_{\theta=0^+} \ge 0. \tag{14}$$

That is,

$$-\sum_{i=1}^k \int_{\mathcal{R}^m} q_i \bar{h}_i(x_\epsilon, z) d\mu_\epsilon(z) \cdot \frac{1}{(2\pi)^{m/2} 2} trace(C_\epsilon^{-1} D)$$

$$-\frac{1}{2}\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x_\epsilon,z)\langle z,Dz\rangle d\mu_\epsilon(z)+\frac{d}{d\theta}W^2(\mu_{\epsilon\theta},\mu^*)|_{\theta=0^+}$$

$$\geq -\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x_\epsilon,z)d\mu_\epsilon(z)\cdot\frac{1}{(2\pi)^{1/2}2}\,trace(C_\epsilon^{-1}C_\epsilon)$$

$$-\frac{1}{2}\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x_\epsilon,z)\langle z,C_\epsilon z\rangle d\mu_\epsilon(z)$$

Thus, for any positive semidefinite matrix R we have

$$-\frac{1}{(2\pi)^{m/2}2}\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x_\epsilon,z)d\mu_\epsilon(z)\cdot\,trace(C_\epsilon^{-1}R)$$

$$-\frac{1}{2}\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x_\epsilon,z)\langle z,Rz\rangle d\mu_\epsilon(z)\geq 0. \tag{15}$$

2. **Limiting Operations**

Let

$$M(\epsilon)=1+K_\epsilon\omega'(f(x_\epsilon))+K_\epsilon\sum_{i=1}^{k}\|\tilde{\omega}'(\bar{h}_i(x_\epsilon,z)+\langle A_i,\bar{y}\rangle)\|_\infty$$

Dividing (10), (13), and (15) by $M(\epsilon)$ and letting $\{\epsilon\}_{>0}$ tend to zero through an appropriate subsequence we obtain

$$\lambda^0\nabla g(x^*)+\lambda^0\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x^*,z)d\mu^*(z)+\lambda_1\nabla f(x^*)$$

$$\sum_{i=1}^{k}\int_{\mathcal{R}^m}\bar{\lambda}_i(z)\nabla_x\bar{h}_i(x^*,z)\exp(-|z|^2)dz=0 \tag{16}$$

$$\lambda^0\left(\sum_{i=1}^{k}q_iA_i+\bar{q}\right)+\sum_{i=1}^{k}\lambda_{2i}A_i\geq 0,$$

$$\lambda_{2i}=\int\bar{\lambda}_i(z)\exp(-|z|^2)dz \tag{17}$$

$$\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x^*,z)C^*(z-r^*)d\mu^*(z)=0 \tag{18}$$

$$\frac{1}{(2\pi)^{m/2}2}\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x^*,z)d\mu^*(z)\cdot trace(C^{*-1}R)$$

$$+\sum_{i=1}^{k}\int_{\mathcal{R}^m} q_i\bar{h}_i(x^*,z)\langle z,Rz\rangle d\mu^*(z)\leq 0. \tag{19}$$

where in (18) and (19)

$$r^* = mean(\mu^*)$$
$$C^* = cov(\mu^*)$$

$$\lambda^0 + \lambda_1 + \sum_{i=1}^{k} \|\bar{\lambda}_i\|_\infty \neq 0, \quad \lambda^0 \geq 0, \quad \lambda_1 \geq 0, \quad \bar{\lambda}_i \geq 0. \tag{20}$$

Theorem 2.1. *Let $(x^*, \bar{y}^*, \mu^*)$ be optimal for problem $(\mathcal{P}_{00})$. Then, there exist multipliers $\lambda^0 \geq 0$, $\lambda_1 \geq 0$, $\bar{\lambda}_i \geq 0$, $i = 1, \ldots, k$ such that (16)–(20) hold.*

3. Second Order Conditions

Let $(x^*, \bar{y}^*, \mu^*)$ be optimal for problem $\mathcal{P}_{00}$. Let

$$Q_{\mu^*}(x) = \sum_{l=1}^{k} \int_{\mathcal{R}^m} q_l \bar{h}_l(x, z) d\mu^*(z) + \sum_{l=1}^{k} q_l \langle q_l, \bar{y}^* \rangle + \langle \bar{q}, \bar{y}^* \rangle$$

Suppose that $I_3(x*) = I_4(x*) = 0$. If we consider the problem

$$(\mathcal{P}_{000}) \ \min\{g(x) + Q_{\mu^*}(x)\}$$

subject to

$$f(x) \leq 0$$

We know problem $(\mathcal{P}_{000})$ has solution x^*. We will obtain second order conditions for problem $(\mathcal{P}_{000})$. For ease of notation we assume that

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \end{pmatrix}. \tag{21}$$

Let

$$\hat{I}(x) = g(x) + Q_{\mu^*}(x)$$
$$I_i = f_i(x), \quad i = 1, 2, 3, 4. \tag{22}$$

Suppose that

$$I_1(x^*) < 0$$
$$I_2(x^*) < 0$$
$$I_3(x^*) = 0$$
$$I_4(x^*) = 0 \tag{23}$$

Following the approach of Wang [7] let

$$\mathcal{P} = \{\mathbf{p} \in \mathcal{R}^\mathbf{n} : \nabla_\mathbf{x} \mathbf{I_3}(\mathbf{x}^*, \mu^*)^\mathbf{T} \mathbf{p} = \nabla_\mathbf{x} \mathbf{I_4}(\mathbf{x}^*, \mu^*)^\mathbf{T} \mathbf{p} = \nabla_\mathbf{x} \hat{\mathbf{I}}(\mathbf{x}^*, \mu^*)^\mathbf{T} \mathbf{p} = \mathbf{0}\} \tag{24}$$

Suppose that for every $p \in \mathcal{P}$ there exists a vector $V(p) \in \mathcal{R}^n$ such that

$$\begin{aligned} p^T \nabla_x^2 I_3(x^*, \mu^*) p + \nabla_x I_3(x^*, \mu^*)^T V(p) &\le 0 \\ p^T \nabla_x^2 I_4(x^*, \mu^*) p + \nabla_x I_4(x^*, \mu^*)^T V(p) &\le 0 \\ p^T \nabla_x^2 \hat{I}(x^*, \mu^*) p + \nabla_x \hat{I}(x^*, \mu^*)^T V(p) &\le 0 \end{aligned} \tag{25}$$

We keep in force that $\{\nabla_x I_i(x^*, \mu^*) : i = 3, 4\}$ is a linearly independent set. Thus the matrix

$$\begin{pmatrix} \frac{\partial I_3}{\partial x_1}(x^*) & \cdots & \frac{\partial I_3}{\partial x_n}(x^*) \\ \frac{\partial I_4}{\partial x_1}(x^*) & \cdots & \frac{\partial I_4}{\partial x_n}(x^*) \end{pmatrix} \tag{26}$$

has rank 2. Suppose that the first two columns in (27) are linearly independent. Set

$$M_1 = \begin{pmatrix} \frac{\partial I_3}{\partial x_1}(x^*) & \frac{\partial I_3}{\partial x_2}(x^*) \\ \frac{\partial I_4}{\partial x_1}(x^*) & \frac{\partial I_4}{\partial x_2}(x^*) \end{pmatrix} \tag{27}$$

Let

$$M_2(w) = \begin{pmatrix} \frac{\partial I_3}{\partial x_3}(w) & \cdots & \frac{\partial I_3}{\partial x_n}(w) \\ \frac{\partial I_4}{\partial x_3}(w) & \cdots & \frac{\partial I_4}{\partial x_n}(w) \end{pmatrix} \tag{28}$$

For $t \ge 0$ let $\alpha(t)$ be a curve in $\mathcal{R}^n$, $\alpha(t)^T = (\alpha_1(t), \cdots, \alpha_n(t))$. Let

$$E(t) = (\alpha(t)^T \nabla_x^2 I_3(x^*, \mu^*) \alpha(t) \alpha(t)^T \nabla_x^2 I_4(x^*, \mu^*) \alpha(t))^T \tag{29}$$

$$\tilde{V}(p) = (V(p_3), \ldots, V(p_n))^T \tag{30}$$

Consider the differential equation

$$\begin{pmatrix} \frac{d^2 \alpha_1(t)}{dt^2} \\ \cdots \\ \frac{d^2 \alpha_n(t)}{dt^2} \end{pmatrix} = \begin{pmatrix} -M_1^{-1}(M_2(\alpha(t)) + E(t)) \\ \tilde{V}(p) \end{pmatrix} \tag{31}$$

$$\alpha(0) = x^* \tag{32}$$

$$\frac{d\alpha}{dt}(0) = p \tag{33}$$

We have

$$\begin{aligned} \begin{pmatrix} I_3(\alpha(t)) \\ I_4(\alpha(t)) \end{pmatrix} &= \begin{pmatrix} I_3(x^*) \\ I_4(x^*) \end{pmatrix} + t \begin{pmatrix} \nabla_x I_3(x^*)^T \alpha'(0) \\ \nabla_x I_4(x^*)^T \alpha'(0) \end{pmatrix} \\ &+ \frac{1}{2} t^2 \begin{pmatrix} \nabla_x I_3(\alpha(\hat{t}))^T \alpha''(\hat{t}) + \alpha'(\hat{t})^T \nabla_x^2 I_3(\alpha(\hat{t})) \alpha'(\hat{t}) \\ \nabla_x I_4 \alpha(\hat{t}))^T \alpha''(\hat{t}) + \alpha'(\hat{t})^T \nabla_x^2 I_4(\alpha(\hat{t})) \alpha'(\hat{t}) \end{pmatrix} \end{aligned} \tag{34}$$

for some $\hat{t}$, $0 \le \hat{t} \le t$.

Using (24), (29) and (31) we can rewrite (34) as

$$\begin{pmatrix} I_3(\alpha(t)) \\ I_4(\alpha(t)) \end{pmatrix} = \begin{pmatrix} I_3(x^*) \\ I_4(x^*) \end{pmatrix} + t \begin{pmatrix} \nabla_x I_3(x^*)^T \alpha'(0) \\ \nabla_x I_4(x^*)^T \alpha'(0) \end{pmatrix}$$

$$+\frac{1}{2}t^2[M_1\begin{pmatrix}\alpha_1''(\hat{t})\\ \alpha_2''(\hat{t})\end{pmatrix}+M_2(\alpha(\hat{t}))+E(\hat{t})] \tag{35}$$

Thus, there exists t'' such that for $0 < t < t''$

$$\begin{pmatrix} I_3(\alpha(t)\\ I_4(\alpha(t)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

Lemma 3.1. *Suppose that $\nabla_x\hat{I}(x^*,\mu^*)^Tp \le 0$, $\nabla_x I_i(x^*,\mu^*)^Tp \le 0$, $i = 1,2,3,4$. Then, the system*

$$\begin{aligned}
\nabla_x\hat{I}(x^*,\mu^*)^TU + p^T\nabla_x^2\hat{I}(x^*,\mu^*)p &< 0\\
\nabla_x I_1(x^*,\mu^*)^TU + p^T\nabla_x^2 I_1(x^*,\mu^*)p &< 0\\
\nabla_x I_2(x^*,\mu^*)^TU + p^T\nabla_x^2 I_2(x^*,\mu^*)p &< 0\\
\nabla_x I_3(x^*,\mu^*)^TU + p^T\nabla_x^2 I_3(x^*,\mu^*)p &\le 0\\
\nabla_x I_4(x^*,\mu^*)^TU + p^T\nabla_x^2 I_4(x^*,\mu^*)p &\le 0
\end{aligned} \tag{36}$$

where we look for a solution U has no solution.

Proof. If the system had a solution U, then we would construct a curve $\alpha(t)$ as we did in (32) where U plays the role of $\tilde{V}(p)$ in (30). Then,

$$\begin{aligned}
\hat{I}(\alpha(t)) &< \hat{I}(x^*),\\
I_1(\alpha(t)) &< I_1(x^*),\\
I_2(\alpha(t)) &< I_2(x^*), \quad 0 \le t \le t^{**}\\
I_3(x^*) &= 0\\
I_4(x^*) &= 0
\end{aligned} \tag{37}$$

Thus, we contradict the optimality of x^*. □

Lemma 3.2 ([7]). *Let A, B, and C be $r\times n$, $p\times n$, and $m\times n$ real matices. Let b_1, b_2, b_3 be r-dimensional, p-dimensional, and m-dimensional real vectors respectively. The system*

$$\begin{aligned}
Az + b_1 &< 0,\\
Bz + b_2 &\le 0,\\
Cz + b_2 &= 0,
\end{aligned}$$

has no solution if and only if there exist $\lambda^0 \neq 0$, $\lambda \ge 0$, and μ such that

$$\lambda^{0^T}A + \lambda^TB + \mu^TC = 0, \quad \lambda^{0^T}b_1 + \lambda^Tb_2 + \mu^Tb_3 \ge 0.$$

Now using this lemma we obtain

Corollary 3.1. *Suppose that $\nabla_x\hat{I}(x^*,\mu^*)^Tp \le 0$, $\nabla_x I_i(x^*,\mu^*)^Tp \le 0$, $i = 1,2,3,4$. Then, there exist mutipliers $\tilde{\lambda}^0 \ge 0$, $\tilde{\lambda}_i \ge 0$, $i = 1,2,3,4$ such that*

(a) $(\tilde{\lambda}^0, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4) \neq (0,0,0,0,0)$
(b) $\tilde{\lambda}^0 \hat{I}(x^*, \mu^*) + \sum_{i=1}^4 \tilde{\lambda}_i \nabla_x I_i(x^*, \mu^*) = 0,$
(c) $p^T(\tilde{\lambda}^0 \hat{I}(x^*, \mu^*) + \sum_{i=1}^4 \tilde{\lambda}_i \nabla_x I_i(x^*, \mu^*))p \geq 0.$

Corollary 3.2. *Suppose that* $\{\nabla_x I_i(x^*, \mu^*)^T,\ i = 1,2,3,4\}$ *is a linearly independent set and* $\nabla_x \hat{I}(x^*, \mu^*)^T p \leq 0$, $\nabla_x I_i(x^*, \mu^*)^T p \leq 0$, $i = 1,2,3,4$. *Then, there exist multpliers* $\zeta_1 \geq 0, \zeta_2 \geq 0, \zeta_3 \geq 0, \zeta_4 \geq 0$ *such that*
(a) $\nabla_x \hat{I}(x^*, \mu^*) + \sum_{i=1}^4 \zeta_i \nabla_x I_i(x^*, \mu^*) = 0,$
(b) $p^T(\nabla_x^2 \hat{I}(x^*, \mu^*) + \sum_{i=1}^4 \zeta_i \nabla_x^2 I_i(x^*, \mu^*))p > 0.$

Remark. One can also construct second order conditions involving the variable $\bar{y}^*$ as well as μ^*.

4. Other Problems

Here we see the essence of "chance constrained" approach. We can apply the above analysis to the following problem

$$\min\{C^T x + Q(x, \mu) \mid Ax = b, x \in \mathcal{R}_+^{n_1}\}$$

where

$$Q(x, \mu) = \int_{\mathcal{R}^n} \Phi(x, \xi) d\mu(\xi)$$

$$\Phi(x, \xi) = \min\{q^T(\xi) y \mid Wy = h(\xi) - T(\xi)x, y \in \mathcal{R}_+^{n_2}\} \tag{38}$$

Let us now consider a more general problem

$$(\mathcal{P}) \ \min\{g(x) + Q(x) \mid f(x) \leq 0\}$$

$$Q(x) = \int_{\Omega} \Phi(x, \omega) dP(\omega)$$

$$\Phi(x, \omega) = \min\{q^T(\omega) y \mid W(\omega) y \geq h(\omega) - T(\omega)x, y \in \tilde{Y}\} \tag{39}$$

Let us write the constraint inequality on y as follows.

$$\begin{aligned} X_1(x, y, \omega) &= \sum_{j=1}^k W_{1j}(\omega) y_j - h_1(\omega) - \sum_{j=1}^n T_{1j}(\omega) x_j \geq 0 \\ X_2(x, y, \omega) &= \sum_{j=1}^k W_{2j}(\omega) y_j - h_2(\omega) - \sum_{j=1}^n T_{2j}(\omega) x_j \geq 0 \end{aligned} \tag{40}$$

Assume W_{ij}, $i = 1,2$ and $j = 1, \ldots, n$ as well as h_1, h_2 are identically distributed independent normal random variables. Thus

$$E(X_i(x, y, \cdot)) = 0, \quad i = 1,2.$$

$$E(h_i) = 0, \quad i = 1,2.$$

$$E(X_i(x, y, \cdot) X_j(x, y, \cdot)) = \delta_{ij}(|y|^2 + |x|^2 + 1)$$

Thus

$$cov(X_1, X_2) = (|y|^2 + |x|^2 + 1)\mathcal{I}$$

where $\mathcal{I}$ is the identity matrix.

An approximate solution for the above problem $(\mathcal{P})$ can be constructed as follows. Given x such that $f(x) \le 0$ and $0 < \epsilon < 1$ we can choose y so that

$$prob\{\omega \mid X_1(x,y,\omega) \ge 0, X_2(x,y,\omega) \ge 0\} \ge 1 - \epsilon$$

We can accomplish this if we choose y so that

$$\frac{1}{2\pi}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$
$$\frac{1}{2\pi}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$

That is we may consider the problem

$$\min_{x,y}\{g(x) + E(q)\cdot y\}$$

subject to

$$\frac{1}{2\pi}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$
$$\frac{1}{2\pi}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$

$$y \in \tilde{Y}$$
$$f(x) \le 0 \tag{41}$$

Let $F(s)$ be the cumulative distribution of the normal random variable with mean 0 and variance 1. Now this problem can be rewritten as

$$\min_{x,y}\{g(x) + E(q)\cdot y\}$$

subject to

$$\theta^2(|y|^2 + |x|^2 + 1) \ge \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right]^2$$
$$\theta^2(|y|^2 + |x|^2 + 1) \ge \left[F^{-1}\left(1 - \sqrt{\frac{\epsilon}{2}}\right)\right]^2$$

$$y \in \tilde{Y}$$
$$f(x) \le 0 \tag{42}$$

Suppose that $M_1 = \max\{|y| : y \in \tilde{Y}\}$ and $M_2 = \max\{|x| : f(x) \le 0\}$. Then θ should be chosen so the $\theta^2 = (F^{-1}(\sqrt{\epsilon})^2/(M_1^2 + M_2^2 + 1)$ to allow all the y and x values to compete. If we choose θ^2 to bigger, then we might eliminate some feasible x and y

values from competing. If we take θ^2 to be smaller, then nonfeasible values of x and y would be considered in our computation unnecessarily.

We now consider a version that is different from the above problem.

$$(\mathcal{P}) \quad \min\{C^T x + Q(x,\mu) \mid Ax = b, x \in \mathcal{R}_+^{n_1}\}$$

where

$$\begin{aligned} Q(x,\mu) &= \int_{\mathcal{R}^n} \Phi(x,\xi) d\mu(\xi) \\ \Phi(x,\xi) &= \min\{q^T(\xi) y \mid Wy = h(\xi) - T(\xi)x,\ y \in \mathcal{R}_+^{n_2}\} \end{aligned} \tag{43}$$

Let us now consider a more general problem

$$(\mathcal{P}_1) \quad \min\{g(x) + Q(x) \mid f(x) \leq 0\}$$

$$\begin{aligned} Q(x) &= \int_\Omega \Phi(x,\omega) d\mu(\omega) \\ \Phi(x,\omega) &= \min\{q^T(\omega) y \mid W(\omega) y \geq h(\omega) - T(\omega)x, y \in \tilde{Y}\} \end{aligned} \tag{44}$$

Without loss of generality we assume that $q \geq 0$. We also assume that $0 \leq q_i \leq M$, $i = 1, \ldots, n$. Henceforth, we assume that q, W, h, and T are independent random variables, and W, h, T are normal random variables with mean 0 and standard deviation 1. Let

$$E_{l_i} = \{\omega : l_i \cdot \delta \leq q_i(\omega) \leq l_i \cdot \delta + \delta\}$$

and

$$\mu\left(\bigcap_{i=1}^n E_{l_i}\right) = \Delta(l_1, \ldots, l_n)$$

$$\int_{\bigcap_{i=1}^n E_{l_i}} q(\omega) \cdot y d\mu(\omega) \approx \sum_{i=1}^n (l_i \cdot \delta) y_i \Delta(l_1, \ldots, l_n)$$

We now give a problem that approximates the problem $\mathcal{P}_1$ to a desired accuracy on the set $\bigcap_{i=1}^n E_{l_i}$.

$$(\mathcal{P}_2) \quad \min\{g(x) + \sum_{j=1}^n (l_j \cdot \delta) y_j \Delta(l_1, \ldots, l_n)\}$$

subject to

$$\begin{aligned} \chi_{\cap_{i=1}^n E_{l_i}}(\omega) W(\omega) y &\geq \chi_{\cap_{i=1}^n E_{l_i}}(\omega)(h(\omega) - T(\omega)x) \\ f(x) &\leq 0 \\ y &\in \tilde{Y} \end{aligned} \tag{45}$$

For ease of notation in $\mathcal{P}_2$ we would take W to be $2 \times k$ matrices and T to be $2 \times n$ matrices. We would also replace the constraint $y \in \tilde{Y}$ by $y \geq 0$. Then we would consider the following problem

$$(\mathcal{P}_3) \quad \min \left\{ g(x) + \sum_{j=1}^{n} (l_j \cdot \delta) y_j \Delta(l_1, \ldots, l_n) \right\}$$

subject to

$$
\begin{aligned}
X(1, l_1, \ldots, l_n, \omega) &= \chi_{\cap_{i=1}^{n} E_{l_i}}(\omega)(W(\omega)y - h_1(\omega) - T(\omega)x) \geq 0 \\
X(2, l_1, \ldots, l_n, \omega) &= \chi_{\cap_{i=1}^{n} E_{l_i}}(\omega)(W(\omega)y - h_2(\omega) - T(\omega)x) \geq 0 \\
f(x) &\leq 0 \\
y &\geq 0
\end{aligned} \tag{46}
$$

$$E(X(1, l_1, \ldots, l_n, \omega)) = 0$$
$$E(X(1, l_1, \ldots, l_n, \omega)) = 0$$
$$E(X(1, l_1, \ldots, l_n, \omega)X(1, l_1, \ldots, l_n, \omega)) = (\Delta(l_1, \ldots, l_n))^2(|y|^2 + |x|^2 + 1)$$
$$E(X(2, l_1, \ldots, l_n, \omega)X(2, l_1, \ldots, l_n, \omega)) = (\Delta(l_1, \ldots, l_n))^2(|y|^2 + |x|^2 + 1)$$

Let $F(s)$ be the cumulative distribution of the normal random variable with mean 0 and variance 1. In $\bigcap_{i=1}^{n} E_{l_i}$, $1 \leq l_i \leq N$ we solve

$$(\mathcal{P}_4) \quad \min_{x,y} \left\{ g(x) + \sum_{j=1}^{n} (l_j \cdot \delta) y_j \Delta(l_1, \ldots, l_n) \right\}$$

subject to

$$\delta^2(\Delta(l_1, \ldots, l_n))^2(|y|^2 + |x|^2 + 1) \geq \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2N}} \right) \right]^2$$
$$\delta^2(\Delta(l_1, \ldots, l_n))^2(|y|^2 + |x|^2 + 1) \geq \left[F^{-1}\left(1 - \sqrt{\frac{\epsilon}{2N}} \right) \right]^2$$
$$f(x) \leq 0$$
$$y \geq 0$$

We are trying to minimize the objective function $g(x) + \sum_{j=1}^{n} l_j \delta y_j \Delta(l_1, \ldots, l_n)$ while enforcing $X(1, l_1, \ldots, l_n, \omega) \geq 0$ and $X(2, l_1, \cdots, l_n, \omega) \geq 0$ hold on a set of measure $\geq \left(1 - \frac{\epsilon}{N}\right)$. The idea is, we divide the range of q_i into N parts and then approximately solve the minimization problem $(\mathcal{P}_1)$ on each set $\bigcap_{i=1}^{n} E_{l_i}$ while enforcing the inequality of the problem on a set of measure $\geq 1 - \frac{\epsilon}{N}$. Finally, we put things together, that is, we consider the problem

$$(\mathcal{P}_5) \quad \min_{x, y_{l_1 \ldots l_n}} \left\{ g(x) + \sum_{(l_1, \ldots, l_n)} \sum_{j=1}^{n} (l_j \cdot \delta) y_{l_1 \cdots l_n j} \Delta(l_1, \ldots, l_n) \mid 1 \leq l_i \leq N, i = 1, \ldots, n \right\}$$

subject to

$$\delta^2(\Delta(l_1,\ldots,l_n))^2(|y|^2+|x|^2+1) \geq \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2N}}\right)\right]^2$$
$$\delta^2(\Delta(l_1,\ldots,l_n))^2(|y|^2+|x|^2+1) \geq \left[F^{-1}\left(1-\sqrt{\frac{\epsilon}{2N}}\right)\right]^2$$
$$1 \leq l_i \leq N, \quad i=1,\ldots,n$$
$$f(x) \leq 0$$
$$y_{l_1\ldots l_n} \geq 0$$

We remark that the second sum is taken over all n-tuples $(l_1,\ldots,l_n)$. Problem $(\mathcal{P}_5)$ is an approximate solution to $(\mathcal{P}_1)$ while enforcing the random constraints of $(\mathcal{P}_1)$ on a set of measure $\geq 1-\epsilon$. Again we remark that we choose δ^2 to go along with the constraints as pointed out above. This problem is a nonlinear programming problem that can be solved using MatLab.

5. **Infinite Dimensional Approach**

5.1. **First Approach.** In the previous section we made assumptions that the random variables are independent. Here we will avoid making the assumption of independence and consider the decision variable y as a measurable function. Consider the problem

$$(\mathcal{P}_6) \quad \min_{x,y}\left\{g(x)+\int_\Omega F(y(\omega),x)d\mu(\omega)\right\}$$
$$f(x) \leq 0$$
$$G(y(\omega),x)) \leq 0 \text{ a.e } [\mu].$$

We assume that

$$G(\xi,x) > 0 \text{ if } |\xi|^2+|x|^2 \geq M.$$

Now our decision variables are x and $y(\omega)$. We impose on f the conditions stated in the list of assumptions in Section 1. For a fixed positive integer n let $E_1,E_2,\ldots,E_n$ be measurable subsets of Ω. We look for a solution of $(\mathcal{P}_6)$ where the $y's$ are simple function of the form $\sum_{i=1}^n \alpha_i\chi_{E_i}$, $\alpha_i \in \mathcal{R}$. Using the procedure in the proof of Lemma 1.1 we can get a set of necessary conditions as in Lemma 5.1 below.

Lemma 5.1. *Suppose x^* and $y^*(\omega)$ are optimal for problem $\mathcal{P}_6$. Then, there exist multipliers $\lambda^0 \geq 0$, $\lambda_1 \geq 0$, $\lambda_2(\omega) \geq 0$ such that*

$$\lambda^0+\lambda_1+\|\lambda_2(\omega)\|_\infty \neq 0$$
$$\lambda_1 f(x^*) = 0.$$
$$\begin{aligned}\lambda^0\nabla g(x^*)+\lambda_1\nabla f(x^*)+\lambda^0\int_\Omega \nabla_x F(y(\omega),\, x^*)d\mu(\omega)&\\ +\int_\Omega \lambda_2(\omega)\nabla_x G(y(\omega),x^*)d\mu(\omega)&\\ =0&\end{aligned}$$

$$\lambda^0 \nabla_y F(y(\omega), x^*) + \lambda_2(\omega) \nabla_y G(y(\omega), x^*) = 0 \ \ a.e. \ [\mu]$$
$$\lambda_2(\omega) G(y(\omega), \, x^*) = 0 \ \ a.e. \ [\mu]$$

The above lemma is not practical, and problem $(\mathcal{P}_6)$ is not, in practice, easy to deal with. It is more practical if we knew the probability distributions of the random variables involved. The next lemma is a prelude to that idea. In the next section we will deal with the practical situation when we know the distributions.

We now consider a problem that is relevant to the material in the next section.

$$(\mathcal{P}_6') \quad \min_{x,y} \left\{ g(x) + \int F(x, \zeta, y(z,\zeta)) e^{-d^2(|z|^2+|\zeta|^2)} dz d\zeta \right\}$$
$$f(x) \leq 0$$
$$G(x, y(z,\zeta)) \leq 0 \text{ a.e.}$$

In this problem we assume that

$$f(x) > 0 \text{ if } |x| \geq M, \quad |y(z,\zeta)| \leq \text{ const.}$$

We also assume that F and G are continuously differentiable and

$$|F(x, \xi_1, \xi_2)| + G(x, \xi_2) \leq \text{ const.}(1 + |\xi_1|^k + |\xi_2|^k)$$

for some k, a positive integer. We also assume that $x \in \mathcal{R}^l$, $\ \xi_1 \in \mathcal{R}^m$ and $\xi_2 \in \mathcal{R}^n$. In this problem our decision variables are $x \in \mathcal{R}^l$ and $y \in \mathcal{H}^1(\mathcal{R}^q)$, $q = m + n$. Here $\mathcal{H}^1(\mathcal{R}^q)$ is the Sobolev space of order one.

Lemma 5.2. *Suppose that $\{y_n\}$ is a sequence in $\mathcal{H}^1(\mathcal{R}^q)$ and y_0 is a fixed element in $\mathcal{H}^1(\mathcal{R}^q)$ such that $\|y_n - y_0\|_{\mathcal{H}^1(\mathcal{R}^q)} \leq \theta$. Then there exists a subsequence of $\{y_n\}$ that converges pointwise almost everywhere to an element of $\mathcal{H}^1(\mathcal{R}^q)$.*

Proof. There exist a subsequence $\{y_{n_i}\}$ and $y^0 \in \mathcal{H}^1(\mathcal{R}^q)$ such that $\{y_{n_i}\}$ converges to y^0 weakly. A further subsequence of $\{y_{n_i}\}$ converges to y^0 strongly in $L^r(B(0;1))$, $1 \leq r < \frac{nq}{nq-1}$ and a further subsequence of the last subsequence converges pointwise to y^0 a.e. on $B(0;1)$. Now taking a subsequence of the sequence that converged to y^0 a.e. on $B(0;1)$ we get a subsequence that converges a.e. to y^0 on $B(0;2)$. We can continue this process taking balls of radius $3, 4, \ldots$ and get a diagonal sequence that converges to y^0 a.e. □

Using the procedure of Lemma 1.1 we can get the following

Lemma 5.3. *Suppose (x^*, y^*) is a solution to $(\mathcal{P}_6')$. Then*

$$\lambda^0 + \lambda_1 + \|\lambda_2(z,\zeta)\|_\infty \neq 0$$
$$\lambda_1 f(x^*) = 0$$
$$\lambda^0 \nabla g(x^*) + \lambda_1 \nabla f(x^*) + \lambda^0 \int \nabla_x F(x^*, \zeta, y^*(z,\zeta)) \cdot e^{-d^2(|z|^2+|\zeta|^2)} dz d\zeta$$

$$+\int \lambda_2(z,\zeta)\nabla_x G(x^*,y^*(z,\zeta))\cdot e^{-d^2(|z|^2+|\zeta|^2)}dzd\zeta = 0$$

$$\lambda^0\nabla_y F(x^*,\zeta,y^*(z,\zeta)) + \lambda_2(z,\zeta)\nabla_y G(x^*,y^*(z,\zeta)) = 0 \quad a.e.$$

$$\lambda_2(z,\zeta)G(x^*,y^*(z,\zeta)) = 0 \quad a.e.$$

5.2. **Second Approach.** Here we follow the spirit of $\mathcal{P}_6'$. We reconsider problem $(\mathcal{P})$ to get additional insight to stochastic programming problems. We keep the assumptions on the random variables q, W, h, T made in problem $(\mathcal{P})$. (See (40) and the equations preceding it.)

In what follows $\zeta = (\zeta_1,\ldots,\zeta_k)^T$, $\xi_i = (\xi_{i1},\ldots,\xi_{ik})^T$, $i = 1,2$. Similarly $w_i = (w_{i1},\ldots,w_{1k})^T$, and $T_i = (T_{i1},\ldots,T_{1n})^T$, $i = 1,2$. Next η_1,η_2 represent real numbers while τ_1,τ_2 represent vectors in $\mathcal{R}^n$. The quantities h_1,h_2, are scalars. Let ν_ζ be the measure defined on $\mathcal{R}^k$ as follows. Let B be a Borel subset of $\mathcal{R}^k$. We define the measure ν_ζ by the formula

$$\nu_\zeta(B) = P(q^{-1}(B))$$

We define the measures μ_{ξ_1} and μ_{ξ_2} in the same way. That is,

$$\mu_{\xi_1}(B) = P(W_1^{-1}(B)), \quad \mu_{\xi_2}(B) = P(W_2^{-1}(B))$$

We define the measures μ_{η_1} and μ_{η_2} as follows. That is, for C a Borel subset of $\mathcal{R}$

$$\mu_{\eta_1}(C) = P(h_1^{-1}(C)), \quad \mu_{\eta_2}(C) = P(h_2^{-1}(C)),$$

Let D be a Borel subset of $\mathcal{R}^n$. We define the measures μ_{τ_1}, μ_{τ_2} as follows

$$\mu_{\tau_1}(D) = P(T_1^{-1}(D)), \quad \mu_{\tau_2}(D) = P(T_2^{-1}(D)).$$

We assume that the measures ν_ζ and μ_{ξ_1} and μ_{ξ_2} are absolutely continuous with respect to the Lebesgue measure on $\mathcal{R}^k$. We assume that the measures μ_{η_1} and μ_{η_2} are absolutely continuous with respect to the Lebesgue measure on the real line. We also assume that the measures μ_{τ_1}, μ_{τ_2} are absolutely continuous with respect to the Lebesgue measure on $\mathcal{R}^n$. We denote by $\pi_1(\zeta)$, $\pi_{21}(\xi_1)$, $\pi_{22}(\xi_2,)$, $\pi_{31}(\eta_1)$, $\pi_{32}(\eta_2)$, $\pi_{41}(\tau_1)$, $\pi_{42}(\eta_2)$ the Radon-Nikodym derivatives with respect to the corresponding Lebesgue measure. In fact, these Radon-Nikodym derivatives are all Gaussian.

We now set

$$d\Theta(\zeta,\xi_1,\xi_2,\eta_1,\eta_2,\tau_1,\tau_2) = \pi_1(\zeta)\pi_{21}(\xi_1)\pi_{22}(\xi_2)\pi_{31}(\eta_1)\pi_{32}(\eta_2)\pi_{41}(\tau_1)\pi_{42}(\tau_2)$$
$$\cdot\, d\zeta d\xi_1 d\xi_2 d\eta_1 d\eta_2 d\tau_1 d\tau_2$$

For ease of notation we will simply write $d\Theta$ for $d\Theta(\zeta,\xi_1,\xi_2,\eta_1,\eta_2,\tau_1,\tau_2)$. Similarly we will write y_l instead of $y_l(\zeta,\xi_1,\xi_2,\eta_1,\eta_2,\tau_1,\tau_2)$. Now problem $(\mathcal{P})$, without the restriction $y \in \tilde{Y}$, becomes

$$(\mathcal{P}_7) \quad \min_{x,y}\left\{g(x) + \sum_{l=1}^{k}\int \zeta_l y_l d\Theta\right\}$$

subject to

$$\sum_{j=1}^{k} \xi_{ij} y_j - \eta_i - \sum_{j=1}^{n} \tau_{ij} x_j \geq 0, \quad i = 1, 2 \tag{47}$$

$$f(x) \leq 0, \quad f(x) > 0 \text{ if } |x| > M$$

Note that

$$\sum_{j=1}^{k} \xi_{ij} y_j - \eta_i - \sum_{j=1}^{n} \tau_{ij} x_j = \langle \xi_i, y \rangle - \eta_i - \langle \tau_i, x \rangle$$

Also note that

$$|y| \leq \text{const.}(1 + |\eta_1| + |\eta_2| + |\tau_1| + |\tau_2|).$$

We remark here that in order to solve $(\mathcal{P}_7)$ we discretize the $\xi_{ij} y_j$, η_i, τ_{ij} space, and in each subdivision we solve the problem. We also know the measure of the particular subdivision. That is we solve a nonlinear programming problem. Let ω be a smooth function that is zero on $(-\infty, 0]$ and positive on $(0, \infty)$. Then the inequality

$$\langle \xi_i, y \rangle - \eta_i - \langle \tau_i, x \rangle \geq 0$$

is the same as

$$\int \omega(-\langle \xi_i, y \rangle + \eta_i + \langle \tau_i, x \rangle) d\Theta = 0.$$

One can see how the formulation in $(\mathcal{P}_7)$ can be useful in approximation.

Set

$$E(\eta, \zeta) = \epsilon \cdot (|\eta_1|^2 + |\eta_2|^2 + |\zeta_1|^2 + |\zeta_2|^2)$$

where $\epsilon > 0$ is sufficiently small so that the $e^{E(\eta,\zeta)} d\Theta$ decreases exponentially fast. Now consider the following problem instead of $(\mathcal{P}_7)$.

$$(\mathcal{P}_{7'}) \quad \min_{x,y} \left\{ g(x) + \sum_{l=1}^{k} \int \zeta_l y_l e^{E(\eta,\zeta)} d\Theta \right\}$$

subject to

$$\sum_{j=1}^{k} \xi_{ij} y_j - e^{-E(\eta,\zeta)} \eta_i - \sum_{j=1}^{n} e^{-E(\eta,\zeta)} \tau_{ij} x_j + e^{-E(\eta,\zeta)} \geq 0, \quad i = 1, 2$$

$$f(x) \leq 0,$$

$$f(x) > 0 \quad \text{if } |x| > M$$

We note $\mathcal{P}_7$ and $\mathcal{P}_{7'}$ are equivalent except that in $\mathcal{P}_{7'}$ we have

$$|y| \leq \text{const.}(1 + |\eta_1| + |\eta_2| + |\tau_1| + |\tau_2|) e^{-E(\eta,\zeta)}.$$

and we can apply the analysis of $\mathcal{P}_{6'}$. Also the formulation in $\mathcal{P}_{7'}$ allows us to consider the decision variable y to come from appropriate Sobolev space as in $\mathcal{P}_{6'}$ and also assert existence of solution provided the admissible set is nonempty. In practice we solve problem $\mathcal{P}_{7'}$ by discretizing the measure $d\Theta$ and approximating the decision variable y by a simple function.

6. Numerical Examples

6.1. **A stochastic problem with recourse.** Consider the following two-stage stochastic program with recourse.

$$\min\left\{c^T x + \int_\Omega Q(x,\xi)d\mu(\xi)\right\}$$

subject to

$$\begin{aligned} Ax &\geq b \\ x &\geq 0 \\ Q(x,\xi) &= \min\{q^T y \mid Wy = h(\xi) + T(\xi)x, y \geq 0\} \end{aligned}$$

To illustrate the ideas presented above let

$$\Omega = \{\xi_1, \xi_2, \dots, \xi_N\}, \quad \mu(\xi_i) = p_i,$$

$$W = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

and

$$h(\xi) = (0\ 0)^T, \text{ and } T = (T_{jk})_{j=1,2;k=1,2,3}.$$

We can consider the measure μ as one of the decision variables. We now reformulate the problem. Set

$$\begin{aligned} q_j(\omega) &= \sum_{i=1}^N q_{ji}\chi_{\xi_i}, \quad j = 1,2,3 \\ y_j(\omega) &= \sum_{i=1}^N y_{ji}\chi_{\xi_i}, \quad j = 1,2,3 \\ T_{jk}(\omega) &= \sum_{i=1}^N \tau_{ijk}\chi_{\xi_i}, \quad j = 1,2; \quad k = 1,2,3. \end{aligned}$$

(48)

Then, we have the following problem

$$\min_{\{x, y_{1i}, y_{2i}, y_{3i}, p_i : i=1,\dots,N\}} \left\{c^T x + \sum_{i=1}^N (q_{1i}y_{1i} + q_{2i}y_{2i} + q_{3i}y_{3i})p_i\right\}$$

subject to

$$Ax \geq b$$

$$x \geq 0$$

$$y_{1i} + y_{2i} = \sum_{k=1}^{3} \tau_{i1k} x_k, \quad i = 1, \ldots, N$$

$$y_{2i} + y_{3i} = \sum_{k=1}^{3} \tau_{i2k} x_k, \quad i = 1, \ldots, N$$

$$x, y_{1i}, y_{2i}, y_{3i} \geq 0, \quad i = 1, \ldots, N$$

$$\sum_{k=1}^{N} p_i = 1, \quad p_1 \geq 0, \ldots, p_N \geq 0$$

This reformulation has the same set-up as in Section 1. We note that this problem is a nonlinear programming problem which can be solved using MatLab, and also illustrates the essence of the above approach. We also note the multistage problem in the next section fits in this approach. In this example, if we take

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

$$b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$(q_{11}\ q_{12}\ q_{13}) = (0.1\ 0.13\ 0.4),$$
$$(q_{21}\ q_{22}\ q_{23}) = (0.3\ 0.4\ 0.23),$$
$$(q_{31}\ q_{32}\ q_{33}) = (0.51\ 0.12\ 0.3)$$

$$(\tau_{111}\ \tau_{112}\ \tau_{113}) = (0.2\ 0.3\ 0.5),$$
$$(\tau_{211}\ \tau_{212}\ \tau_{213}) = (0.1\ 0.6\ 0.3),$$
$$(\tau_{311}\ \tau_{312}\ \tau_{313}) = (0.4\ 0.3\ 0.3)$$

$$(\tau_{121}\ \tau_{122}\ \tau_{123}) = (0.1\ 0\ 0.9),$$
$$(\tau_{221}\ \tau_{222}\ \tau_{223}) = (0.8\ 0.1\ 0.1),$$
$$(\tau_{321}\ \tau_{322}\ \tau_{323}) = (0.2\ 0.4\ 0.4)$$

then, using MatLab, the values of the decision variable and the corresponding objective functional value is presented in the following table.

x_1	0
x_2	0.4038
x_3	0.5962
y_{11}	0.4192
y_{21}	0.4211
y_{31}	0.3
y_{12}	0
y_{22}	0
y_{32}	0
y_{13}	0.5366
y_{23}	0.1
y_{33}	0.4
p_1	0
p_2	1
p_3	0
Objectve function	1

6.2. **A general stochastic programming.** We may consider the problem

$$\min_{x,y}\{g(x) + E(q)\cdot y\}$$

subject to

$$\frac{1}{2\pi}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}}\int_{-\infty}^{-\theta\sqrt{|y|^2+|x|^2+1}} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$

$$\frac{1}{2\pi}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty}\int_{\theta\sqrt{|y|^2+|x|^2+1}}^{\infty} \exp\left(-\frac{1}{2}(\eta_1^2+\eta_2^2)\right) d\eta_1 d\eta_2 \le \epsilon/2,$$

$$y \in \tilde{Y}$$

$$f(x) \le 0 \tag{49}$$

Let $F(s)$ be the cumulative distribution of the normal random variable with mean 0 and variance 1. Now this problem can be rewritten as

$$\min_{x,y}\{g(x) + E(q)\cdot y\}$$

subject to

$$\theta^2(|y|^2+|x|^2+1) \ge \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right]^2$$

$$\theta^2(|y|^2+|x|^2+1) \ge \left[F^{-1}\left(1-\sqrt{\frac{\epsilon}{2}}\right)\right]^2$$

$$y \in \tilde{Y}$$

$$f(x) \le 0 \tag{50}$$

Suppose that $M_1 = \max\{|y| : y \in \tilde{Y}\}$ and $M_2 = \max\{|x| : f(x) \le 0\}$. Then θ should be chosen so the $\theta^2 = (F^{-1}(\sqrt{\epsilon})^2/(M_1^2 + M_2^2 + 1)$ to allow all the y and x values to compete. If we choose θ^2 to bigger, then we might eliminate some feasible x and y values from competing. If we take θ^2 to be smaller, then nonfeasible values of x and y would be considered in our computation unnecessarily.

Let us consider now a specific version of this problem

$$\tilde{Y} = \{y \mid y_1^2 + y_2^2 \le 4\}, \quad f(x_1, x_2) = x_1^2 + x_2^2 - 1,$$
$$g(x) = 3x_1^2 + 10x_2, \quad E(q) = (2, -1)^T, \quad \epsilon = .01.$$

Then, we solve

$$\min_{x,y}\{3x_1^2 + 10x_2 + 2y_1 - y_2\}$$
$$\text{subject to}$$
$$x_1^2 + x_2^2 - 1 \le 0$$
$$y_1^2 + y_2^2 \le 4$$
$$\theta^2(y_1^2 + y_2^2 + x_1^2 + x_2^2 + 1) \ge \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right]^2$$
$$y \;\in\; \tilde{Y}$$
$$f(x) \;\le\; 0$$

$\frac{\epsilon}{2} = .005$, $\sqrt{\frac{\epsilon}{2}} = .0707$, $\left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right] = 1.47$, $\left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right]^2 = 2.1609$, $2.1609/4 = .540225$. We choose $\theta^2 = \left[F^{-1}\left(\sqrt{\frac{\epsilon}{2}}\right)\right]^2/5$. This problem can be quickly solved using MatLab. The values of the decision variables and the corresponding objective functional value are presented in the following table.

x_1	0.0000
x_2	-1.0000
y_1	-1.7889
y_2	0.8944
Objectve function	-14.4721

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