Neural, Parallel, and Scientific Computations 22 (2014) 205-222

# ROLE OF MULTIPLE DELAYS IN RATIO-DEPENDENT PREY-PREDATOR SYSTEM WITH PREY HARVESTING UNDER STOCHASTIC ENVIRONMENT

DEBALDEV JANA[1,†] AND G. P. SAMANTA[2,‡]

[1]Ecological Modelling Laboratory, Department of Zoology
Visva-Bharati University, Santiniketan - 731 235, India
[2]Department of Mathematics
Indian Institute of Engineering Science and Technology
Shibpur, Howrah-711103, India

**ABSTRACT.** In the present article, we have studied a multi-delayed predator-prey model where the prey species is subject to harvesting under stochastic environment. Predator's interference in predator-prey relationship provides better descriptions of predator's feeding over a range of predator-prey abundances, so the predator's functional response is considered to be Type II ratio-dependent here. A constant time delay is incorporated in the logistic growth of the prey to represent a delayed density dependent feedback mechanism. A second time delay is considered to account for the length of the gestation period of the predator. Along with these delays harvesting of prey species plays a significant role to better description of the system. Stochastic stability is measured by second order moment terms by calculating the non-equilibrium fluctuation of the non-delayed system and Fourier transform technique depicts the fluctuation of stochastic stability by introducing time lag. Different dynamical behaviors for both situations have been illustrated numerically also. The biological implications of the analytical and numerical findings are discussed critically.

**Key words:** Delay; Gaussian white-noise; Ratio dependent functional response; Harvesting; Moment equations; Fourier transform

## 1. Introduction

In ecology, understanding the prey-predator relationship is the central goal and a very significant component of this is the predators rate of feeding upon prey. Predator's functional response, defined as the amount of prey catch per predator per unit of time. After the pioneering studies of [1], i.e., linear, only prey dependent and ratio dependent functional response, response functions of predators which depends on both prey and predator abundance can provide better descriptions of predator feeding over a range of predator-prey abundances because of predator interference. More recent theoretical work has demonstrated that the mathematical form of the

‡Corresponding author's E-mail: g_p_samanta@yahoo.co.uk, gpsamanta@math.becs.ac.in

†Research is supported by UGC (Dr. D. S. Kothari Postdoctoral Fellowship), India; No. F.4-2/2006(BSR)/13-1004/2013(BSR).
Received April 15, 2014 1061-5369 $15.00 

feeding rate can influence the distribution of predators through space [2], the stability of enriched predator-prey systems [3, 6] correlations between nutrient enrichment and the biomass of higher trophic levels [3], and the length of food chains [7]. Predators may or may not kill their prey prior to feeding on them, but the act of predation often results in the death of its prey and the eventual absorption of the prey's tissue through consumption. Predators can have profound impacts on the dynamics of their prey that depend on how predator consumption is affected by prey density (the predator's functional response). Predator's functional response is affected by the structure of prey habitat and predator's hunting ability [8, 9]. Predator's functional response is considered as Type II ratio-dependent [10, 11, 12, 13, 14, 15, 16, 17] because a ratio-dependent predator-prey model does not show the so called paradox of enrichment [10, 18, 19] and biological control paradox [20]. In general, in our ecosystem, harvesting is a very frequently used process to exploit biological resources for the necessity of human beings and the society. There are different ways of harvesting have been used in the ecosystem and the most simple and common way to harvest the ecological resources is when the resource population is harvested at a constant rate and mathematically it is represented by $h(t) = h$, where $h$ being a constant. The drawback of the constant rate harvesting is that it is independent of the density of the harvesting stock. Another important harvesting strategy is based on the catch-per-unit-effort (CPUE) hypothesis and mathematically it is written as $h(t) = qEx(t)$, where $q$ is the catchability coefficient, $E$ is the constant external effort and $x(t)$ is the density of the harvested species at time $t$. CPUE based harvesting strategy is supposed to be more realistic and productive than its constant rate harvesting counterpart regarding the cause that it is proportional to the density of the harvested stock [21]. In this paper, we have first modified such a system, where the prey population is harvested following the CPUE based harvesting rate and normalizing the unit of effort by setting $q = 1$, we write the model with prey harvesting as follows:

$$\begin{aligned} \frac{dx}{dt} &= x\left[r\left(1-\frac{x}{K}\right)-\frac{\alpha y}{ay+x}-E\right], \\ \frac{dy}{dt} &= y\left[-d_0+\frac{\alpha b_0 x}{ay+x}\right] \end{aligned} \tag{1}$$

where, $x$ and $y$ are densities of prey and predator populations respectively. Here we assume that prey population grows logistically with intrinsic growth rate $r$ and grows up to the environmental carrying capacity $K$ in absence of predator. $\alpha$ is the maximal per capita prey consumption rate of predator, $a$ is the amount of prey necessary for the relative biomass growth rate of the predator to be half its maximum, $d_0$ is the food independent death rate of predator population. Notice that $b_0$ is the conversion efficiency of the predator, which measures the efficiency of the predator of converting prey biomass into predator biomass. Thus, a predator is said to be productively inefficient if $b_0 < \frac{1}{a}$, or simply $ab_0 < 1$. Similarly, a predator is said to be productively neutral or productively efficient according as $ab_0 = 1$ or $ab_0 > 1$.

It is well understood that many of the processes, both natural and manmade, in biology and medicine involve time-delays. Time-delays occur so often, in almost every situation, that to ignore them is to ignore reality. Kuang [22] mentioned that animals must take time to digest their food before further activities and responses take place, and hence any model of species dynamics without delays is an approximation at best. Now it is beyond doubt that in an improved analysis, the effect of time-delay due to the time required in going from egg stage to the adult stage, gestation period etc has to be taken into account. Detailed arguments on the importance and usefulness of time-delays in realistic models may be found in the classical books of MacDonald [23], Gopalsamy [24], and Kuang [22]. In a review paper of predator-prey models with discrete delay, Ruan [25] discussed different types of delays and the dynamics of the corresponding models. Xiao et al. [26] observed different kinds of bifurcations of the system (1) with delayed predator specific growth rate and constant harvesting of either the prey or the predator population. In [14] Jana et al. studied the system (1) with feedback delay in the prey specific growth rate and with Michaelis-Menten type harvesting of the predator species. Recently, researchers [27, 28, 29, 30, 31] have been incorporating multiple time delays in the differential equations to obtain more realistic models. In the present work, we introduce two delay parameters in the model (1), where one parameter $\tau_1$ is introduced in the growth rate of prey to account for the effect of density dependent feedback mechanisms [32] and the second delay parameter $\tau_2$ is introduced in the predator's response function which is regarded as a gestation period or reaction time of the predator [22]. We thus obtain the following multiple time delayed predator-prey model with prey harvesting:

$$\begin{aligned} \frac{dx}{dt} &= x\left[r\left(1 - \frac{x(t-\tau_1)}{K}\right) - \frac{\alpha y}{ay + x} - E\right], \\ \frac{dy}{dt} &= y\left[-d_0 + \frac{\alpha b_0 x(t-\tau_2)}{ay(t-\tau_2) + x(t-\tau_2)}\right]. \end{aligned} \tag{2}$$

Deterministic models in ecology do not usually incorporate environmental fluctuation; they are often justified by the implicit assumption that in large populations, stochastic deviations are small enough to be ignored. Deterministic model will prove ecologically useful only if the dynamical patterns they reveal are still in evidence when stochastic effects are introduced. For terrestrial system, the environmental variability is large at both short and long time periods and could be expected to develop internal mechanisms to the system which would cope with short term variability and minimize the effects of long term variations, hence analysis of the system with white noise gives better results. Uncertain growth of populations is usually considered as an effect of environmental stochasticity. Reproduction of species depends on various factors, such as temperature, humidity, parasites and pathogens, environmental pollution etc. [33]. Since physical and biological environments of populations are not totally predictable, the growth of populations should be considered as a stochastic process rather than a deterministic one [34]. In spite of some shortcomings, Gaussian white noise has been proved extremely useful to model rapidly fluctuating phenomena [35, 36]. Therefore, model system (2) by introducing the environmental stochasticity in the form of Gaussian white noise is represented by:

$$\begin{aligned} \frac{dx}{dt} &= x\left[r\left(1-\frac{x(t-\tau_1)}{K}\right)+\eta_1(t)\right]-\frac{\alpha xy}{ay+x}-Ex, \\ \frac{dy}{dt} &= y\Big[-d_0+\eta_2(t)\Big]+\frac{\alpha b_0 x(t-\tau_2)y}{ay(t-\tau_2)+x(t-\tau_2)} \end{aligned} \tag{3}$$

where the perturbed terms $\eta_1(t)$ and $\eta_2(t)$ are assumed to be the independent Gaussian white noise satisfying the conditions:

$$\langle\eta_j(t)\rangle = 0 \text{ and } \langle\eta_j(t_1)\eta_j(t_2)\rangle = \epsilon_j\delta(t_1-t_2) \text{ for } j=1,2.$$

Here $\epsilon_j > 0$ $(j=1,2)$ are the intensities or strengths of the random perturbations, $\delta$ is the Dirac delta function defined by

$$\begin{cases} \delta(x)=0, \text{ for } x\neq 0, \\ \int_{-\infty}^{\infty}\delta(x)dx = 1, \end{cases}$$

and $\langle\cdot\rangle$ represents the ensemble average of the stochastic process.

Though Gaussian white noises are so very irregular, these are extremely useful to model rapidly fluctuating phenomena. Of course true white noises do not occur in nature. However, as can be seen by studying their spectra, thermal noises in electrical resistance, the force acting on a Brownian particle, climate fluctuations, disregarding the periodicity of astronomical origin etc. are white to a very good approximation. These examples support the usefulness of the white noise idealization in applications to natural systems. Furthermore it can be proved that the process $(x,y)^T$, solution of (3), is Markovian if and only if the external noises are white. These results explain the importance and appeal of the white-noise idealization [5]. It is noted that $\eta_i(t)$ are not defined in the ordinary sense. It can be proved that $\eta_i(t)$ are the derivatives of the Wiener process $W_i(t)$ in the generalized functions sense.

## 2. **Stochastic scenario of non-delayed system**

Assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the natural growth rate of the prey and in the natural mortality rate of the predator since these are the main terms subject to coupling of a prey-predator pair with its environment [4]. Thus the behaviour of the prey-predator pair (3) without delay in a random environment will be considered within the framework of the following model:

$$\frac{dx}{dt} = x\left[r\left(1-\frac{x}{K}\right)+\eta_1(t)\right]-\frac{\alpha xy}{ay+x}-Ex, \qquad \frac{dy}{dt} = y\left[-d_0+\eta_2(t)\right]+\frac{\alpha b_0 xy}{ay+x}. \tag{4}$$

To study the behavior of the system (4) about the steady state $E^*$, let us substitute $x' = \ln x$, $y' = \ln y$; $x = u + x^*$ and $y = v + y^*$. Then the system (4) reduces to the following Itô type stochastic differential equations in terms of deviation variables $(u, v)$:

$$\frac{du}{dt} = a_1u + b_1u^2 + c_1v + d_1v^2 + e_1uv + \eta_1(t), \qquad \frac{dv}{dt} = a_2u + b_2u^2 + c_2v + d_2v^2 + e_2uv + \eta_2(t), \tag{5}$$

where

$$\begin{aligned} &a_1 = -\frac{rx^*}{k} + \frac{\alpha x^*y^*}{(ay^*+x^*)^2}, && b_1 = -\frac{r}{k} + \frac{a\alpha y^{*2})}{(ay^*+x^*)^3},\\ &c_1 = -\frac{\alpha x^{*2}}{(ay^*+x^*)^2}, && d_1 = \frac{a\alpha x^{*2}}{(ay^*+x^*)^3},\\ &e_1 = -\frac{2a\alpha x^*y^*}{(ay^*+x^*)^3}, && a_2 = \frac{\alpha b_0 ay^{*2}}{(ay^*+x^*)^2},\\ &b_2 = \frac{\alpha b_0 y^*(a^2x^*-x^*-ay^*)}{(ay^*+x^*)^3}, && c_2 = -\frac{\alpha ab_0x^*y^*}{(ay^*+x^*)^2},\\ &d_2 = \frac{\alpha b_0 x^*(y^*-ax^*-a^2y^*)}{(ay^*+x^*)^3}, && \text{and } e_2 = \frac{2\alpha b_0 ax^*y^*)}{(ay^*+x^*)^3}. \end{aligned} \tag{6}$$

The solutions $\{u(t), v(t)\}$ of (5) subject to known initial values $\{u(t_0), v(t_0)\}$ determine the statistical behavior of the model system (4) near the steady state $E^*$ at time $t > t_0$.

2.1. **Statistical linearization: moment equations.** The statistical linearization of the system (5) are represented by the following system of linear equations:

$$\frac{du}{dt} = \alpha_1u + \beta_1v + f_1 + \eta_1(t), \qquad \frac{dv}{dt} = \alpha_2u + \beta_2v + f_2 + \eta_2(t), \tag{7}$$

where the errors in the above linearization are given by

$$e_1 = a_1u + b_1u^2 + c_1v + d_1v^2 + e_1uv - \alpha_1u - \beta_1v - f_1, \qquad e_2 = a_2u + b_2u^2 + c_2v + d_2v^2 + e_2uv - \alpha_2u - \beta_2v - f_2. \tag{8}$$

The unknown coefficients $\alpha_i$, $\beta_i$ and $f_i$ $(i = 1, 2)$ of the equations (7) are to be determined from the minimization of the averages of the squares of errors in (8). The unknown coefficients are determined by demanding that [35, 39, 40]:

$$\frac{\partial}{\partial\alpha_i}\langle e_i^2\rangle = \frac{\partial}{\partial\beta_i}\langle e_i^2\rangle = \frac{\partial}{\partial f_i}\langle e_i^2\rangle = 0, \quad i = 1, 2. \tag{9}$$

Let us express $\langle u^3\rangle, \langle u^4\rangle, \langle u^2v\rangle, \langle uv^2\rangle, \langle u^2v^2\rangle$ and $\langle u^3v\rangle$ in terms of the first two moments of each of the variables and the correlation coefficients using a bivariate Gaussian distribution [35]. Since we are interested only in the first few moments, it is convenient to use the characteristic function:

$$\chi(\nu_1,\nu_2) = \exp\left[i\langle u\rangle\nu_1 + i\langle v\rangle\nu_2 - \frac{1}{2}\left\{\sigma_1^2\nu_1^2 + \sigma_2^2\nu_2^2 + 2\rho_{12}\sigma_1\sigma_2\nu_1\nu_2\right\}\right],$$

$$\sigma_1^2 = \langle u^2\rangle - \langle u\rangle^2,$$
$$\sigma_2^2 = \langle v^2\rangle - \langle v\rangle^2,$$
$$\rho_{12} = \frac{\langle uv\rangle - \langle u\rangle\langle v\rangle}{\sigma_1\sigma_2}.$$

So, we have

$$\langle u^n v^m\rangle = (-1)^{n+m}\frac{\partial^{n+m}}{\partial\nu_1^n\partial\nu_2^m}[\chi(\nu_1,\nu_2)]|_{\nu_1=\nu_2=0},$$

$$\begin{aligned}
\langle u^4\rangle &= 3\langle u^2\rangle^2 - 2\langle u\rangle^4,\\
\langle u^2v^2\rangle &= \langle u^2\rangle\langle v^2\rangle + 2\langle uv\rangle^2 - 2\langle u\rangle^2\langle v\rangle^2,\\
\langle u^3v\rangle &= 3\langle u^2\rangle\langle uv\rangle - 2\langle u\rangle^3\langle v\rangle,\\
\langle u^3\rangle &= 3\langle u\rangle\langle u^2\rangle - 2\langle u\rangle^3,\\
\langle v^3\rangle &= 3\langle v\rangle\langle v^2\rangle - 2\langle v\rangle^3,\\
\langle u^2v\rangle &= 2\langle u\rangle\langle uv\rangle - 2\langle u\rangle^2\langle v\rangle + \langle u^2\rangle\langle v\rangle,\\
\langle uv^2\rangle &= 2\langle u\rangle\langle uv\rangle - 2\langle u\rangle\langle v\rangle^2 + \langle u\rangle\langle v^2\rangle.
\end{aligned} \tag{10}$$

Then expressions for $\alpha_i$, $\beta_i$ and $f_i$ $(i=1,2)$ are given by

$$\begin{aligned}
&\alpha_i = a_i + 2b_i\langle u\rangle + e_i\langle v\rangle, \quad \beta_i = c_i + 2d_i\langle v\rangle + e_i\langle u\rangle,\\
&f_i = b_i(\langle u^2\rangle - 2\langle u\rangle^2) + d_i(\langle v^2\rangle - 2\langle v\rangle^2) + e_i(\langle uv\rangle - 2\langle u\rangle\langle v\rangle).
\end{aligned} \tag{11}$$

The coefficients are the functions of the parameters involved within the model system and also of the different moments involving $u$ and $v$. After some algebraic manipulations, we obtain the following system of first two moments:

$$\begin{aligned}
\frac{d\langle u\rangle}{dt} &= a_1\langle u\rangle + b_1\langle u^2\rangle + c_1\langle v\rangle + d_1\langle v^2\rangle + e_1\langle uv\rangle,\\
\frac{d\langle v\rangle}{dt} &= a_2\langle u\rangle + b_2\langle u^2\rangle + c_2\langle v\rangle + d_2\langle v^2\rangle + e_2\langle uv\rangle,\\
\frac{d\langle u^2\rangle}{dt} &= 2[a_1\langle u^2\rangle + b_1\langle u^3\rangle + c_1\langle uv\rangle + d_1\langle uv^2\rangle + e_1\langle u^2v\rangle] + 2\epsilon_1,\\
\frac{d\langle v^2\rangle}{dt} &= 2[a_2\langle uv\rangle + b_2\langle u^2v\rangle + c_2\langle v^2\rangle + d_2\langle v^3\rangle + e_2\langle uv^2\rangle] + 2\epsilon_2,\\
\frac{d\langle uv\rangle}{dt} &= a_1\langle uv\rangle + b_1\langle u^2v\rangle + c_1\langle v^2\rangle + d_1\langle v^3\rangle + e_1\langle uv^2\rangle + a_2\langle u^2\rangle\\
&\quad + b_2\langle u^3\rangle + c_2\langle uv\rangle + d_2\langle uv^2\rangle + e_2\langle u^2v\rangle,
\end{aligned} \tag{12}$$

where the following relations are used:

$$\langle u\eta_1\rangle = \epsilon_1, \quad \langle u\eta_2\rangle = \langle v\eta_1\rangle = 0, \quad \langle v\eta_2\rangle = \epsilon_2. \tag{13}$$

Let me now assume that the system size expansion is valid such that the correlations $\epsilon_i$ $(i = 1, 2)$ given by (13) decrease with the increase of the population size and they are assumed to be of the order of the inverse of the population size $N$ [35, 41, 40]:

$$\epsilon_i \propto o[\frac{1}{N}], \quad i = 1, 2. \tag{14}$$

Therefore, using the expressions (10), (13) and keeping the lowest order terms and replacing the averages $\langle u\rangle$ and $\langle v\rangle$ by their steady state values $\langle u\rangle = \langle v\rangle = 0$ [42], we get the following reduced equations for second order moments:

$$\begin{aligned} [D - 2a_1]\langle u^2\rangle &= 2c_1\langle uv\rangle, \\ [D - 2c_2]\langle v^2\rangle &= 2a_2\langle uv\rangle, \\ [D - a_1 - c_2]\langle uv\rangle &= a_2\langle u^2\rangle + c_1\langle v^2\rangle, \end{aligned} \tag{15}$$

where $D$ stands for the operator $\frac{d}{dt}$.

2.2. **Non-equilibrium fluctuation and stability analysis.** Eliminating $\langle u^2\rangle$ and $\langle v^2\rangle$ from the equations of (15), we get the following third order linear ordinary differential equation in $\langle uv\rangle$:

$$[D^3 + 3AD^2 + 3BD + C]\langle uv\rangle = 0. \tag{16}$$

Let $\langle uv\rangle = e^{mt}$ be a trial solution of (16) and the auxiliary equation is given by

$$m^3 + 3Am^2 + 3Bm + C = 0, \tag{17}$$

where

$$A = -(a_1 + c_2), \quad B = \frac{2}{3}\{(a_1 + c_2)^2 + 2(a_1c_2 - a_2c_1)\}, \quad C = -4(a_1 + c_2)(a_1c_2 - a_2c_1).$$

Let $H = A^2 - B$. Then the nature and structure of the roots of (17) will solely be determined by the quantities $A$ and $H$. We discuss the following two cases.

**Case 1.** $H < 0$

In this case, roots of (17) are given by

$$m_1 = -A, \quad m_{2,3} = -A \pm i\sqrt{3H_0}, \quad \text{where } H_0 = -H(> 0).$$

The solutions of the linear system (15) are then given by

$$\begin{aligned} \langle uv\rangle &= A_1e^{-At} + e^{-At}[A_2\cos(\sqrt{3H_0}t) + A_3\sin(\sqrt{3H_0}t)], \\ \langle v^2\rangle &= B_1e^{-At} + e^{-At}[B_2\cos(\sqrt{3H_0}t) + B_3\sin(\sqrt{3H_0}t)] + P_1e^{2a_1t}, \\ \langle u^2\rangle &= C_1e^{-At} + e^{-At}[C_2\cos(\sqrt{3H_0}t) + C_3\sin(\sqrt{3H_0}t)] + P_2e^{2c_2t} \end{aligned} \tag{18}$$

where $A_i$, $B_i$, $C_i$, $(i = 1, 2, 3)$, $P_1$ and $P_2$ are constants. Thus, each of $\langle uv\rangle$, $\langle u^2\rangle$, $\langle v^2\rangle$, given by (18), converge with increasing time if $A > 0$, i.e., if $a_1 + c_2 < 0$, depicting the

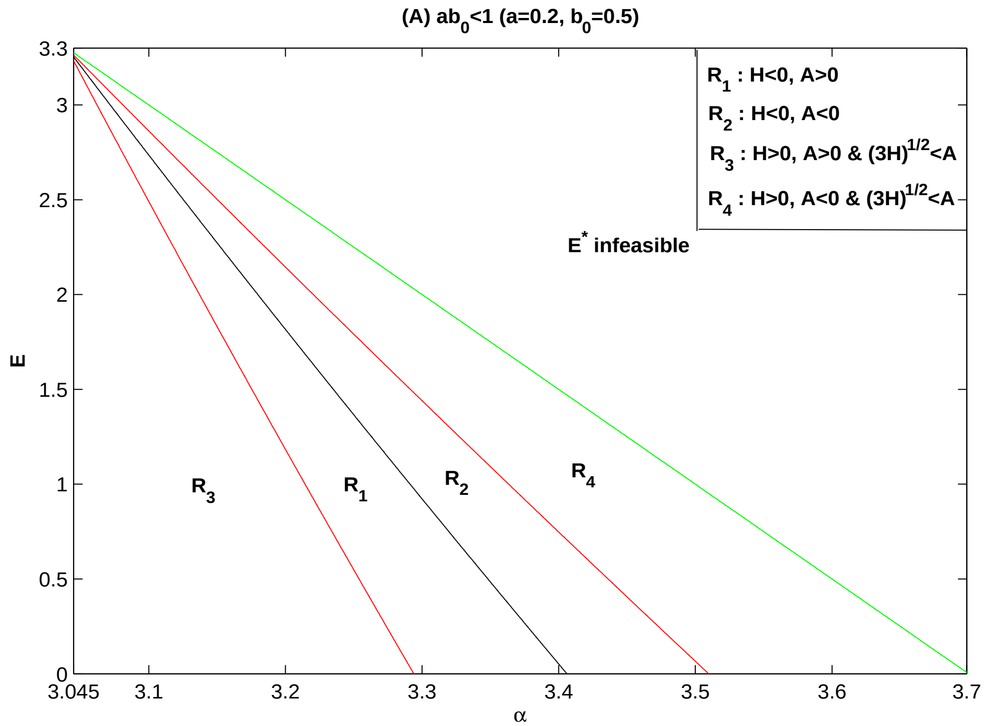


FIGURE 1. Stability region w.r.t second order moment of system (5) in $\alpha E$-plane for $ab_0 < 1$. Parameters: $r = 3.3$, $k = 898$, $a = 0.2$, $b_0 = 0.5$, $d_0 = 1.52$.

stochastically stable situation of the system (4) in the sense of second order moments (Arnold, 1974). On the other hand, if $a_1 + c_2 > 0$, then $A < 0$ and the system is stochastically unstable.

**Case 2.** $H > 0$

In this case, roots of (17) are given by

$$m_1 = -A, \quad m_{2,3} = -A \pm \sqrt{3H}.$$

The solutions of the linear system (15) are then given by

$$\begin{aligned} \langle uv \rangle &= F_1 e^{-At} + F_2 e^{(-A+\sqrt{3H})t} + F_3 e^{(-A-\sqrt{3H})t}, \\ \langle v^2 \rangle &= G_1 e^{-At} + G_2 e^{(-A+\sqrt{3H})t} + G_3 e^{(-A-\sqrt{3H})t} + Q_1 e^{2a_1 t}, \\ \langle u^2 \rangle &= K_1 e^{-At} + K_2 e^{(-A+\sqrt{3H})t} + K_3 e^{(-A-\sqrt{3H})t} + Q_2 e^{2c_2 t}, \end{aligned} \tag{19}$$

where $F_i$, $G_i$, $K_i$, $(i = 1, 2, 3)$, $Q_1$, $Q_2$ are constants. In this case, we observe that each of $\langle uv \rangle$, $\langle u^2 \rangle$, $\langle v^2 \rangle$, given by (19), converge with increasing time when $A > 0$ (i.e., $a_1 + c_2 < 0$) and $\sqrt{3H} < A$ hold simultaneously and the system will said to be stochastically stable in the sense of second order moments. The system, on the other hand, is unstable when $a_1 + c_2 > 0$ or $\sqrt{3H} < A$ or both of them hold. Thus, exchange of stability occurs when $a_1 + c_2 = 0$ and $\sqrt{3H} = A$.

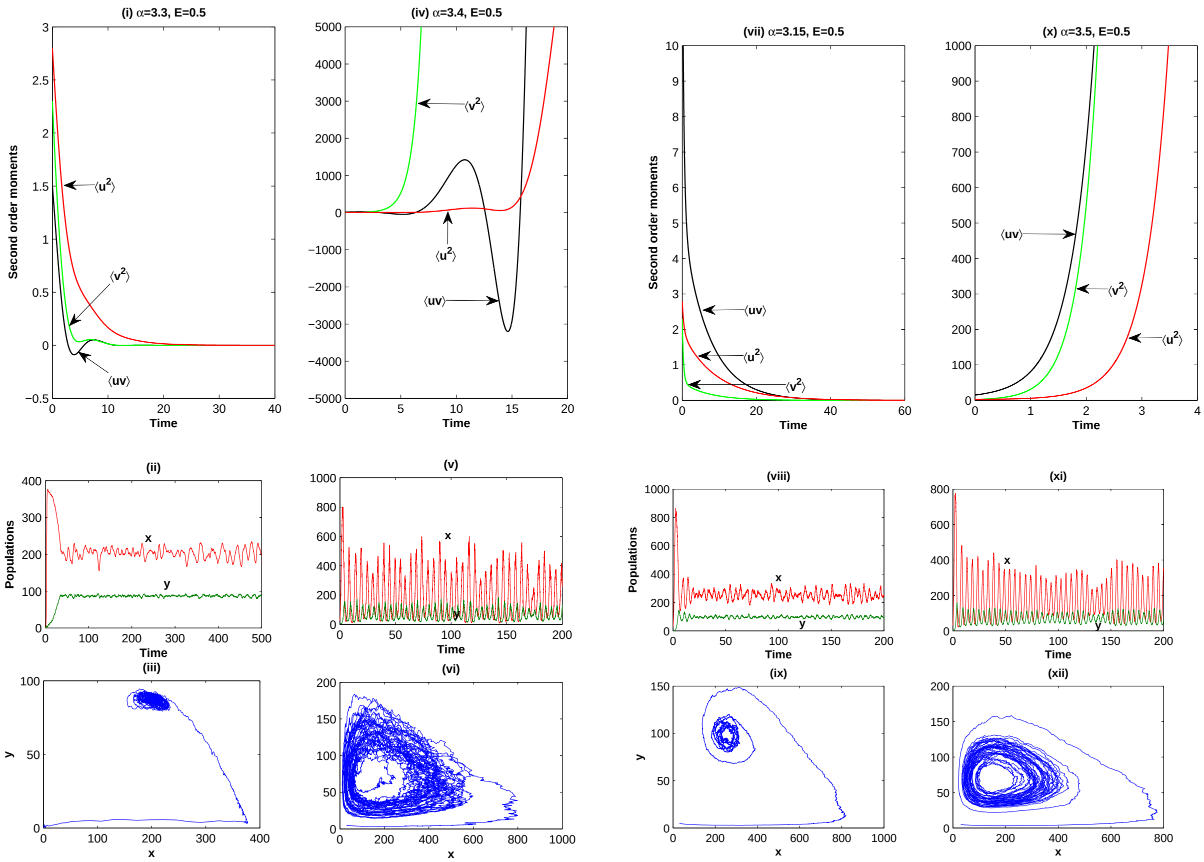


FIGURE 2. Fig. (i,iv,vii,x) are second order moment w.r.t time of system (5) and Fig. (ii,v,viii,xi) and (iii,vi,ix,xii) are respectively time evolution and phase plane diagram of system (4). $(\alpha, E)$ are from stability domain $R_i$ $(i = 1, 2, 3, 4)$ of Fig. 1, where 1st and 3rd panel depict the stochastic stability, where 2nd and 4th panel depict the stochastic instability. Other parameters are as in Fig. 1.

## 3. **Stochastic scenario of delayed system**

$$\begin{aligned}
\frac{dx}{dt} &= x\left[r\left(1-\frac{x(t-\tau_1)}{K}\right)+\eta_1(t)\right]-\frac{\alpha xy}{ay+x}-Ex, \\
\frac{dy}{dt} &= y\Big[-d_0+\eta_2(t)\Big]+\frac{\alpha b_0 x(t-\tau_2)y}{ay(t-\tau_2)+x(t-\tau_2)}.
\end{aligned} \tag{20}$$

Again using the transformations: $x' = \ln x$, $y' = \ln y$; $x = u + x^*, y = v + y^*$ and assuming the delay to be very small, the system (20) (to a first approximation) can be written as

$$\begin{aligned}
\frac{du}{dt} &= \bar{a}_1 u + c_1 v + A_1 u(t-\tau_1) + \eta_1, \\
\frac{dv}{dt} &= a_2 u(t-\tau_2) + c_2 v(t-\tau_2) + \eta_2.
\end{aligned} \tag{21}$$

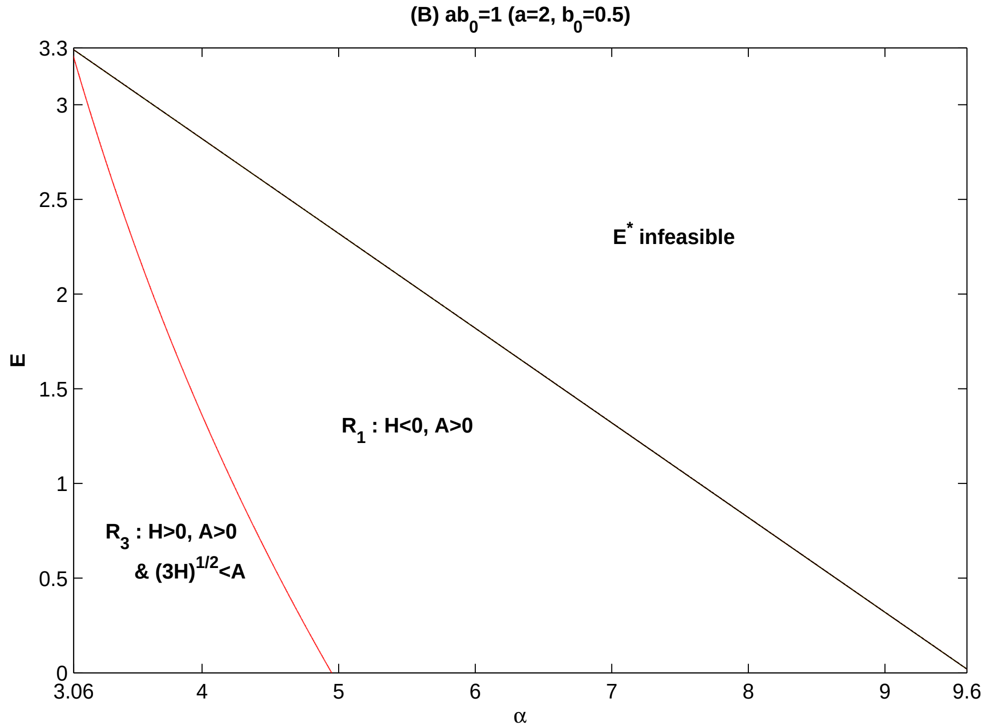


FIGURE 3. Stability region w.r.t second order moment of system (5) in $\alpha E$-plane for $ab_0 = 1$. Parameters: $r = 3.3$, $k = 898$, $a = 2$, $b_0 = 0.5$, $d_0 = 1.52$.

where, $\bar{a}_1 = \frac{\alpha x^* y^*}{(ay^*+x^*)^2}$ and $A_1 = -\frac{rx^*}{k}$.

3.1. **Fourier transforms: spectral density.** Taking Fourier transform of both sides of each of the equations in (21) and following [20, 26], we obtain

$$\begin{aligned} \bar{\eta}_1(s) &= is\bar{u}(s) - \bar{a}_1\bar{u}(s) - c_1\bar{v}(s) - A_1\bar{u}(s)e^{-is\tau_1}, \\ \bar{\eta}_2(s) &= is\bar{v}(s) - a_2\bar{u}(s)e^{-is\tau_2} - c_2\bar{v}(s)e^{-is\tau_2}, \end{aligned} \tag{22}$$

where $\bar{n}(s) = \int_{-\infty}^{+\infty} n(t)e^{-ist}dt$.

The system of equations (22) can be written in matrix form as

$$AX = B, \tag{23}$$

where

$$A = \begin{pmatrix} -\bar{a}_1 - A_1 e^{-is\tau_1} + is & -c_1 \\ -a_2 e^{-is\tau_2} & -c_2 e^{-is\tau_2} + is \end{pmatrix}, \quad X = \begin{pmatrix} \bar{u}(s) \\ \bar{v}(s) \end{pmatrix} \quad \text{and } B = \begin{pmatrix} \bar{\eta}_1 \\ \bar{\eta}_2 \end{pmatrix}.$$

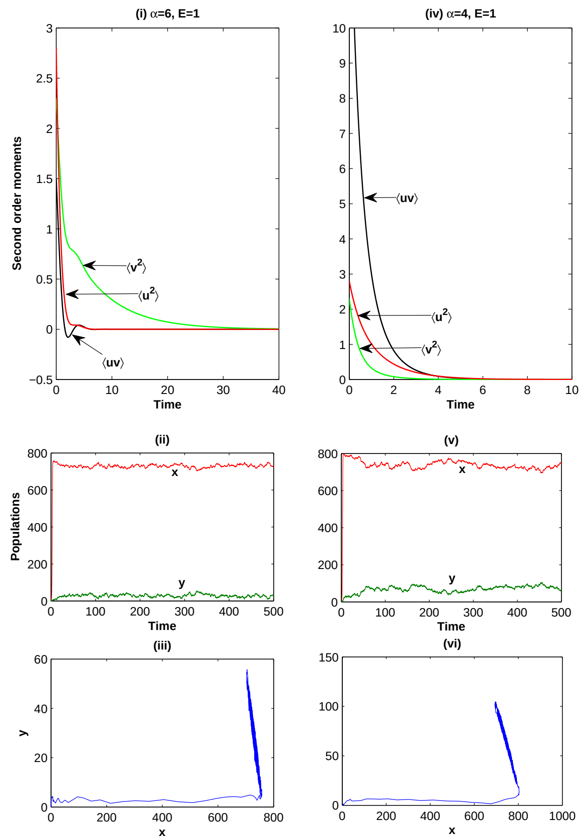


FIGURE 4. Fig. (i,iv) are second order moment w.r.t time of system (5) and Fig. (ii,v) and (iii,vi) are respectively time evolution and phase plane diagram of system (4). $(\alpha, E)$ are from stability domain $R_i$ $(i = 1, 3)$ of Fig. 3, all figures depict the stochastic stability of the system. Other parameters are as in Fig. 3.

Now

$$\begin{aligned} M = \det A = \{&(\bar{a}_1c_2 - c_1a_2)\cos(s\tau_2) + A_1c_2\cos(s(\tau_1+\tau_2)) \\ &- sA_1\sin(s\tau_1) - sc_2\sin(s\tau_2) - s^2\} \\ &- i\{(\bar{a}_1c_2 - a_2c_1)\sin(s\tau_2) + A_1c_2\sin(s(\tau_1+\tau_2)) \\ &+ sA_1\cos(s\tau_1) + sc_2\cos(s\tau_2) + s\bar{a}_1\}. \end{aligned}$$

Assuming $A^{-1}$ exists, we have $A^{-1} = (a_{ij})_{2\times 2}$, where

$$\begin{aligned} a_{11} &= \tfrac{-c_2\cos(s\tau_2)+i\{s+c_2\sin(s\tau_2)\}}{M}, & a_{12} &= \tfrac{c_1}{M}, \\ a_{21} &= \tfrac{a_2(\cos(s\tau_2)-i\sin(s\tau_2))}{M}, & a_{22} &= \tfrac{-\bar{a}_1-A_1\cos(s\tau_1)+i\{s+A_1\sin(s\tau_1)\}}{M}. \end{aligned}$$

Then the solution of (23) can be written as

$$\bar{u}(s) = \sum_{j=1}^{2} a_{1j}\eta_j, \quad \bar{v}(s) = \sum_{j=1}^{2} a_{2j}\eta_j \tag{24}$$

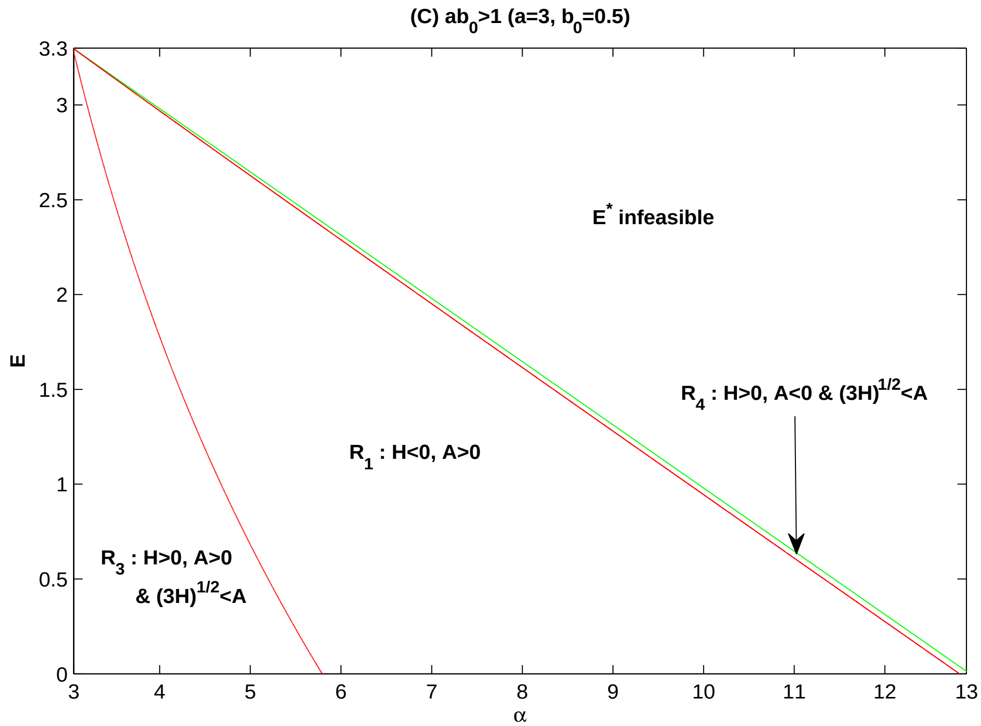


FIGURE 5. Stability region w.r.t second order moment of system (5) in $\alpha E$-plane for $ab_0 > 1$. Parameters: $r = 3.3$, $k = 898$, $a = 3$, $b_0 = 0.5$, $d_0 = 1.52$.

Now following [20, 26] and using (24), the spectral density of $u$ is given by

$$S_u(\omega) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \langle u(t)u(t')\rangle \exp\{i\omega(t'-t)\}dtdt' = \sum_{j=1}^{2} |a_{1j}|^2 S_{\eta_j}(\omega).$$

Similarly the spectral density of $v$ is given by

$$S_v = \sum_{j=1}^{2} |a_{2j}|^2 S_{\eta_j}(\omega).$$

Therefore the fluctuation intensity (variance) of $u$ is given by

$$\sigma_u^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{1j}|^2 S_{\eta_j}(\omega)d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{1j}|^2 d\omega,$$

since $S_{\eta_j}(\omega) = 1$.

Similarly the fluctuation intensity of $v$ is given by

$$\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{2j}|^2 d\omega.$$

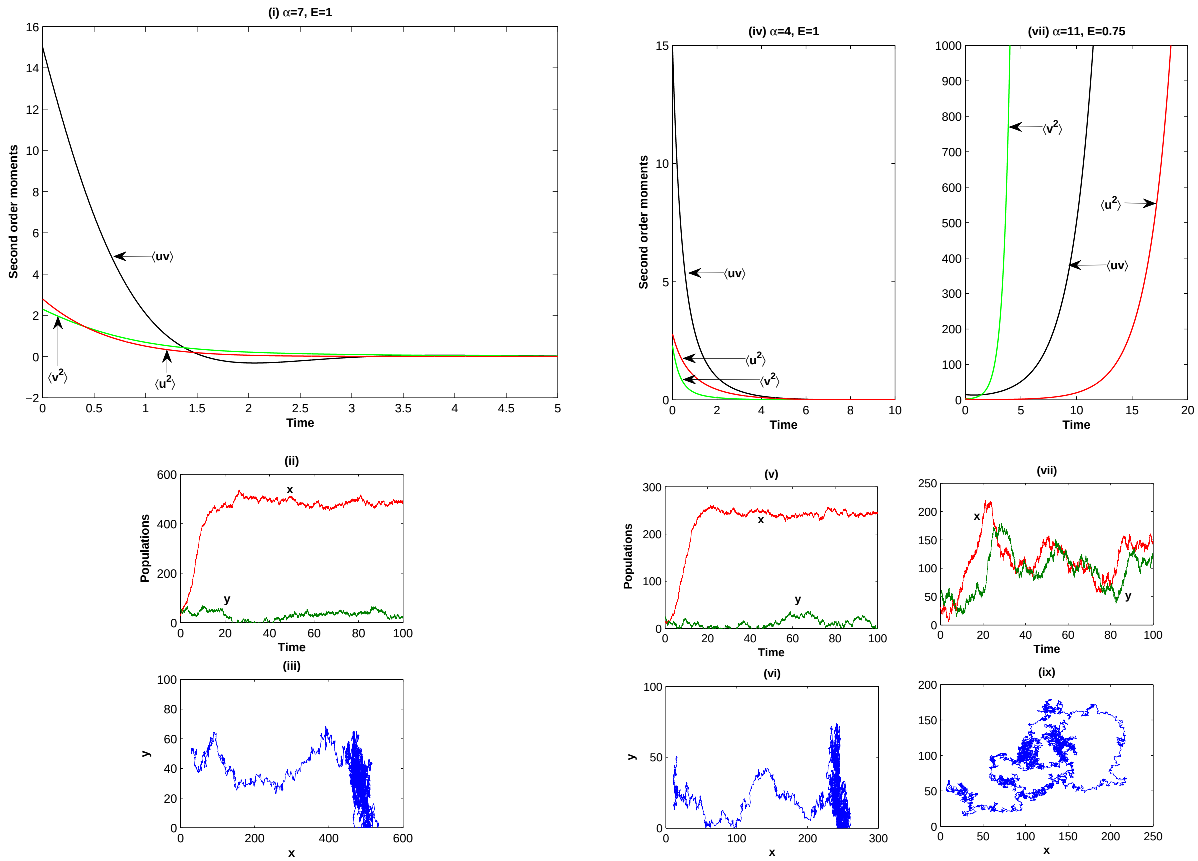


FIGURE 6. Fig. (i,iv,vii) are second order moment w.r.t time of system (5) and Fig. (ii,v,viii) and (iii,vi,ix) are respectively time evolution and phase plane diagram of system (4). $(\alpha, E)$ are from stability domain $R_i$ $(i = 1, 3, 4)$ of Fig. 5, panel 1 and 2 depict the stochastic stability of the system and panel 3 depicts the stochastic instability of the system. Other parameters are as in Fig. 5.

## 4. Numerical computations

We perform numerical computations to observe various dynamics of the systems, so we consider the parameters are as $r = 3.3$, $k = 898$, $a = 3$, $b_0 = 0.5$, $d_0 = 1.52$ and $(\alpha, E)$ are varying for different figures and approximate the solutions of the system by Euler-Maruyama method. First we consider the non-delayed case, i.e. $\tau_1 = \tau_2 = 0$. First we plot the stability region of the system (5) w.r.t second order moment in $\alpha E$-plane (Figs. 1, 3, 5). Figs. 1, 3 and 5 are respectively for productively inefficient ($ab_0 < 1$), productively nutrient ($ab_0 = 1$) and productively efficient ($ab_0 > 1$) predators. Figs. 2, 4, 6 are the status of the system w.r.t second order moments, time evolutions and phase plane diagrams. Here also Figs. 2, 4 and 6 are respectively for productively inefficient, productively nutrient and productively efficient predators. In the figures of stability region (Figs. 1, 3, 5), $R_i$ $(i = 1, 2, 3, 4)$

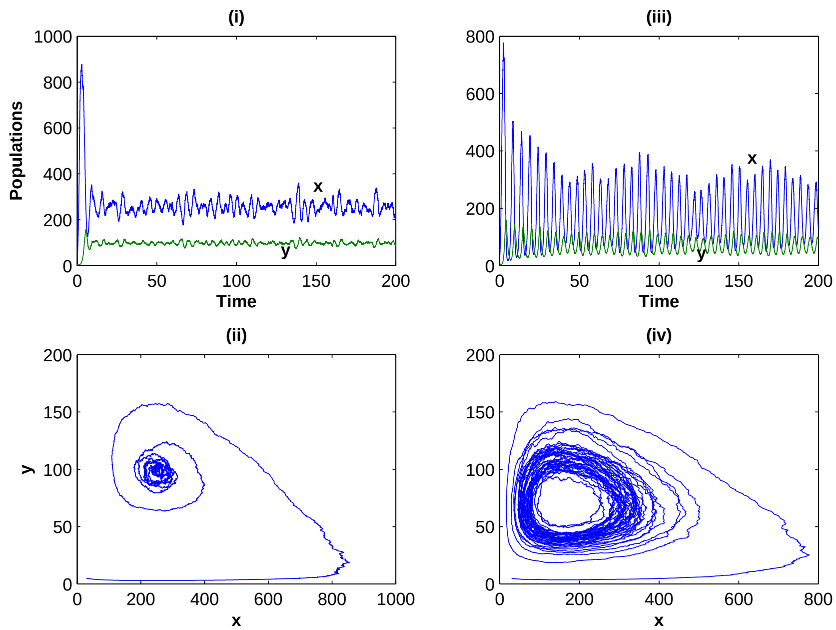


FIGURE 7. Figures (i, iii) and (ii, iv) depict, respectively, the time evolution and corresponding phase trajectory of the delay-stochastic model system (20), when the predator population is productively inefficient ($ab_0 < 1$). For $\tau_1 = 0.48$, $\tau_2 = 0.4$, system is stochastically stable (Fig. (i, ii)) and for $\tau_1 = 0.54$, $\tau_2 = 0.4$, system is stochastically unstable (Fig. (iii, iv)). Other parameters are in Fig. 1.

stands for the stochastic stability conditions $H < 0$, $A > 0$; $H < 0$, $A < 0$; $H > 0$, $A > 0$, $\sqrt{3H} < A$ and $H > 0$, $A < 0$, $\sqrt{3H} < A$.

In the numerical experiments of the delay-stochastic system (20), consider the set of parameter values for the case when the predators are productively inefficient ($ab_0 < 1$). Delay-stochastic system (20) is stable for $\tau_1 = 0.48$ and $\tau_2 = 0.4$ (time series in Fig. 7(i) and phase plane in Fig. 7(ii)) and unstable for $\tau_1 = 0.54$ and $\tau_2 = 0.4$ (time series in Fig. 7(iii) and phase plane in Fig. 7(iv)).

## 5. Conclusions and Discussions

Understanding the relationship between predator and prey is central goal in ecology. Predators rate of feeding upon prey is one of the most significant components in the study of predator-prey relationship. Predator's interference in prey-predator interaction is supposed to have significant role in the stability of the their interaction. Also the time lag between prey capture and its corresponding positive feedback to predator's growth rate is a very significant part to realize the system. The purpose of this work is to observe the extent to which Predator's interference and gestation delay drive the population dynamics of a predator-prey interaction under fluctuating environment. Results shows that Predator's interference plays a significant role to change the stochastic stability of the system. To study the effect of environmental fluctuation on the time-delayed predator-prey system (20), I have superimposed Gaussian white noises on (2) and then study non-equilibrium fluctuation and stability of the resulting stochastic model (20) by using Fourier transform technique. Following the criteria of stability in the stochastic environment [43], it is seen that the environmental noises have a destabilizing effect on the system. Also the deterministic system and the noise-induced stochastic system may behave alike with respect to stability. It is well known that natural populations of plants and animals neither increase indefinitely to blanket the world nor become extinct (except in some rare cases due to some rare reasons). Hence, in practice, we often want to keep the prey and predator population to an acceptable level in finite time. In order to accomplish this we strongly suggest that in realistic field situations (where effect of time-delay and environmental fluctuation can never be violated), the parameters of the system should be regulated in such a way that $E^*$ is deterministically stable.

## REFERENCES

[1] C. S. Holling, Some characteristics of simple types of predation and parasitism. Canadian Entomologist, 91, 385–398, 1959.

[2] J. van der Meer, B. J. Ens, Models of interference and their consequences for the spatial distribution of ideal and free predators. Journal of Animal Ecology, 66, 846–858, 1997.

[3] D. L. DeAngelis, R. A. Goldstein, R. V. ONeill, A model for trophic interaction. Ecology, 56, 881–892, 1975.

[4] Dimentberg, M.F., *Statistical Dynamics of Nonlinear and Time-Varying Systems*, John Wiley and Sons, New York, 1988.

[5] Horsthemke, W. and Lefever, R., *Noise Induced Transitions*, Springer-Verlag, Berlin, 1984.

[6] G. Huisman, R. J. De Boer, A formal derivation of the "Beddington" functional response. Journal of Theoretical Biology, 185, 389–400, 1997.

[7] Schmitz, O. J., 1992. Exploitation in model food chains with mechanistic consumerresource dynamics. Theoretical Population Biology, 41, 161–183.

[8] Alstad, D., 2001. Basic Populas Models of Ecology. Prentice Hall, Inc., NJ.

[9] Anderson, O., 1984. Optimal Foraging by largemouth bass in structured environments. Ecology. 65, 851–861.

[10] Kuang, Y., Beretta, E., 1998. Global qualitative analysis of a ratio-dependent predator-prey system. J. Math. Biol., 36, 389–406.

[11] Martin, A., Ruan, S., 2001. Predator-prey models with delay and prey harvesting. J. of Math. Biol. 43(3), 247–267.

[12] Jost, C., Arino, O., Arditi, R., 1999. About deterministic extinction in ratio-dependent predator-prey models. Bull. Math. Biol. 61, 19–32.

[13] Xiao, D., Ruan, S., 2001. Global dynamics of a ratio-dependent predator-prey system. J. Math. Biol. 43, 268–290.

[14] Jana, D., Chakraborty, S., Bairagi, N., 2012. Stability, Nonlinear Oscillations and Bifurcation in a Delay-Induced Predator-Prey System with Harvesting. Engineering Letters, 20(3), 238–246.

[15] Xiao, D., Jennings, L. S., Bifurcations of a ratio-dependent predator-prey system with constant rate harvesting. *SIAM J. Appl. Math. Biol.*, 659(3), 737–753, (2005).

[16] Xiao, D., Li, W., Han, M., 2006. Dynamics of a ratio-dependent predator-prey model with predator harvesting. J. Math. Anal. Appl. 324, 14–29.

[17] Chakraborty, S., Pal, S. and Bairagi, N. (2012). Predator-prey interaction with harvesting - mathematical study with biological ramifications. Applied Mathematical Modelling, 36, 4044–4059.

[18] Hairston, N.G., Smith, F.E., Slobodkin, L.B., 1960. Community structure, population control and competition. American Naturalist 94, 421–425.

[19] Rosenzweig, M.L., 1969. Paradox of Enrichment: destabilization of exploitation systems in ecological time. Science 171, 385–387.

[20] Luck, R.F., 1990. Evaluation of natural enemies for biological control: a behavior approach. Trends in Ecology and Evolution 5, 196–199.

[21] Clark C.W. (1976) Mathematical Bioeconomics: The optimal Management of Renewable Resources. John Wiley & Sons, New York.

[22] Kuang Y., Delay Differential Equations with Applications in Population Dynamics, Academic Press, New York, 1993.

[23] Macdonald N., Biological Delay Systems: Linear Stability Theory, Cambridge University Press, Cambridge, 1989.

[24] Gopalsamy K., Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht,1992.

[25] Ruan, S. (2009) On Nonlinear Dynamics of Predator-Prey Models with Discrete Delay. Math. Model. Nat. Phenom., 4(2), 140–188.

[26] Xia, J., Liu, Z. Yuan, R. and Ruan, S. (2009) The effects of harvesting and time delay on predator-prey systems with Holling Type II functional response. SIAM J. Appl. Math., 70(4), 1178–1200.

[27] Hu, G.P., Li, W.T. and Yan, X.P. (2009) Hopf bifuractation in a delayed predator-prey system with multiple delays. Chaos Soli. Frac., 42, 1273–1285.

[28] Liao, M., Tang, X. and Xu, C. (2012) Bifurcation analysis for a three-species predator-prey system with two delays. Commun Nonlinear Sci Numer Simulat., 17, 183–194.

[29] Meng, X.Y., Huo, H.F. and Zhang, X.B. (2011) Stability and global Hopf bifurcation in a delayed food web consisting of a prey and two predators. Commun Nonlinear Sci Numer Simulat., 16, 4335–4348.

[30] Song, Y.L., Peng, Y.H. and Wei, J.J. (2008) Bifurcations for a predator-prey system with two delays. J. Math. Anal. Appl., 337, 466–79.

[31] Yan, X.P. and Chu, Y.D. (2006) Stability and bifurcation analysis for a delayed Lotka-Volterra predator-prey system. Journal of Computational and Applied Math., 196, 198–210.

[32] Freedman, H.I. (1987) Deterministic Mathematical Models in Population Ecology. HIFR Consulting Ltd., Edmonton.

[33] Y. Iwasa, H. Hakoyama, M. Nakamaru, J. Nakanishi, *Estimate of population extinction risk and its application to ecological risk management*, Popul Ecol 42 (2000), pp. 73–80.

[34] M. Turelli, *Stochastic Community Theory: A Partially Guided Tour*, In Mathematical Ecology, edited. by Hallman TG, Levin S, Springer-Verlag, Berlin, 1986.

[35] M. C. Valsakumar, K. P. N. Murthy, G. Ananthakrishna, *On the linearization of non-linear Langevin type stochastic differential equations*, J. Stat. Phys 30 (1983), pp. 617–631.

[36] G. P. Samanta, A. Maiti, *Stochastic Gomatam model of interacting species: non-equilibrium fluctuation and stability*, Syst. Anal. Model. Sim. 43 (2003), pp. 683–692.

[37] Y. Kuang, H. I. Freedman, *Uniqueness of limit cycles in Gause-type models of predator-prey systems*, Math. Biosci. 88 (1988), pp. 67–84.

[38] J. E. Marsden, M. McCracken, *The Hopf Bifurcation and its application*, Springer-Verlag, New York, 1976.

[39] N. G. Van Kampen, *Stochastic process in Physics and Chemistry*, North Holland Publishing Co, Amsterdam, 1981.

[40] M. Bandyopadhyay and C. G. Chakraborti, *Deterministic and stochastic analysis of a non-linear prey-predator system*, J. Biol. Syst. 11 (2003), pp. 161–172.

[41] M. C. Baishya and C. G. Chakraborti, *Non-equilibrium fluctuation in Lotka-Volterra system*, B. Math. Biol. 49 (1987), pp. 125–131.

[42] G. Nikolis, L. Prigogine, *Self-Organization in Non-Equilibrium Systems*, Wiely, New York, 1977.

[43] R. M. May, Stability and Complexity in Model Ecosystems. Princeton University Press, Princeton, 1974.