Neural, Parallel, and Scientific Computations 22 (2014) 239-248

A NONSTANDARD VARIABLE STEP LENGTH METHOD FOR THE SOLUTION OF FIRST ORDER INITIAL VALUE PROBLEMS IN ODES

P. K. PANDEY

Department of Mathematics, Dyal Singh College (University of Delhi)
Lodhi Road, New Delhi-110003, India
Present address: Department of Information Technology,
College of Applied Sciences, Ministry of Higher Education,
P.O. Box No.347, PC-215, Saada, Salalah, Sultanate of Oman
pramod_10p@hotmail.com

ABSTRACT. In this article, we have presented the development and implementation of a nonstandard variable step length numerical method for solving first order initial value problems in ordinary differential equations. The method is convergent and stable. The method applied to find the numerical solution of several model problems. The theoretical conclusions of the proposed method are confirmed by the numerical results obtained for these model problems with known solution. The computation results obtained for these model problems suggest that method is accurate and efficient but theoretical order of the convergence is reduced in computation.

KEYWORD. Variable step length, Nonstandard method, Initial value problems, Implicit method, Absolute stability, Finite difference method, Maximum absolute error, Stiff initial value problem.

AMS (MOS) Subject Classification. 65L10, 65L12.

1. Introduction

The numerical solution of initial value problems in ordinary differential equations, in general have following form

$$\frac{dy}{dx} = f(x, y(x)), \quad x \in [a, b], \quad y(a) = y_0, \quad \text{and} \quad y, f(x, y) \in R \tag{1}$$

However it is disappointing if we think about solution of these problems. Relatively few differential equations have analytical solutions. Even if $f(x, y)$ is sufficiently smooth, it is challenging problem in computational mathematics to achieve desired accurate numerical solutions [1, 2]. We have not considered any specific assumption on the source function $f(x, y)$ to ensure existence and uniqueness of the solution. So we assume that problem (1) possess unique solution in a domain of interest.

Many physical problems for example in study of decay of radioactive material or climatic change in natural sciences, economic growth, logistic support distribution

Received April 22, 2014 1061-5369 $15.00

in social sciences and growth of bacteria in study of medical sciences, are modeled mathematically in form of either ordinary differential equations or partial differential equations. In general, to find an exact analytical solution to these problems are not possible, a challenging task to obtain an approximate solutions being faced by the scientist. Thus a natural interest is to develop an appropriate numerical method, to obtain an approximate solution to this modeled problem.

In numerical solution of the differential equations a step length plays a critical role. For example small step length there are too many round off error in comparison to large step length, hence it causes a numerical instability. Thus the introducing concept of variable step length method may be useful in solving differential equation and some literary work can be found in [3, 4]. The motivation of variable step length in nonstandard method arises from work in [5, 6].

In this article, we have considered initial value problems and the solution of these initial value problems can be obtained by moving away from the given specified initial condition and approximating the derivative by discrete expressions. We propose a numerical method that is efficient and reliable for solving these initial value problems (1) and solution is obtained by moving non uniformly away from specified initial condition. For developing propose method, we have assume that the solution of the problems (1) depends differential on the initial condition only.

The present work is organized in six sections. Section 2 deals with development of the method while local truncation error estimated in section 3. The sections 4 and 5, we have discussed convergence and stability of the method. Numerical experiments on model test problem discussed in final section 6. A discussion on the performance of the method are presented as a conclusion.

2. **The Derivation of the method**

We define the nodes which are non-uniformly spaced throughout in the interval of interest $[a, b] : x_{i+1} = x_i + h_{i+1},\ i = 0, 1, 2, \ldots, N,$ where h_{i+1} is step length. So $\triangle = \{a = x_0, x_1, x_2, \ldots, x_N, x_{N+1} = b\}$ be a set a set of non-uniform nodes in interval $[a, b]$. Let us assume y_i represent an approximate value of of the theoretical solution $y(x)$ of the problem (1) at the node $x = x_i$ and f_i represent $f(x_i, y_i)$ at node $x = x_i$. Suppose we have numerically solved problem at node x_i and obtained numerical value y_i, an approximate value of $y(x_i)$. We are interested in finding an approximate value y_{i+1} of $y(x_{i+1})$. Let us assume a local assumption as in [7] that no previous truncation errors have been made i.e. $y(x_i) = y_i$ and following the ideas in [8, 9, 10], we propose an approximation to the analytical solution $y(x_{i+1})$ of the problem (1) at node $x = x_{i+1}$ as:

$$y(x_{i+1}) = y(x_i) + \frac{h_{i+1}y'(x_i)}{\Phi(x_i + h_{i+1}) + y(x_i)} \tag{2}$$

where Φ is differentiable function of x_i which is to be determined. Let us define a function $F_i(h, x, y, y')$ as:

$$F_i(h, x, y, y') \equiv (y(x_i + h_{i+1}) - y(x_i))(\Phi(x_i + h_{i+1}) + y(x_i)) - h_{i+1}y'(x_i) \tag{3}$$

Thus from (2) and (3), we have

$$F_i(h, x, y, y') = 0 \tag{4}$$

If we write $F_i(h, x, y, y')$ in Taylor series about node $x = x_i$, and from (4), we have

$$\begin{aligned} h_{i+1}y'(x_i)(\Phi(x_i) + y(x_i) - 1) + \frac{h_{i+1}^2}{2}(y''(x_i)(\Phi(x_i) + y(x_i)) \\ + 2y'(x_i)\Phi'(x_i)) + O(h^3) = 0 \end{aligned} \tag{5}$$

To determine function Φ, compare the coefficients of h_{i+1} and h_{i+1}^2 both side in (5), we have

$$\begin{aligned} \Phi(x_i) + y(x_i) - 1 = 0 \\ (\Phi(x_i) + y(x_i))y''(x_i) + 2y'(x_i)\Phi'(x_i) = 0 \end{aligned} \tag{6}$$

To determine Φ and Φ', solve the system of equations (6), we have

$$\begin{aligned} \Phi(x_i) &= 1 - y(x_i) \\ \Phi'(x_i) &= -\frac{1}{2}\frac{y''(x_i)}{y'(x_i)}, \quad y'(x_i) \neq 0 \end{aligned} \tag{7}$$

Thus from (7), we approximate $\Phi(x_i + h_{i+1})$ as:

$$\begin{aligned} \Phi(x_i + h_{i+1}) &= \Phi(x_i) + h_{i+1}\Phi'(x_i) + O(h^2) \\ &= 1 - y(x_i) - \frac{h_{i+1}}{2}\frac{y''(x_i)}{y'(x_i)} + O(h_{i+1}^2) \end{aligned} \tag{8}$$

Neglecting the terms $O(h_{i+1}^2)$ and higher in (8), substitute $\Phi(x_i + h_{i+1})$ in (2), we have

$$y(x_{i+1}) = y(x_i) + \frac{2h_{i+1}(y'(x_i))^2}{2y'(x_i) - h_{i+1}y''(x_i)} \tag{9}$$

Thus, using notations as defined above in (9) and from (1), we have our proposed nonstandard variable step length method as:

$$y_{i+1} = y_i + \frac{2h_{i+1}f_i^2}{2f_i - h_{i+1}f_i'} \tag{10}$$

where $f_i' = (\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}f)_{(x_i, y_i)}$.

Thus we have developed non uniform step length method of the form $y_{i+1} = y_i + G(h, f, f')$, where G is an increment function depends on variables h_{i+1}, f and f'. If we replace f_i' in (10) by first order difference approximation, we have

$$y_{i+1} = y_i + \frac{2h_{i+1}f_i^2}{3f_i - f_{i+1}} \tag{11}$$

If $3f_i - f_{i+1}$ becomes zero in the course of numerical integration, we may replace $\frac{f_i^2}{3f_i - f_{i+1}}$ in (11) by $\frac{3f_i + f_{i+1}}{9}$ in computation of numerical solution. Thus method (11) reduces to following expression:

$$y_{i+1} = y_i + \frac{2h_{i+1}}{9}(3f_i + f_{i+1})$$

Thus method (11) is an implicit method. Method (11) is linear if we have source function $f(x)$ otherwise it is nonlinear. We have applied Newton-Raphson and quasi linearization technique while solving nonlinear method (11) otherwise it was computed by direct method.

3. **Local Truncation Error**

The local truncation error in method (10) at the node $x = x_i$ using the exact arithmetic, is given as:

$$\begin{aligned} T_{i+1} &= y(x_i + h_{i+1}) - y(x_{i+1}) \\ &= y(x_i + h_{i+1}) - y_i - \frac{2h_{i+1} f_i^2}{2f_i - h_{i+1} f_i'} \\ &= \frac{h_{i+1}^3}{12}\left(2y_i'''(\xi) - 3\frac{(y_i'')^2}{y_i}\right) + O(h_{i+1}^4) \end{aligned} \tag{12}$$

where $\xi \in (x_i, x_{i+1})$. If we define $M = \max\left|\left(2y'''(x) - \frac{3(y''(x))^2}{y'(x)}\right)\right|$ all $a \le x \le b$, then from (12) we have

$$|T_{i+1}| \le \frac{h_{i+1}^3}{12} M + O(h_{i+1}^4) \tag{13}$$

Thus from (13), we conclude that method is at least second order accurate.

4. **Convergence Analysis**

To include the effect of the rounding errors, following the ideas in [4], we introduce a new approximation $\overline{y}_i$, which is determined by same method, except that rounding errors are allowed. Thus we have

$$\overline{y}_{i+1} = \overline{y}_i + \frac{2h_{i+1}(\overline{f}_i)^2}{2\overline{f}_i - h_{i+1}\overline{f}_i'} \tag{14}$$

where $\overline{f}_i = f(x_i, \overline{y}_i)$ and $\overline{f}_i' = \overline{y}_i''$. The rounding error R_{i+1} is the amount by which method (10) not satisfied by $\overline{y}_i$. Apply method (10) and (14) to test equation $y' = \lambda y$, where $\lambda = \frac{\partial f}{\partial y}$ at some point $x_i \in [a, b]$ and subtracting, we have

$$y(x_{i+1}) - \overline{y}_{i+1} = y(x_i) - \overline{y}_i + h_{i+1}\frac{2\lambda(y_i - \overline{y}_i)}{2 - h_{i+1}\lambda} + T_{i+1} + R_{i+1} \tag{15}$$

Let we have error $\epsilon_i = y_i - \overline{y}_i$ at node $x = x_i$. Substitute ϵ_i in (15), we have

$$\epsilon_{i+1} = \epsilon_i + \lambda h_{i+1}\epsilon_i \left(1 - \frac{\lambda h_{i+1}}{2}\right)^{-1} + T_{i+1} + R_{i+1} \tag{16}$$

$$\epsilon_{i+1} = \epsilon_i \left(1 + \lambda h_{i+1} + \frac{(\lambda h_{i+1})^2}{2} + \cdots\right) + T_{i+1} + R_{i+1}$$

$$\epsilon_{i+1} = \exp(\lambda h_{i+1})\epsilon_i + B$$

where $\exp(\lambda h_{i+1}) \approx 1 + \lambda h_{i+1} + \frac{(\lambda h_{i+1})^2}{2}$ and $B \geq T_{i+1} + R_{i+1}$. Let us introduce a difference equation, so (16) can be written as:

$$E_{i+1} = \exp(\lambda h_{i+1})E_i + B \tag{17}$$

So if $|\epsilon_0| \leq E_0$ then $|\epsilon_i| \leq E_i$. Thus we obtain

$$E_i = \exp(\lambda h_i + h_{i-1} + \cdots + h_1)E_0 + \chi(h_i)B \tag{18}$$

where $\chi(h_i) = \exp(\lambda h_i)\chi(h_{i-1}) + 1$, $\chi(h_0) = 0$, $i = 0, 1, 2, \ldots$. Also we know that $h_1 + h_2 + \cdots + h_N = b - a$, so (18) can be written as:

$$\begin{aligned} E_i = {} & \exp(\lambda(b-a))E_0 + (\exp(\lambda(b-a-h_1)) \\ & + \exp(\lambda(b-a-h_1-h_2)) + \cdots + \exp(\lambda h_i))B \end{aligned} \tag{19}$$

Substitute $E_0 = |\epsilon_0|$ in (19). Thus we have if $\lambda > 0$,

$$|\epsilon_i| \leq \exp(\lambda(b-a))\,|\epsilon_0| + \frac{i\exp(\lambda(b-a))}{\exp(\lambda h_1)}\,|T + R| \tag{20}$$

and if $\lambda < 0$,

$$|\epsilon_i| \leq \exp(\lambda(b-a))\,|\epsilon_0| + i\,|T + R| \tag{21}$$

where $R = \max|R_{i+1}|$ and $T = \max|T_{i+1}|$for all $i = 0, 1, 2, \ldots$. Let $\epsilon_0 = 0$ and $R_0 = 0$ then from (20) and (21), we have

$$|\epsilon_i| \leq \max\{i, i\exp(\lambda(b-a-h_1)\}\,|T| \tag{22}$$

Thus from (13), we have

$$|\epsilon_i| \leq \max\{i, i\exp(\lambda(b-a-h_1))\}O(h_{i+1}^2) \tag{23}$$

From (23) we have $|\epsilon_i| \to 0$ as $h_{i+1} \to 0$. Thus method (10) is convergent. If $\epsilon_0 = 0$ but $R_0 \neq 0$ then

$$|\epsilon_i| \leq \max\{i, i\exp(\lambda(b-a))\}\left|\left(\frac{R}{\exp(\lambda h_1)} + \frac{T}{\exp(\lambda h_1)}\right)\right| \tag{24}$$

Since $T = O(h_{i+1}^2)$, we see bound decrease if h_1 decrease until the contribution due to R becomes dominant, at which further decrease in h_1 will increase bound. Thus bound on error depends on h_1 i.e. first step length from specified condition.

5. **Stability Analysis**

If we solve test equation $y' = \lambda y$ by proposed method (10), we have

$$(25) \qquad \begin{aligned} y_{i+1} &= y_i + \frac{2h_{i+1}\lambda^2 y_i^2}{2\lambda y_i - h_{i+1}\lambda^2 y_i} \\ y_{i+1} &= y_i + \lambda h_{i+1}(1 - \frac{\lambda h_{i+1}}{2})^{-1} y_i \\ y_{i+1} &= (1 + \lambda h_{i+1} + \frac{(\lambda h_{i+1})^2}{2} + \cdots) y_i \\ y_{i+1} &\leq \exp(\lambda h_{i+1}) y_i \\ y_{i+1} &= E(\lambda h_{i+1}) y_i \end{aligned}$$

where $E(\lambda h_{i+1})$ is second order approximation of the $\exp(\lambda h_{i+1})$. Solving (25), we find that $-2 < \lambda h_{i+1} < 0$ for all $i = 0, 1, 2, \ldots$ Thus proposed method (10) is absolutely stable.

6. **Numerical Experiments**

In this section, we have reported the computational performance of the method (11) when applied to solve numerically several initial value problems in ODEs. We have computed maximum absolute error on non-uniform step length nodes in the interval of integration. In tables, we have shown MAU the maximum absolute error and ERR, the error in numerical solution of the problem at the end point of the interval of integration using following formulas:

$$MAY = \max_{1 \leq i \leq N} |y(x_i) - y_i|$$

and

$$ERR = |y(b) - y_{N+1}|$$

Let us define the step lengths ratio between two adjacent nodes in the interval as:

$$r_{i+1} = \frac{h_{i+1}}{h_i}, \quad i = 1, 2, 3, \ldots, N$$

So in computation, we have consider r_{i+1} as fixed and in tables it is written as r. We have used iterative Newton-Raphson method and quasi linearization technique to solve nonlinear equation/system of equations and applied Gauss Seidel method to solve linear equation/system of linear equations. All the computations in the experiment were performed on MS Window 2007 professional operating system in the GNU FORTRAN environment version -99 compiler (2.95 of gcc) running on Intel Duo core 2.20 Ghz PC. The stopping condition for iteration was either error of order 10^{-6} or number of iterations 10^3 .

Problem 1. Consider a nonlinear initial value problem [9] which, when solving consists of

$$\frac{dy}{dx} = 1 + y^2(x), \quad 0 \le x \le 1$$

with the specified initial condition $y(0) = 0$ in $[0, 1]$. The exact known analytical solution of the problem is $y(x) = \tan(x)$. In table 1, we have presented the computed MAY and ERR for different values of N and r.

Problem 2. Consider a stiff linear initial value problem [11] which, when solving consists of

$$\frac{dy}{dx} = -100y + 99\exp(2x), \quad 0 \le x \le 1$$

with the specified initial condition $y(0) = 0$ in $[0, 1]$. The exact known analytical solution of the problem is $y(x) = \frac{33}{34}(\exp(2x) - \exp(-100x))$. In table 2, we have presented the computed MAY and ERR for different values of N and r.

Problem 3. Consider a linear initial value problem [12] which, when solving consists of

$$\frac{dy}{dx} = y(x)\cos(x), \quad 0 \le x \le 1$$

with the specified initial condition $y(0) = 1$ in $[0, 1]$. The exact known analytical solution of the problem is $y(x) = \exp(\sin(x))$. In table 3, we have presented the computed MAY and ERR for different values of N and r.

Problem 4. Consider a linear initial value problem [12] which, when solving consists of

$$\frac{dy}{dx} = \sin(5x) - 0.4y(x), \quad 0 \le x \le 1$$

with the specified initial condition $y(0) = 5$ in $[0, 1]$. The exact known analytical solution of the problem is

$$y(x) = \frac{1}{629}\left(3270\exp\left(\frac{-2x}{5}\right) - 125\cos(5x) + 10\sin(5x)\right)$$

In table 4, we have presented the computed MAY and ERR for different values of N and r.

Problem 5. Consider a stiff system of nonlinear initial value problems [13] which, when solving consist of

$$\frac{dy}{dx} = -1002y(x) + 1000(z(x))^2$$
$$\frac{dz}{dx} = y(x) - z(x)(1 + z(x)), \quad 0 \le x \le 1$$

with the specified initial conditions $y(0) = 1.0$ and $z(x) = 1.0$ in $[0, 1]$. The exact known analytical solution of the problem is $y(x) = \exp(-2x)$ and $z(x) = \exp(-x)$. In tables 5-6, we have presented the computed MAY and MAZ for different values of N and r.

TABLE 1. Maximum absolute error and ERR in $y(x) = \tan(x)$ for problem 1.

N	Errors			
	$r = 0.8$		$r = 0.9$	
	MAY	ERR	MAY	ERR
16	.24703383(-1)	.24703383(-1)	.11529207(-1)	.11529207(-1)
32	.21677494(-1)	.21677494(-1)	.53838491(-2)	.53838491(-2)
64	.21597147(-1)	.21597147(-1)	.46275854(-2)	.46275854(-2)
128	.21597862(-1)	.21597862(-1)	.46033859(-2)	.46033859(-2)

TABLE 2. Maximum absolute error and ERR in $y(x) = \frac{33}{34}(\exp(2x) - \exp(-100x))$ for problem 2.

N	Errors			
	$r = 1.1$		$r = 1.9$	
	MAY	ERR	MAY	ERR
16	....	....	.47334686(-1)	.47334686(-1)
32	.25728671(-1)	.20617316(-1)	.47332413(-1)	.47332413(-1)
64	.18685706(-2)	.18685706(-2)	.47332566(-1)	.47332566(-1)
128	.18594265(-2)	.18594265(-2)	.47332399(-1)	.47332399(-1)

TABLE 3. Maximum absolute error and ERR in $y(x) = \exp(\sin(x))$ for problem 3.

N	Errors			
	$r = 0.8$		$r = 2.9$	
	MAY	ERR	MAY	ERR
16	.37977695(-2)	.37977695(-2)	.86135864(-1)	.86135864(-1)
32	.36189556(-2)	.36189556(-2)	.86135864(-1)	.86135864(-1)
64	.36139488(-2)	.36139488(-2)	.86134911(-1)	.86134911(-1)
128	.36141872(-2)	.36137104(-2)	....	....

TABLE 4. Maximum absolute error and ERR in $y(x) = \frac{1}{629}(3270\exp(\frac{-2x}{5}) - 125\cos(5x) + 10\sin(5x))$ for problem 4.

N	Errors			
	$r = 0.8$		$r = 1.9$	
	MAY	ERR	MAY	ERR
16	.47373772(-1)	.41676283(-1)	.84107161(-1)	.84107161(-1)
32	.43942451(-1)	.38583040(-1)	.81969976(-1)	.81969976(-1)
64	.43847561(-1)	.38498163(-1)	.81969500(-1)	.81969500(-1)
128	.43847084(-1)	.38497448(-1)	.81966400(-1)	.81966400(-1)

TABLE 5. Maximum absolute error and ERR in $y(x) = \exp(-2x)$ for problem 5.

N	Errors			
	$r = 0.8$		$r = 1.9$	
	MAY	ERR	MAY	ERR
8	.52329898(-2)	.23737401(-2)	.93970448(-2)	.93970448(-2)
16	.36774576(-2)	.14648288(-2)	.92419237(-2)	.92419237(-2)
32	.34802258(-2)	.13534874(-2)	.92389882(-2)	.92389882(-2)
64	.44892371(-1)	.43899119(-1)	.92389882(-2)	.92389882(-2)

TABLE 6. Maximum absolute error and ERR in $z(x) = \exp(-x)$ for problem 5.

N	Errors			
	$r = 0.8$		$r = 1.9$	
	MAZ	ERR	MAZ	ERR
8	.40714741(-2)	.31358898(-2)	.12899548(-1)	.12899548(-1)
16	.28175712(-2)	.20039976(-2)	.12681425(-1)	.12681425(-1)
32	.26341677(-2)	.18441677(-2)	.12680054(-1)	.12680054(-1)
64	.26290417(-2)	.18346310(-2)	.12680054(-1)	.12680054(-1)

7. Conclusion

In this article, formulation and a study of the variable step length nonstandard finite difference method for solving initial value problems in ordinary differential equations is presented. A comprehensive study of the proposed method shows that method is absolutely stable and converges. The performance of the present method in solving stiff and non-stiff differential equations is considered. The computational results obtained for the model problem is in good agreement to the estimated order of the accuracy of the method. This result may be even better in absence of the second order derivative of the variable in method. This fact creates some difficulties in implementation and performance of the method accurately. Our future work will deal with extension of the present method to solve higher order boundary value problems and to improve the computational performance; work in this specific direction is in progress.

REFERENCES

[1] E. Hairer, S. P. Norsett, and G. Warner, Solving Ordinary Differential Equation I : Nonstiff Problems. Springer-Verlag, Berlin, 1987.

[2] J. C. Butcher, Numerical Methods for Ordinary Differential Equations. Wiley Chichester, 2008.

[3] C. W. Gear, Numerical Initial Value Problems in Ordinary Differential Equations. Prentice-Hall,Eaglewood Cliffs NJ, 1971.

[4] M. K. Jain, Numerical Solution of Differential Equations(2/eds.). Wiley Eastern Ltd. New Delhi, 1984.

[5] H. Ramos, and R. Garcia-Rubio, Numerical solution of nonlinear singularly perturbed problem by using nonstandard algorithm on variable step size implementation. J.Math. Chem. 48,98–108, 2010.

[6] G. Y. Kulikov, Solving the order reduction phenpmenon in variable step size quasi consistent Nordesieck methods. Comput. Math. and Math. Phys. Vol.52,No.11,1547–1564, 2012.

[7] J. D. Lambert, Numerical Methods for Ordinary Differential Systems. John Wiley, England, 1991.

[8] H. Ramos, A non-standard explicit integration scheme for initial value problems. Applied Mathematics and Computation. 189, no.1,710–718, 2007.

[9] F. D. Van Niekerk, Rational one step method for initial value problem. Comput. Math. Applic. Vol.16, No.12, 1035–1039, 1988.

[10] P. K. Pandey, An Implicit Rational Method for Solution of Second Order Initial Value Problems in Ordinary differential Equation. International Journal of Computer Applications, November 57(2), 30–33, 2012.

[11] R. R. Ahmad, N. Yacoob and A. H. Mohd. Murid, Explicit method in solving stiff ordinary differential equations. Int. J. Comput. Math. 81, 1407–1415, 2004.

[12] S. Qureshi, Z. N. Menon, A. A. Shaikh and M. S. Chandio, On the construction and comparision of an explicit iterative algorith with nonstandard finite difference method. Mathematical Theory and Modeling, Vol.3, No.13, 78–87, 2013.

[13] X. Wu and J. Xia, Two low accuracy method for stiff system. Appl. Math. Comput.123, 141–153, 2001.