VARIABLE VISCOSITY CONDITION IN THE MODELING OF A SLIDER BEARING

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ABSTRACT. To reduce tear and wear of machinery lubrication is essential. Lubricants form a layer between two surfaces preventing direct contact and reduce friction between moving parts and hence reduce wear. In this short letter the lubrication of two slider bearings with parallel and nonparallel is studied. First, we show that bearings with parallel plates cannot support any load. For bearings with nonparallel plates we are interested on how constant and temperature dependent viscosity affects the properties of the bearings. Also, a critical temperature for which the bearings would fail due to excess in temperature is found for both latter cases. If the viscosity is constant, the critical temperature is given by an explicit formula, while for the non-constant viscosity the critical temperature can be always found from a closed form formula involving Weber functions.

Keywords: Lubrication, Slider Bearing, Whittaker, Weber, Navier-Stokes

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1. Slider bearing with parallel plates

The incompressible Navier-Stokes equations

$$(1.1) \qquad \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} + \frac{\nabla p}{\rho} = \nu \Delta \vec{v} + \vec{f}$$

together with the continuity equation

$$(1.2) \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

describe the motion of a fluid with velocity field $\vec{v} = (u, w)$, pressure p, density ρ, kinematic viscosity ν, and body forces $\vec{f}$ that may arise from friction [2]. Let the horizontal x and vertical z velocity components of the fluid be $u(x, z, t)$ and $w(x, z, t)$, and take incompressible fluid $\rho = const$ which gives $\nabla \cdot \vec{v} = 0$, respectively. Then

from (1.1) and (1.2) with $\vec{f} = \vec{0}$, we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

(1.3)
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

(1.4)
$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

We impose the following boundary conditions

$$(u, w) = (0, 0) \text{ at } z = 0 \text{ and } (u, w) = (u_p, 0) \text{ at } z = l$$

where u_p is the horizontal velocity of the top plate, l the separation distance between the plates along the z-axis, L is the length of the plates, and the bottom plate is fixed, see Fig.1.

Top plate   Velocity $u_p \rightarrow$
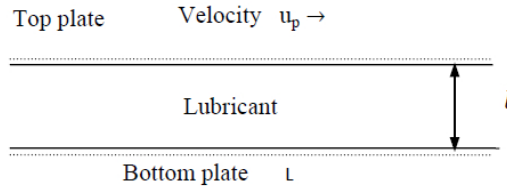
Lubricant l

Bottom plate   L

FIGURE 1. Bearing with parallel plates

Let a typical set of parameters for the bearing plate be $L = 5 \cdot 10^{-2}\,m$, $u_p = 1\,m/sec$, $\rho = 10^3\,kg/m^3$, $\mu = 10^{-4}\,m^2/sec$. To non- dimensionalize equations (1.3)-(1.4) we use new scaled parameters

$$x = \bar{x}L, \; z = \bar{z}l, \; u = \bar{u}u_p, \; w = \bar{w}\epsilon u_p, \; t = \bar{t}L/u_p, \; p = \bar{p}P,$$

where P is the undecided scaling factor for pressure and will be determined later. By eliminating the terms which have small coefficients as compared to $1/\epsilon$, from (1.3) and (1.4) we choose P such that

(1.5)
$$P = \frac{\mu u_p L}{l^2},$$

with $\mu = \rho\nu$.

Then using (1.5) we obtain

$$(1.6) \qquad \frac{\partial \overline{p}}{\partial \overline{x}} = \frac{\partial^2 \overline{u}}{\partial \overline{z}^2}$$

$$(1.7) \qquad \frac{\partial \overline{p}}{\partial \overline{z}} = 0$$

and from (1.4) we also have

$$(1.8) \qquad \frac{\partial \overline{u}}{\partial \overline{x}} + \frac{\partial \overline{w}}{\partial \overline{z}} = 0.$$

The new boundary conditions for $\overline{u}$ and $\overline{w}$ are

$$\overline{u} = 0 \text{ at } \overline{z} = 0 \text{ and } \overline{u} = 1 \text{ at } \overline{z} = 1$$

$$\overline{w} = 0 \text{ at } \overline{z} = 0 \text{ and } \overline{w} = 0 \text{ at } \overline{z} = 1$$

One integration of (1.7) gives

$$\overline{p} = \phi(\overline{x})$$

and from (1.6) we obtain $\overline{u}_\zeta$ and $\overline{u}$ by two successive integrations

$$(1.9) \qquad \frac{\partial \overline{u}}{\partial \overline{z}} = \frac{\partial \phi}{\partial \overline{x}} \overline{z} + c_1,$$

and

$$(1.10) \qquad \overline{u} = \frac{\partial \phi}{\partial \overline{x}} \frac{\overline{z}^2}{2} + c_1 \overline{z} + c_2.$$

c_1 and c_2 are constants that depend on the boundary conditions $\overline{u} = 0$ at $\overline{z} = 0$ and $\overline{u} = 1$ at $\overline{z} = 1$ which give $c_2 = 0$ and $c_1 = 1 - \frac{1}{2} \frac{d\phi}{d\overline{x}}$. Using all of these the velocity in the x direction is

$$(1.11) \qquad \overline{u} = \left(\frac{\overline{z}^2 - \overline{z}}{2} \right) \frac{d\phi}{d\overline{x}} + \overline{z}.$$

In the z direction $\overline{w}$ is obtained easily from (1.8) which gives

$$(1.12) \qquad \frac{\partial \overline{w}}{\partial \overline{z}} = -\frac{\partial \overline{u}}{\partial \overline{x}}.$$

Using (1.11) in the above, by one integration we get

$$(1.13) \qquad \overline{w} = \left(\frac{\overline{z}^2}{4} - \frac{\overline{z}^3}{6} \right) \frac{d^2 \phi}{d\overline{x}^2} + c_3.$$

As before, using the boundary conditions $\overline{w} = 0$ at $\overline{z} = 0$ and $\overline{z} = 1$, we have $c_3 = 0$, and $c_3 = -\frac{1}{12} \frac{d^2 \phi}{d\overline{x}^2}$, which gives $\phi(\overline{x}) = c_4 \overline{x} + c_5$. But since $\phi = 0$ then $\overline{p} = 0$ and the velocity field is $\vec{v} = (z, 0)$. Since the pressure is zero, the bearing with parallel plates cannot support any load, see [3, 4]. Therefore, we will consider next the case of nonparallel plates.

2. Slider bearing with nonparallel plates

For the nonparallel plates, we have that bottom plate is flat and we assume that the top plate is a linear function $z = h(x)$, see Fig. 2.
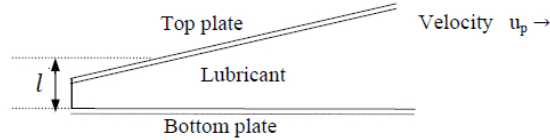

FIGURE 2. Bearing with nonparallel plates

Scaling the equations (1.3)-(1.4) as before together with

$$h(x) = l\bar{h}(x)$$

gives the same equations (1.6)- (1.8) but subject to new boundary conditions

$$\bar{u} = 0 \text{ at } \bar{z} = 0 \text{ and } \bar{u} = 1 \text{ at } \bar{z} = \bar{h}(\bar{x})$$

From (1.10) $c_2 = 0$ and

$$c_1 = \frac{1}{\bar{h}(\bar{x})} - \frac{1}{2}\frac{d\phi}{d\bar{x}}\bar{h}(\bar{x}).$$

Thus

$$(2.1) \qquad \bar{u} = \frac{d\phi}{d\bar{x}}\frac{\bar{z}^2 - \bar{z}\bar{h}}{2} + \frac{\bar{z}}{\bar{h}}$$

From (1.8) then

$$(2.2) \qquad \frac{\partial \bar{w}}{\partial \bar{z}} = -\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\phi''(\bar{x}) + \frac{\bar{z}}{2}\frac{d\bar{h}}{d\bar{x}}\phi'(\bar{x}) + \frac{\bar{z}}{\bar{h}^2}\frac{d\bar{h}}{d\bar{x}}$$

Integrating we obtain the velocity in the z direction

$$(2.3) \qquad \bar{w} = -\left(\frac{\bar{z}^3}{6} - \frac{\bar{z}^2\bar{h}}{4}\right)\phi''(\bar{x}) + \frac{\bar{z}^2}{4}\frac{d\bar{h}}{d\bar{x}}\phi'(\bar{x}) + \frac{\bar{z}^2}{2\bar{h}^2}\frac{d\bar{h}}{d\bar{x}} + c_3$$

By imposing the boundary conditions

$$\bar{w} = 0 \text{ at } \bar{z} = 0$$

$$\bar{w} = 0 \text{ at } \bar{z} = \bar{h}(\bar{x})$$

we have $c_3 = 0$ and

(2.4)
$$\frac{\bar{h}^3(\bar{x})}{12}\phi''(\bar{x}) + \frac{\bar{h}^2(\bar{x})}{4}\frac{d\bar{h}(\bar{x})}{d\bar{x}}\phi'(\bar{x}) + \frac{1}{2}\frac{d\bar{h}(\bar{x})}{d\bar{x}} = 0$$

which implies

(2.5)
$$\frac{d}{d\bar{x}}\left[\frac{\bar{h}^3(\bar{x})}{12}\phi'(x)\right] + \frac{1}{2}\frac{d\bar{h}}{d\bar{x}} = 0.$$

Assuming a linear profile [3, 4] $\bar{h}(\bar{x}) = k_1\bar{x} + k_2$, and after simplification

(2.6)
$$\frac{(k_1\bar{x} + k_2)^3}{12}\phi'(\bar{x}) + \frac{1}{2}(k_1\bar{x} + k_2) = r_2$$

then

(2.7)
$$\phi'(\bar{x}) = \frac{-6}{(k_1\bar{x} + k_2)^2} + \frac{r_1}{(k_1\bar{x} + k_2)^3}.$$

One integration gives

(2.8)
$$\phi(\bar{x}) = \frac{6k_1}{k_1\bar{x} + k_2} + \frac{k_1 r_1}{(k_1\bar{x} + k_2)^2} + r_2$$

The constants r_1 and r_2 are obtained by using boundary conditions $\phi = 0$ at $\bar{x} = 0$ and $\bar{x} = 1$. Hence,

(2.9)
$$\begin{cases} r_1 + r_2 = -6 \\ r_1 + 4r_2 = -12 \end{cases}$$

The linear system (2.9) has solution $r_1 = -4, r_2 = -2$. Therefore the pressure

$$\bar{p} = \phi(\bar{x}) = \frac{2\bar{x}(1 - \bar{x})}{(1 + \bar{x})^2}$$

and is positive for $\bar{x} \in (0, 1)$. Hence, the pressure is developed inside the fluid and the bearing supports a load given by

(2.10)
$$\text{load} = \int_0^1 \frac{2\bar{x}(1 - \bar{x})}{(1 + \bar{x})^2}d\bar{x} = 6\ln(2) - 4$$

Substituting the pressure ϕ in (2.1) and (2.3) one can obtain an analytic expression for the velocity field $\vec{v}$ with components

$$u = \frac{\bar{z}}{(\bar{x} + 1)^3}[\bar{x}(4\bar{x} - 3\bar{z} + 4) + \bar{z}]$$

(2.11)
$$w = \frac{2\bar{z}^2}{(\bar{x} + 1)^4}(\bar{x} - 1)(\bar{x} - \bar{z} + 1).$$

The velocity field $\vec{v}$ for the normalized scales, along with the streamlines are depicted in Fig.3.
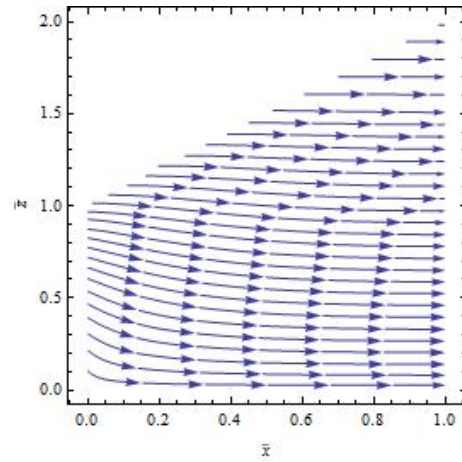
FIGURE 3. Velocity field $\vec{v}$

2.1. **Constant viscosity.** In the following subsection we wish to find a critical temperature T_c before the lubricant catches fire. Suppose that the plates are kept at a constant temperature T_1 and lubricant catches fire at critical temperature T_c. We rescale the temperature according to

$$(2.12) \qquad T = T_1 + \bar{\theta}(T_c - T_1)$$

where $\bar{\theta}$ is the non-dimensional variable $\in [0,1]$ and represents a critical parameter that will be determined.

The energy equation is

$$\rho c_p \left(\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} \right) =$$

$$(2.13) \qquad = \mu \left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)^2 + 2 \left(\frac{\partial u}{\partial x} \right)^2 + 2 \left(\frac{\partial w}{\partial z} \right)^2 \right] + k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

where c_p is the specific heat of the viscous fluid and k is its thermal conductivity.

The boundary conditions for $\bar{\theta}$ are

$$\bar{\theta} = 0 \text{ at } \bar{z} = 0 \text{ and } \bar{z} = \bar{h}(\bar{x})$$

As before, by neglecting the terms that are small as compared to $1/\epsilon$, it yields to the the new energy equation

$$(2.14) \qquad \mu \frac{u_p^2}{l^2} \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 = -\frac{k}{l^2} \frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2} (T_c - T_1)$$

from which

$$(2.15) \qquad \left(\frac{\partial \bar{u}}{\partial \bar{z}} \right)^2 = -B \frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}$$

where

$$(2.16) \qquad B = \frac{k}{\mu u_p^2} (T_c - T_1).$$

Substituting $\bar{u}$, from (2.1) then

(2.17)
$$\left[\frac{\partial}{\partial \bar{z}}\left(\frac{\bar{z}^2 - \bar{z}\bar{h}}{2}\frac{d\phi}{d\bar{x}} + \frac{\bar{z}}{\bar{h}}\right)\right]^2 = -B\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}$$

Integrating once and using boundary conditions for $\bar{\theta}$ we get

(2.18)
$$-B\bar{\theta} = \phi'(\bar{x})^2\frac{\bar{z}^4 - \bar{h}^2\bar{z}}{12} + \left(\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\phi'(\bar{x})\right)^2\frac{\bar{z}^2 - \bar{h}\bar{z}}{2} + \phi'(\bar{x})\left(\frac{1}{\bar{h}} - \frac{\bar{h}}{2}\phi'(\bar{x})\right)\frac{\bar{z}^3 - \bar{h}^2\bar{z}}{3}$$

Hence, for the linear profile $h(\bar{x}) = \bar{x} + 1$, and using (2.7) the critical parameter becomes

(2.19)
$$\bar{\theta} = \frac{z}{3B(x+1)^6}[24x^2(x+1)^2z - (3x-1)^2z^3 - 8x(x+1)(3x-1)z^2 + (x+1)^2(x^2-14x+1)]$$

from which one can find the critical temperature T_c of the lubricant before catches fire via (2.16).

2.2. **Variable viscosity.** Finally, we will consider the lubricant in which the viscosity decreases with the temperature. So (1.3) and (1.4) are

(2.20)
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial x}\right)$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = \frac{\partial}{\partial x}\left(\nu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial w}{\partial x}\right)$$

Using dimensionless viscosity $\nu = \nu_0\bar{\nu}$ in

(2.21)
$$\frac{\partial \bar{p}}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}\left(\nu\frac{\partial \bar{u}}{\partial \bar{z}}\right)$$

we obtain

(2.22)
$$\frac{\partial \bar{p}}{\partial \bar{z}} = \nu_0\frac{\partial}{\partial \bar{z}}\left(\bar{\nu}\frac{\partial \bar{u}}{\partial \bar{z}}\right),$$

and the energy equation reduces to

(2.23)
$$\bar{\nu}\left(\frac{\partial \bar{u}}{\partial \bar{z}}\right)^2 = -B\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2}$$

with B given by (2.16).

As the viscosity changes with temperature we will assume $\bar{\nu} = \alpha/\bar{\theta}$. From (2.21) as $p = \phi(\bar{x})$ we have

(2.24)
$$\frac{\partial \bar{u}}{\partial \bar{z}} = \frac{\phi'(\bar{x})\bar{z} + c_1(\bar{x})}{\bar{\nu}}$$

or in terms of $\bar{\theta}$

(2.25)
$$\frac{\partial^2 \bar{\theta}}{\partial \bar{z}^2} = -\frac{\bar{\theta}}{\alpha B}\left(\phi'(\bar{x})\bar{z} + c_1(\bar{x})\right)^2$$

We recognize (2.25) as a parabolic differential equation which will be solved using Weber functions. To see that, let us the the transformation

$$(2.26) \qquad \zeta = \frac{\sqrt{2}\phi(\bar{x})}{\sqrt[4]{\alpha\beta}}\left(\bar{z} + \frac{c_1(\bar{x})}{\phi'(\bar{x})}\right)$$

which leads to the special case of Weber's equation [1]

$$(2.27) \qquad \bar{\theta}_{\zeta\zeta} + \left(n + \frac{1}{2} - \frac{1}{4}\zeta^2\right)\bar{\theta} = 0$$

provided that $n = -\frac{1}{2}$. Since we have two independent solutions

$$\bar{\theta}_1 = e^{-\frac{\zeta^2}{4}}\,_1F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{\zeta^2}{2}\right)$$

$$(2.28) \qquad \bar{\theta}_2 = \zeta e^{-\frac{\zeta^2}{4}}\,_1F_1\left(\frac{3}{4}, \frac{3}{2}; \frac{\zeta^2}{2}\right)$$

where $_1F_1(a, b; \zeta) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{\zeta^n}{n!}$ is the confluent hypergeometric function that satisfies

$$(2.29) \qquad \zeta y'' + (b - \zeta)y' - ay = 0$$

one can construct the auxiliary functions [5] from (2.29)

$$Y_1 = \frac{\sqrt{\pi}}{\Gamma(1/4)}\bar{\theta}_1$$

$$(2.30) \qquad Y_2 = \frac{2\sqrt{\pi}}{\Gamma(1/4)}\bar{\theta}_2$$

which in turn yield to the Whittaker functions

$$U(\zeta) = \frac{\sqrt{2}}{2}(Y_1 - Y_2)$$

$$(2.31) \qquad V(\zeta) = \frac{1}{\Gamma(1/2)}\frac{\sqrt{2}}{2}(Y_1 + Y_2)$$

Using all of the above, the solution to (2.25) may be written as

$$(2.32) \qquad \bar{\theta}(\zeta) = \frac{\sqrt{\pi}e^{-\frac{\zeta^2}{4}}}{\sqrt[4]{2}\Gamma(3/4)}\left(\,_1F_1\left(\frac{1}{4}, \frac{1}{2}; \frac{\zeta^2}{2}\right) - \sqrt{2}\zeta\,_1F_1\left(\frac{3}{4}, \frac{3}{2}; \frac{\zeta^2}{2}\right)\right)$$

which upon using the substitution again to go back to the $\bar{z}$ variable using (2.26), together with the boundary conditions for $\bar{\theta}$ that determine $c_1(\bar{x})$ would yield the critical temperature before the bearing will fail due to excess in heath.

3. **Conclusion**

In this short letter we studied two models of slider bearings with nonparallel plates and two types of viscous fluids between the plates. One of the plates is fixed and other is moving horizontally with a constant velocity. The work obtained due to the motion of the top plate on lubricant makes the temperature of the viscous layer change, which yields in the properties of the viscous fluid to be affected. We found the conditions under which the safe operation of the bearing is ensured. That is, we found two critical temperatures before the lubricant fails. The first temperature was based on the simpler case of constant viscosity, while the second case was found from the temperature dependent viscosity. Closed form solutions in terms of Weber functions were found for the non-constant case. These findings can be extended to slider bearings with different geometries $h(\bar{x})$, and viscosities that depend on temperature differently from the discussion here.

ACKNOWLEDGMENTS

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