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# ON A TWO STAGE QUEUEING-INVENTORY SYSTEM WITH REJECTION OF CUSTOMERS

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**ABSTRACT.** In this paper we analyze a two-stage queueing-inventory system, wherein the first stage is an infinite capacity, queue. Customers arrive in a Poisson fashion; service rendered in batches of maximum size $K$. The second stage has a finite capacity buffer. Each stage is managed by one single server. If the number of customers in the first queue is less than $K$ at a service completion epoch, still service is provided to as many as is available. The service time of customers is assumed to follow exponential distribution. On service completion at the first stage, only a fraction of customers are eligible to move to the second stage. The customers who move to the second stage are those who have successfully completed their service in the first stage and others leave the system. Let $p$ be the probability that a customer completes service successfully in the first stage. Customers arrive to the in batches of random size $r$, $1 \leq r \leq K$, on completion of service from stage 1. They are served one at a time according to an exponential distribution. Successful customers join second stage only if at least one item is in the inventory. At the end of his service, a customer is provided exactly one item from the inventory with probability $p'$ or with complementary probability $1 - p'$, he leaves without getting the item. The maximum inventory is limited to S. Inventory is replenished according to $(s, S)$ policy, with lead time following exponential distribution. Further all customers are flushed out from the second stage of the system once the inventory level drops to zero. Thus the second stage of the tandem queue is always stable. The system performance measures like the expected time by which the inventory vanishes, the expected number of customers in the system are computed. We computed the optimal values of $p$ and $p'$. We also observe that a suitable cost function of the model turns out to be convex in $s$ and $S$ when one of these is varied and the other is kept fixed. Some illustrative numerical examples are presented.

**AMS (MOS) Subject Classification.** 60K25, 68M20, 90B05, 90B15.

## 1. INTRODUCTION

In many real life situations, often one can see multiple job classes, one after the other and an arriving customer undergoes each job class before leaving the system. Tandem queueing models are now-a-days very popular and the literature on such

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models is vast. Jackson ([3], [4]) pioneered the network queueing model. Jackson [4] studied a queueing process in which customers arrive at random and are served in order of arrival at each of a number of counters arranged in series. Each counter of the series is permitted to have several servers; the distribution of service time for a customer at each phase of service is supposed to be negative exponential. The output process in a queueing process is studied by Burke [2].

Serfozo [11] presents conditions under which a point process of certain jump times of a Markov process is a Poisson process, and discusses several applications for queueing systems with batch arrivals and batch services and also for network of queues. He also extends these results to functionals of non-Markovian process. Schwarz et al. [10] investigate a new class of stochastic networks that exhibit a product form steady-state distribution. The stochastic networks developed here are integrated models for networks of service stations and inventories. In this they investigate a server with attached inventory under $(r, Q)$ or $(r, S)$ policy into Jackson or Gordon-Newell networks. Replenishment lead time are non-zero and random and depend on the load of the system. While the inventory is depleted to zero the server with attached inventory does not accept new customers (lost sale regime); however, they assume that the lost sales are not lost to the system. The authors pursue three different approaches to handle routing with respect to a node during the time the inventory is empty there.

The present paper combines Serfozo [11] on the one hand in the first stage and on the other, the classical queueing-inventory model with customer flush out in the second stage when items are not available in the inventory.

Sigman and Levi [12] were the first to introduce inventory models with positive service time. In that model processing of inventory requires a positive random amount of time which leads to the formation of queue of demands. Berman and Sapna [1] study an inventory control problem of a service facility which requires one item of the inventory. They assume Poisson arrivals of demands, arbitrarily distributed service times and zero lead time. Schwarz et al. [9] discuss $M/M/1$ queueing system with inventory where the lead time is exponentially distributed. They analyzed the problem for both $(s, Q)$ and $(s, S)$ inventory policies and derived product form solution for these models by assuming that no customer joins the queue when inventory level is zero.

Saffari et al. [8] consider an $M/M/1$ queueing system with inventory under the $(s, Q)$ policy and with lost sales in which demands occur according to a Poisson process and service times are exponentially distributed. Replenishment lead times are random variables with known probability distribution and independent of on-hand inventory and number of customers in system, which ultimately leads them to a product form solution for the long run system state distribution. There had been numerous other studies on inventory models with positive service time. For further details about various inventory models with positive service time we refer to a survey paper by Krishnamoorthy et al. [5].

In all the above works, on completion of service customers will get an item from the inventory. Nevertheless, there are several situations where the customers may not be served the item on completion of service. Krishnamoorthy et al. [6] consider a queueing-inventory system, with the items given with probability $\gamma$ to a customer at his service completion epoch. They analyze such type of situations under Poisson demand, exponentially distributed service and lead time. Also the condition that no customer joins when the on-hand inventory is zero is imposed. In this paper we also consider such type of situation under exponentially distributed service time and lead time. We further impose the condition that customers are flushed-out from the system when the on-hand inventory is zero.

A motivating example of the present model in human resources management is the following. Consider the first stage as a training process to job aspiring candidates. These candidates are provided training in groups of finite size (maximum size is $K$). A fraction $p$ of candidates successfully complete the training programme and are sent to the second stage where interview for some jobs related to the training, is held. Unsuccessful candidates go out of the system forever. Jobs are created in bulk so that the maximum number of positions vacant at any given time is $S$. On completion of interview a candidate is selected for the job/accepts the job with probability $p'$ and is rejected/ candidate declines the offer of the job with probability $1-p'$. When there is no job available, candidates will not queue up and so all those who are present when a candidate takes away the job leaving no position vacant, the queue of customers vanish from the second stage.

This paper is arranged as follows. In Section 2, we describe the mathematical model under study and in Section 3, the system state distribution is computed. Some important performance measures and a cost function depending on these measures are analyzed in Section 4. Finally, in Section 4.2 we provide some numerical examples for analyzing different aspects of the system under study.

Notations used in the sequel are:
$\mathcal{L}(t)$ : level of the system, it contains $n$ customers at time $t$.
$\mathcal{N}_1(t)$ : number of customers in the first stage at time $t$.
$\mathcal{N}_2(t)$ : number of customers in the second stage at time $t$.
$\mathcal{I}(t)$ : inventory level in the system at time $t$.
$I_k$ : identity matrix of order $k$.
$\boldsymbol{e}$: $(1, 1, \ldots, 1)'$ a column vector of 1's of appropriate order.
CTMC : Continuous time Markov chain.
LIQBD: Level independent Quasi birth and death process.

## 2. MATHEMATICAL FORMULATION

The model under study is described as follows. Consider a two-stage tandem queueing model, where customers are served in batches at the first stage and served individually at the second stage, with the second stage providing an inventoried item with probability $p'$ at the end of a service. Each stage is managed by one server. In the first stage, it is assumed that the customers arrive in a Poisson fashion with rate $\lambda$. The units are served in batches according to a general bulk service rule such that the maximum batch size is $K$. If the number of customers in the queue is less than $K$, still service is provided with as many as is available. The service time of customers is assumed to follow exponential distribution with parameter $\mu_i$, $1 \leq i \leq K$, such that $\mu_K < \mu_{K-1} < \cdots < \mu_1$. Waiting room capacity of the first stage is infinite but the second stage has finite capacity $N$.

After the completion of service in the first stage, only a fraction of customers are eligible to move to the second stage; the rest are to leave the system. The customers who move to the second stage are those who have successfully completed their service in the first stage and others leave the system forever. Let $p$ be the probability that a customer completes service successfully in the first stage. Thus the number of customers eligible for selection of second service is a binomial random variable with parameter $(i, p)$, $1 \leq i \leq K$ when $i$ customers are served in a batch in first stage. Then out of the $i$ customers, exactly $r$ successfully complete service in the first stage with probability $P_i^{(r)} = \binom{i}{r} p^r (1-p)^{i-r}$, for $0 \leq r \leq i$; $1 \leq i \leq K$.

The second stage has inventory attached with it. Here customers arrive in batches of random size $r$, $0 \leq r \leq K$, from the first stage and are served one at a time according to an exponential distribution with parameter $\mu'$. The arrival of customers can be characterized by truncated compound Poisson process. Successful customers join the second stage only if at least one item is in the inventory. At the end of his service a customer is provided exactly one item from the inventory with probability $p'$ and with complementary probability $1 - p'$, he leaves the system without getting the item. Service is provided only when at least one item is available in the inventory. The maximum inventory is limited to $S$. When the on-hand inventory reaches $s \geq 0$, a replenishment order is placed such that at the time of replenishment as much items as necessary to bring the level to $S$, is added. The lead time follows exponential distribution with parameter $\beta$. After completing service in the first stage the customers go to the second stage only if there is at least one inventory; else they leave the system forever. Further all customers leave from the second stage of the system once the inventory level drops to zero. More specifically, the customers are flushed out from the system when the system reach "stock out" state. Thus the second stage of the tandem queue is always stable.

## 3. STEADY-STATE ANALYSIS

In this section, we examine the movement of customers in the tandem queue where server at the second stage has an attached inventory under $(s, S)$ policy. When the inventory level depletes to zero, all customers present in the second stage move out of the system (lost forever).

Define $\mathcal{L}(t)$ as the level of the system, then $\mathcal{L}(t) = l$, $\ell \geq 1$ of the system represents the system containing $\{n = (\ell - 1)K + n_1; n_1 = 1, 2, \ldots, K\}$ customers. That is, the level $\ell$ reaches to 0 means that, there is no customers in the first stage, the level $\ell = 1$ means the system contains $n = 1, 2, \ldots, K$ customers and the level $\ell = 2$ represent the system has $n = K + 1$ up to $n = 2K$ customers and so on, $\mathcal{I}(t)$ be the inventory level, $\mathcal{N}_1(t)$ be the number of customers in the first stage and $\mathcal{N}_2(t)$ be the number of customers in the second stage at time $t$. Then by the above assumption $\Omega = \{(\mathcal{L}(t), \mathcal{N}_1(t), \mathcal{I}(t), \mathcal{N}_2(t)), t \geq 0\}$ is a CTMC which is LIQBD, with state space is given by $E_1 = \{(0,0,0,0)\} \bigcup \{(0, 0, i, n_2), 1 \leq i \leq S, 0 \leq n_2 \leq N\} \bigcup \{(\ell, n_1, i, n_2),$ $\ell \geq 1$, $1 \leq n_1 \leq K$, $1 \leq i \leq S$, $0 \leq n_2 \leq N\} \bigcup \{(\ell, n_1, 0, 0), \ell \geq 1, 1 \leq n_1 \leq K, \}$, and has infinitesimal generator

$$\boldsymbol{Q} = \begin{bmatrix} C_{00} & C_{01} & & & & \\ C_{10} & C_{11} & C_0 & & & \\ & C_2 & C_1 & C_0 & & \\ & & C_2 & C_1 & C_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.1}$$

where $C_0, C_1, C_2$ are square matrices of order $K[S(N+1)+1]$. Dimension of the matrices $C_{00}, C_{01}, C_{10}$ are $[S(N+1)+1] \times [S(N+1)+1]$, $[S(N+1)+1] \times K[S(N+1)+1]$ and $K[S(N+1)+1] \times [S(N+1)+1]$ respectively. The sub-matrices are given by

$$C_{00} = \begin{bmatrix} -(\lambda+\beta) & & & & & & & B^0 \\ M' & D & & & & & & M^0 \\ & M & D & & & & & M^0 \\ & & \ddots & \ddots & & & & \vdots \\ & & & M & D & & & M^0 \\ & & & & M & D' & & \\ & & & & & \ddots & \ddots & \\ & & & & & & M & D' \end{bmatrix},$$

$$C_{01} = \begin{bmatrix} \lambda I_{S(N+1)+1} & 0 & \cdots & 0 \end{bmatrix}, \; C_{10} = [Y_1, \ldots, Y_K]^T,$$

$$C_{11} = \begin{bmatrix} F_1 & \lambda I_{S(N+1)+1} & & & \\ & F_2 & \lambda I_{S(N+1)+1} & & \\ & & \ddots & & \\ & & & F_{K-1} & \lambda I_{S(N+1)+1} \\ & & & & F_K \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ \lambda I_{(N+1)S} & \dots & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} F_K & \lambda I_{(N+1)S} & & & \\ & F_K & \lambda I_{(N+1)S} & & \\ & & \ddots & & \\ & & & F_K & \lambda I_{(N+1)S} \\ & & & & F_K \end{bmatrix}, C_2 = \begin{bmatrix} Y_K & & & \\ & Y_K & & \\ & & \ddots & \\ & & & Y_K \end{bmatrix}$$

with

$$B^0 = \begin{bmatrix} \beta & 0 & \cdots & 0 \end{bmatrix},\ M' = [0 \ \ p'\mu' \ \ \cdots \ \ p'\mu']^T,\ M^0 = diag(\beta, \dots, \beta),$$

$$M = \begin{bmatrix} 0 & & & \\ p'\mu' & 0 & & \\ & \ddots & \ddots & \\ & & p'\mu' & 0 \end{bmatrix}, Y_j = diag(\mu_j, U_j, \dots, U_j),$$

$$D' = \begin{bmatrix} -\lambda & & & \\ (1-p')\mu' & -(\lambda+\mu') & & \\ & & \ddots & \\ & & (1-p')\mu' & -(\lambda+\mu') \end{bmatrix},$$

$$D = \begin{bmatrix} -(\beta+\lambda) & & & \\ (1-p')\mu' & -(\beta+\lambda+\mu') & & \\ & & \ddots & \\ & & (1-p')\mu' & -(\beta+\lambda+\mu') \end{bmatrix},$$

$$U_j = \begin{bmatrix} w_j^0 & w_j^1 & \cdots & w_j^j & & & & \\ & w_j^0 & \cdots & w_j^{j-1} & w_j^j & & & \\ & & \ddots & & & & & \\ & & & w_j^0 & w_j^1 & \cdots & w_j^j & \\ & & & & w_j^0 & \cdots & w_j^{j-1} & \\ & & & & & \ddots & & \\ & & & & & & w_j^0 & \end{bmatrix},$$

where $w_j^i = P_j^{(i)}\mu_j,\ 0 \le i \le j.$

$$F_j = \begin{bmatrix} h'_j & & & & & & & & B^0 \\ M' & D_j & & & & & & & M^0 \\ & M & D_j & & & & & & M^0 \\ & & \ddots & \ddots & & & & & \vdots \\ & & & M & D_j & & & & M^0 \\ & & & & M & D'_j & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & M & D'_j \end{bmatrix},$$

$$D_j = \begin{bmatrix} h'_j & & & & & & \\ h_0 & h_j & & & & & \\ & \ddots & \ddots & & & & \\ & & h_0 & h_j & & & \\ & & & h_0 & h_j^{(j-1)} & & \\ & & & & \ddots & \ddots & \\ & & & & & h_0 & h_j^{(0)} \end{bmatrix}, \quad 1 \le j \le K,$$

where $h_0 = (1-p')\mu'$, $h'_j = -(\beta+\lambda+\mu_j)$, $h_j = -(\beta+\lambda+\mu'+\mu_j)$, $h_j^{(i)} = -(\beta+\lambda+\mu'+\sum_{l=0}^{i} P_j^{(l)}\mu_j)$, $0 \le i \le j-1$, $1 \le j \le K$.

$$D'_j = \begin{bmatrix} t'_j & & & & & & \\ h_0 & t_j & & & & & \\ & \ddots & \ddots & & & & \\ & & h_0 & t_j & & & \\ & & & h_0 & t_j^{(j-1)} & & \\ & & & & \ddots & \ddots & \\ & & & & & h_0 & t_j^{(0)} \end{bmatrix}, \quad 1 \le j \le K,$$

where $h_0 = (1-p')\mu'$, $t'_j = -(\lambda+\mu_j)$, $t_j = -(\lambda+\mu'+\mu_j)$, $t_j^{(i)} = -(\lambda+\mu'+\sum_{l=0}^{i} P_j^{(l)}\mu_j)$, $0 \le i \le j-1$, $1 \le j \le K$. Matrices $D, D', D_j, D'_j, M, M^0, U_j$ and $F_j, Y_j$ are square matrices of order $N+1, S(N+1)+1$ respectively, $1 \le j \le K$.

Next we perform the steady-state analysis of the system by first deriving the stability condition.

3.1. **STABILITY CONDITION.** Let $\boldsymbol{\phi}$ denote the steady-state probability vector of the generator $\boldsymbol{C} = C_0 + C_1 + C_2$. That is, $\boldsymbol{\phi C} = 0, \boldsymbol{\phi e} = 1$. The QBD structure of the model indicates that the queueing system is stable if and only if

$$\boldsymbol{\phi} C_0 \boldsymbol{e} < \phi C_2 \boldsymbol{e}. \tag{3.2}$$

Partitioning the vector $\boldsymbol{\phi}$ as:

$$\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_k),$$

where $\phi_i = (\phi_i(j,k); 1 \leq j \leq S, 0 \leq k \leq N)$, $0 \leq i \leq K$ and $\sum_{j=1}^{S} \sum_{k=0}^{N} \phi_i(j,k) = \frac{1}{K}$, $0 \leq i \leq K$. Using the structure of $C_0$ and $C_2$, relation (3.2) reduces to

$$\lambda < K\mu_K\{1 + S[(N+1-K) + K(1-P_K^{(K)}) - \sum_{i=1}^{K-1} iP_K^{(i)}]\}. \tag{3.3}$$

3.2. **COMPUTATION OF STEADY-STATE PROBABILITY VECTOR.** We find the steady-state probability vector of the Markov Chain $\Omega$. Let $\boldsymbol{\zeta}$ be the steady-state probability vector of the generator $\boldsymbol{Q}$ given in (3.1). That is,

$$\boldsymbol{\zeta}\boldsymbol{Q} = 0, \quad \boldsymbol{\zeta}\boldsymbol{e} = 1. \tag{3.4}$$

Partitioning $\boldsymbol{\zeta}$ as:

$$\boldsymbol{\zeta} = (\boldsymbol{\zeta}_0, \boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \ldots),$$

where $\boldsymbol{\zeta}_\ell = (\zeta_\ell(n_1,0,0), \zeta_\ell(n_1,i,n_2); 1 \leq n_1 \leq K, 1 \leq i \leq S, 1 \leq n_2 \leq K)$; $\ell \geq 1$,

$$\boldsymbol{\zeta}_0 = (\zeta_0(0,0,0), \zeta_0(0,i,n_2); 1 \leq i \leq S, 0 \leq n_2 \leq N).$$

The subvectors satisfy the equations

$$\begin{aligned}
&\boldsymbol{\zeta}_0 C_{00} + \boldsymbol{\zeta}_1 C_{10} = 0,\\
&\boldsymbol{\zeta}_0 C_{01} + \boldsymbol{\zeta}_1 C_{11} + \zeta_2 C_2 = 0,\\
&\boldsymbol{\zeta}_{i-1} C_0 + \boldsymbol{\zeta}_i C_1 + \boldsymbol{\zeta}_{i+1} C_2 = 0, \quad i \geq 2.
\end{aligned}$$

From the matrix-geometric method, we have

$$\boldsymbol{\zeta}_{i+1} = \boldsymbol{\zeta}_i \Re, \quad i \geq 1 \tag{3.5}$$

where $\Re$, the rate matrix, is the minimal non-negative solution of the matrix quadratic equation

$$\Re^2 C_2 + \Re C_1 + C_0 = 0, \tag{3.6}$$

and the vector $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}_1$ are the unique solution of the boundary equations and the normalizing condition (3.4):

$$\begin{aligned}
&\boldsymbol{\zeta}_0 C_{00} + \boldsymbol{\zeta}_1 C_{10} = 0,\\
&\boldsymbol{\zeta}_0 C_{01} + \boldsymbol{\zeta}_1 [C_{11} + \Re C_2] = 0,\\
&\boldsymbol{\zeta}_0 \mathbf{e} + \boldsymbol{\zeta}_1 (I - \Re)^{-1} \boldsymbol{e} = 1.
\end{aligned}$$

We have, $\boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_0 V_1(0)$ where $V_1(0) = -C_{01}[C_{11} + \Re C_2]^{-1}$. Using the normalizing condition $\boldsymbol{\zeta}\boldsymbol{e} = 1$, we get

$$\boldsymbol{\zeta}_0 [I + V_1(0)(I - \Re)^{-1}] \boldsymbol{e} = 1. \tag{3.7}$$

We now turn to some important performances characteristics of the system and their numerical illustration.

## 4. PERFORMANCE MEASURES

1. The expected number of customers in the system is given by

$$E_C = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{K}[(\ell-1)K+n_1]\sum_{i=1}^{S}\sum_{n_2=1}^{N} n_2 \quad \zeta_\ell(n_1,i,n_2).$$

2. The expected inventory level is given by

$$E_I = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{K}\sum_{i=1}^{S}\sum_{n_2=0}^{N} i\ \zeta_\ell(n_1,i,n_2) + \sum_{i=1}^{S}\sum_{n_2=0}^{N} i\ \zeta_0(0,i,n_2).$$

3. The expected loss rate of customers when the inventory level becomes zero is given by

$$E_{L1} = \sum_{m=1}^{K}\sum_{j=m}^{K} P_j^{(m)}\mu_j \left\{\sum_{\ell=1}^{\infty}\sum_{n_1=1}^{K} \zeta_\ell(n_1,0,0) + \zeta_0(0,0,0)\right\}.$$

4. The expected loss rate of customers due to the finite capacity of the second stage is given by

$$E_{L2} = \sum_{m=1}^{K}\sum_{j=m}^{K} P_j^{(m)}\mu_j \left\{\sum_{i=1}^{S}[\zeta_0(0,i,N) + \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{K} \zeta_\ell(n_1,i,N)]\right\}.$$

5. The expected reorder rate is given by

$$E_R = p'\mu'\left[\sum_{\ell=1}^{\infty}\sum_{n_1=1}^{K}\sum_{n_2=1}^{N} \zeta_\ell(n_1,s+1,n_2) + \sum_{n_2=1}^{N} \zeta_0(0,s+1,n_2)\right].$$

4.1. **EXPECTED TIME TO EMPTINESS OF INVENTORY.** Now we compute the expected time by which the inventory vanishes. Consider the Markov Chain $\{(\mathcal{I}(t),\mathcal{N}_2(t)), t \geq 0\}$ where $\mathcal{I}(t)$ denotes the inventory level and $\mathcal{N}_2(t)$ denotes the number of customers in the second stage at time t. The maximum inventory level is $S$. $\mathcal{I}(t)$ gradually reduces to 1 from $S$ if each customer receives an inventory on completion of his service. The state space of the system is given by $E_2 = \{S, S-1,\dots,1\}\times\{0,1,2,\dots,N\}\bigcup\{0\}$. The absorbing state of the Markov chain is $\{0\}$ which denotes the customers are flushed out from the second stage when inventory level is drops to zero. Thus the generator matrix $\tilde{\boldsymbol{Q}}$ of the Markov chain is of the form

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} \Delta' & \Delta'^0 \\ \mathbf{0} & 0 \end{bmatrix}, \tag{4.1}$$

where $\Delta'^0$ is an $S(N+1)+1 \times 1$ matrix given by

$$\Delta'^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ M' \end{bmatrix}, \Delta' = \begin{bmatrix} M_1' & M & & & & & \\ & \ddots & \ddots & & & & \\ & & M_1' & M & & & \\ M^0 & & & M_1 & M & & \\ \vdots & & & & \ddots & \ddots & \\ M^0 & & & & & M_1 & M \\ M^0 & & & & & & M_1 \end{bmatrix},$$

where

$$M_1' = \begin{bmatrix} t_K & g^{(1)} & \dots & g^{(K)} & & & & & \\ h_0 & t_K' & g^{(1)} & \cdots & g^{(K)} & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & h_0 & t_K' & g^{(1)} & \cdots & \cdots & g^{(K)} \\ & & & h_0 & t_{K-1}' & g^{(1)} & \cdots & g^{(K-1)} \\ & & & & \ddots & \ddots & & \vdots \\ & & & & & h_0 & t_1' & g^{(1)} \\ & & & & & & h_0 & t_0' \end{bmatrix}$$

with $h_0 = (1-p')\mu'$, $t_K = -(g^{(1)}+g^{(2)}+\cdots+g^{(K)})$, $t_j' = -(\mu'+g^{(1)}+g^{(2)}+\cdots+g^{(j)})$; $1 \le j \le$K, $t_0' = -\mu'$ and

$$g^{(i)} = \sum_{j=i}^{K} P_j^{(i)}\mu_j, \quad 1 \le i \le K;$$

$$M_1 = \begin{bmatrix} h_K & g^{(1)} & \dots & g^{(K)} & & & & & \\ h_0 & h_K' & g^{(1)} & \cdots & g^{(K)} & & & & \\ & \ddots & \ddots & \ddots & & & & & \\ & & h_0 & h_K' & g^{(1)} & \cdots & \cdots & g^{(K)} \\ & & & h_0 & h_{K-1}' & g^{(1)} & \cdots & g^{(K-1)} \\ & & & & \ddots & \ddots & & \vdots \\ & & & & & h_0 & h_1' & g^{(1)} \\ & & & & & & h_0 & h_0' \end{bmatrix},$$

where $h_0 = (1-p')\mu'$, $h_K = -(\beta + g^{(1)} + g^{(2)} + \cdots + g^{(K)})$, $h_j' = -(\beta + \mu' + g^{(1)} + g^{(2)} + \cdots + g^{(j)})$; $1 \le j \le K$, $h_0' = -(\beta + \mu')$. Matrices $M, M^0$ and $M'$ are as given in Section 3.

Thus, we have to get the expected time by which the inventory vanishes, it is denoted by: $E_T = -\gamma \Delta'^{-1}\boldsymbol{e}$, where $\gamma = (1, 0, 0, \ldots 0)$ is a row vector of order $S(N+1)+1$.

4.2. **NUMERICAL ILLUSTRATIONS.** Now we proceed to provide numerical illustrations of the system performance with variation in values of underlying parameters.

**Effect of the reorder level $s$ on various performance measures.** From Table-1, we observe that an increase in the reorder level s makes an increase in measures like expected number of customers, expected inventory level, expected reorder rate and expected time by which the inventory vanishes, whereas the expected loss rates decrease.

TABLE 1. Effect of the reorder level $s$ on various performance measures: Take $K = 3,\ N = 8,\ \beta = 1,\ \mu_1 = 9,\ \mu_2 = 8,\ \mu_3 = 7,\ \lambda = 3,\ \mu' = 6,$ $p' = 0.5,\ p = 0.5,\ S = 8$

| s | $E_C$ | $E_I$ | $E_{L1}$ | $E_{L2}$ | $E_R$ | $E_T$ |
|---|---|---|---|---|---|---|
| 1 | 0.3575 | 4.3263 | 1.5998 | 0.1201 | 0.1052 | 0.4650 |
| 2 | 0.3582 | 4.7026 | 1.0483 | 0.1176 | 0.1080 | 0.4658 |
| 3 | 0.3590 | 5.0786 | 0.7732 | 0.1147 | 0.1162 | 0.4665 |
| 4 | 0.3597 | 5.4529 | 0.6199 | 0.1114 | 0.1314 | 0.4671 |
| 5 | 0.3604 | 5.8282 | 0.5216 | 0.1074 | 0.1572 | 0.4675 |
| 6 | 0.3613 | 6.2075 | 0.4462 | 0.1017 | 0.2011 | 0.4679 |

**Effect of the maximum inventory level $S$ on various performance measures.** Table-2 shows that the behavior of the performance measures with increase in value of $S$ is similar to that in $s$, except that the trend is reversed for expected number of customers, expected reorder rate and expected loss rate due to finite capacity.

TABLE 2. Effect of the maximum inventory level $S$ on various performance measures: Take $K = 3,\ N = 8,\ \beta = 1,\ \mu_1 = 9,\ \mu_2 = 8,\ \mu_3 = 7,$ $\lambda = 3,\ \mu' = 6,\ p' = 0.5,\ p = 0.5,\ s = 2$

| S | $E_C$ | $E_I$ | $E_{L1}$ | $E_{L2}$ | $E_R$ | $E_T$ |
|---|---|---|---|---|---|---|
| 3 | 0.3619 | 2.0096 | 2.3626 | 0.0883 | 0.2871 | 0.3048 |
| 4 | 0.3606 | 2.5599 | 1.8661 | 0.1005 | 0.2133 | 0.3338 |
| 5 | 0.3597 | 3.1005 | 1.5531 | 0.1077 | 0.1705 | 0.3599 |
| 6 | 0.3591 | 3.6367 | 1.3349 | 0.1123 | 0.1426 | 0.3845 |
| 7 | 0.3586 | 4.1705 | 1.1733 | 0.1154 | 0.1227 | 0.4112 |
| 8 | 0.3582 | 4.7026 | 1.0483 | 0.1176 | 0.1080 | 0.4658 |

**Effect of replenishment rate $\beta$ on various performance measures.** Table-3 shows that an increase in $\beta$ makes the expected time by which the inventory vanishes increase because, as $\beta$ increases more orders will be placed resulting in longer duration with positive inventory.

TABLE 3. Effect of replenishment rate $\beta$ on various performance measures: Fix $K = 3$, $N = 8$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\lambda = 3$, $\mu' = 6$, $p' = 0.5$, $p = 0.5$, $s = 2$, $S = 5$

| $\beta$ | $E_C$ | $E_I$ | $E_{L1}$ | $E_{L2}$ | $E_R$ | $E_T$ |
|---|---|---|---|---|---|---|
| 1 | 0.3597 | 3.1005 | 1.5531 | 0.1077 | 0.1705 | 0.3599 |
| 2 | 0.3583 | 3.3480 | 0.6302 | 0.1184 | 0.1916 | 0.4054 |
| 3 | 0.3577 | 3.4329 | 0.3582 | 0.1223 | 0.2012 | 0.4401 |
| 4 | 0.3573 | 3.4744 | 0.2388 | 0.1242 | 0.2069 | 0.4672 |
| 5 | 0.3570 | 3.4984 | 0.1750 | 0.1253 | 0.2109 | 0.4887 |
| 6 | 0.3567 | 3.5139 | 0.1368 | 0.1259 | 0.2138 | 0.5061 |

4.2.1. *COST ANALYSIS.* For finding an optimal value for system cost, we introduce expected total cost as

$$E_{TC} = \left[C_F + \sum_{i=0}^{s}(S-i)C_P\right] E_R + C_{HI}E_I + C_{HC}E_C + C_{L1}E_{L1} + C_{L2}E_{L2},$$

where $C_F$ is the fixed ordering cost, $C_P$ is the procurement cost per unit, $C_{HI}$ is the holding cost of inventory per unit per unit time, $C_{HC}$ is the holding cost of customer per unit per unit time and $C_{L1}, C_{L2}$ are the cost due to the loss of customers per unit per unit time. The problem of optimizing the cost for various parameter values are carried out.

We assign the following values to the parameters: $K = 3$, $N = 8$, $\lambda = 3$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$. We obtain the following two tables (Tables 4 and 5) which provide the optimal pairs (s, S) corresponding to the input parameter rates and also the corresponding minimum cost(in Dollars). Here $p'$ is varied from 0.1 to 1, each time increasing it by 0.1 unit and $p = 0.5$. The optimal pairs $(s, S)$ and the corresponding minimum cost are given in Table-4. Table-5 contains optimal pairs $(s, S)$ and the corresponding minimum costs when $p$ is varied from 0.1 to 1 and $p' = 0.5$.

4.3. **CHARACTERISTICS OF THE MEASURES OF EFFECTIVENESS.** Next we investigate the effects of variation in $\lambda$, $p$, $\mu'$, $p'$, $\beta$, $s$, $S$ and $N$ on expected total cost.

TABLE 4. Optimal $(s, S)$ pair and corresponding minimum cost

| $p'$ | Optimal $(s, S)$ pair | minimum cost (in \$) |
|---|---|---|
| 0.1 | (2,5) | 37.1792 |
| 0.2 | (2,6) | 49.5549 |
| 0.3 | (2,7) | 60.2819 |
| 0.4 | (2,8) | 69.8303 |
| 0.5 | (2,8) | 78.3641 |
| 0.6 | (1,11) | 85.3267 |
| 0.7 | (1,12) | 90.6190 |
| 0.8 | (1,12) | 94.9899 |
| 0.9 | (1,13) | 98.6380 |
| 1 | (1,13) | 101.6168 |

TABLE 5. Optimal $(s, S)$ pair and corresponding minimum cost

| $p$ | Optimal $(s, S)$ pair | minimum cost (in \$) |
|---|---|---|
| 0.1 | (1,3) | 24.0796 |
| 0.2 | (1,5) | 39.5936 |
| 0.3 | (2,6) | 53.3026 |
| 0.4 | (2,7) | 66.0595 |
| 0.5 | (2,8) | 78.3641 |
| 0.6 | (2,11) | 90.4566 |
| 0.7 | (1,12) | 100.5811 |
| 0.8 | (1,13) | 109.9994 |
| 0.9 | (1,14) | 118.6404 |
| 1 | (1,15) | 127.2814 |

**i. Effect of variation in $\lambda$:** In order to study the variation in $\lambda$ on expected total cost we allow $K = 3$, $N = 8$, $\beta = 1$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p = 0.5$, $p' = 0.5$, $S = 5$, $s = 2$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $\lambda$, the expected total costs are calculated, and are presented in Table-6.

| $\lambda$ | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|
| $E_{TC}$ | 136.4648 | 148.2945 | 158.0604 | 166.1812 | 172.9983 | 178.7613 |

TABLE 6. Effect of variation in $\lambda$

**ii. Effect of variation in $p$.** In order to study the variation in $p$ on expected total cost $K = 3$, $N = 8$, $\beta = 1$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p' = 0.5$, $\lambda = 3$, $S = 5$, $s = 2$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_L = \$20$. and for different values of $p$, the expected total costs are calculated, and are presented in Table-7.

TABLE 7. Effect of variation in $p$

| $p$ | $E_{TC}$ |
|---|---|
| 0.1 | 28.5813 |
| 0.2 | 40.3404 |
| 0.3 | 53.9133 |
| 0.4 | 68.8301 |
| 0.5 | 84.4797 |
| 0.6 | 100.2883 |
| 0.7 | 115.7896 |
| 0.8 | 130.6451 |
| 0.9 | 144.6466 |
| 1 | 153.6306 |

**iii. Effect of variation in $\mu'$.** In order to study the variation in $\mu'$ on expected total cost we assume that $K = 3$, $N = 8$, $\beta = 1$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $p' = 0.5$, $p = 0.5$, $\lambda = 3$, $S = 5$, $s = 2$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $\mu'$, the expected total costs are calculated in Table-8.

TABLE 8. Effect of variation in $\mu'$

| $\mu'$ | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|
| $E_{TC}$ | 84.4797 | 85.2146 | 85.7575 | 86.1774 | 86.5140 | 86.7919 |

**iv. Effect of variation in $p'$.** In order to study the variation in $p'$ on expected total cost $K = 3$, $N = 8$, $\beta = 1$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p = 0.5$, $\lambda = 3$, $S = 5$, $s = 2$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $p'$, the expected total costs are calculated, as they are presented in Table-9.

**v. Effect of variation in $\beta$.** In order to study the variation in $\beta$ on expected total cost take $K = 3$, $N = 8$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p' = 0.5$, $p = 0.5$, $\lambda = 3$, $S = 5$, $s = 2$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $\beta$, the expected total costs are calculated, and are presented in Table-10.

From Table-6, 7, 8, 9 and 10 we notice that the expected total cost increases with the increase in values of $\lambda, p, \mu', p'$ and $\beta$.

**vi. Effect of variation in $s$.** In order to study the variation in $s$ on expected total cost we set $K = 3$, $N = 8$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p' = 0.5$, $p = 0.5$, $\lambda = 3$, $S = 8$, $\beta = 1$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_L = \$20$. and

TABLE 9. Effect of variation in $p'$

| $p'$ | $E_{TC}$ |
|---|---|
| 0.1 | 37.1792 |
| 0.2 | 50.1529 |
| 0.3 | 62.4976 |
| 0.4 | 73.9631 |
| 0.5 | 84.4797 |
| 0.6 | 94.0553 |
| 0.7 | 102.7483 |
| 0.8 | 110.6448 |
| 0.9 | 117.8400 |
| 1 | 124.4263 |

TABLE 10. Effect of variation in $\beta$

| $\beta$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $E_{TC}$ | 84.4797 | 94.9090 | 117.2979 | 142.5994 | 169.0303 | 196.2787 |

for different values of $s$, the expected total costs are calculated, and are presented in Table-11.

From Table-11 we observe that for fixed values of other parameters, the minimum expected total cost is at $s = 2$.

TABLE 11. Effect of variation in $s$

| $s$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $E_{TC}$ | 80.3423 | **78.3641** | 83.0818 | 92.6343 | 107.5829 |

**vii. Effect of variation in $S$.** In order to study the variation in $S$ on expected total cost $K = 3$, $N = 8$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p' = 0.5$, $p = 0.5$, $\lambda = 3$, $s = 3$, $\beta = 1$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $S$, the expected total costs are calculated, as they are presented in Table-12.

From Table-12 we observe that for fixed values of other parameters, the minimum expected total cost is obtained at $S = 8$.

TABLE 12. Effect of variation in $S$

| $S$ | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|
| $E_{TC}$ | 90.1865 | 85.8585 | 83.8055 | **83.0818** | 83.2099 | 83.9193 |

**viii. Effect of variation in $N$.** In order to study the variation in $N$ on expected total cost $K = 3$, $\mu_1 = 9$, $\mu_2 = 8$, $\mu_3 = 7$, $\mu' = 6$, $p' = 0.4$, $p = 0.6$, $s = 2$, $S = 10$, $\beta = 1$, $C_F = \$100$, $C_P = \$10$, $C_{HI} = \$5$, $C_{HC} = \$0.5$, $C_{L1} = \$20$, $C_{L2} = \$2$ and for different values of $N$ and $\lambda$, the expected total costs are calculated, as they are presented in Table-13.

Table 13. Effect of variation in $N$

| $N$ \ $\lambda$ | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|
| 3 | 146.5228 | 150.9562 | 156.2438 | 162.9461 | 170.4848 | 176.0741 |
| 4 | 151.1054 | 155.3091 | 159.6428 | 164.7363 | 171.0361 | 178.1386 |
| 5 | 153.5317 | 157.7827 | 161.8761 | 166.3589 | 171.7280 | **178.1870** |
| 6 | 154.5431 | 158.7925 | 162.7681 | **166.9338** | **171.7464** | 177.5477 |
| 7 | 154.9015 | **159.0945** | **162.9755** | 166.9335 | 171.3730 | 176.6534 |
| 8 | **154.9362** | 159.0300 | 162.8127 | 166.6081 | 170.7682 | 175.6338 |
| 9 | 154.8110 | 158.7762 | 162.4495 | 166.0999 | 170.0302 | 174.5506 |
| 10 | 154.6239 | 158.4434 | 161.9972 | 165.5104 | 169.2413 | 173.4662 |

## CONCLUSION

In this paper we studied a two-stage tandem queue with attached inventory where all customers are flushed out from the second stage of the system once the inventory level drops to zero. Motivation for this model was the problem of training programme of candidates for certain positions. Several parameters on the system performance measures are investigated. We also provided the expected time to emptiness of the inventory. In a future paper we extend the present work to the case of the tandem queue with more than two stages.

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