Neural, Parallel, and Scientific Computations 23 (2015) 169-178

APPROXIMATE SOLUTIONS OF ABEL'S EQUATION USING RATIONAL INVERSION OF THE LAPLACE TRANSFORM

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ABSTRACT. In this paper, we present a new numerical approximation to the solution of the vector-valued Abel integral equation. We first convert the integral equation into an algebraic equation using Laplace transformations. Then, we use rational inversion formulas for the Laplace transform introduced by Jara et. al. [5] in 2012. We apply our method to several examples and present error estimates which illustrate a fast rate of convergence.

AMS (MOS) Subject Classification. 39A10.

1. Introduction

Integral equations evolve naturally in many applications including those in biology, physics, engineering and chemistry [2]. Some of the earliest integral equations were the Abel's integral equations ([2], [8]). These equations were derived by Neils Abel in 1823 [1] when he generalized the tautochrone problem which was first solved by Huygens in 1659 [4]. Traditionally, there are two general forms for Abel's integral equations: the first kind given by

$$f(x) = \int_0^x \frac{u_1(t)}{(x-t)^\alpha}\, dt \tag{1.1}$$

and second kind given by

$$u_2(x) = f(x) + \int_0^x \frac{u_2(t)}{(x-t)^\alpha}\, dt, \tag{1.2}$$

where $0 < \alpha < 1$, $f(x) \in C[0,1]$ and is a known function, $0 \le x, t \le 1$, and $u(x)$ is an unknown function to be determined. Throughout the years, there have been several methods for solving these integral equations using a variety of techniques including Chebyshev [10] and orthogonal polynomials [16], wavelets ([8], [17], [18]), Abomian decomposition ([11], [12], [13]), and Laplace decomposition [15] just to name a few.

In more recent years, Yang [7] and Huang et. al. [6] use Taylor expansion to get approximate solutions to Abel's integral equations of the first and second kind.

Received September 15, 2015

Our study will use rational inversion formulas for the Laplace transform, first introduced by Jara et. al. [5], to get numerical approximations for solutions to Abel's integral equations. These formulas only require the solution to be continuous and exponentially bounded and come with error estimates for sufficiently smooth functions. Therefore, the method used in this paper will not assume the existence of the Taylor expansion for the solution. Furthermore, the method works for functions with values in a Banach space. We then present several examples to show a fast rate of convergence.

2. $\mathcal{A}$- stable Padé approximants to the exponential and Laplace transform

We begin this section by taking the Laplace transform of Abel's integral equations of the first and second kind. We then provide an outline of the inversion formula inversion formula for the Laplace transform presented in [5]. For further details of the inversion formula, the reader is encouraged to see [5] or [9].

2.1. Laplace transform and convolution.

Similar to [7], in order to solve Abel's integral equations of the first and second kind, we first apply the Laplace transform to equations (1.1) and (1.2). Doing this, we get

$$\mathcal{L}[f(x)] = \mathcal{L}\left[\int_0^x \frac{u_1(t)}{(x-t)^\alpha}\,dt\right] \tag{2.1}$$

and

$$\mathcal{L}[u_2(x)] = \mathcal{L}[f(x)] + \mathcal{L}\left[\int_0^x \frac{u_2(t)}{(x-t)^\alpha}\,dt\right]. \tag{2.2}$$

By the convolution property and linearity of the Laplace transform, equations (2.1) and (2.2) become

$$\mathcal{L}[f(x)] = \mathcal{L}\left[u_1(x)\right]\mathcal{L}\left[x^{-\alpha}\right] \tag{2.3}$$

and

$$\mathcal{L}[u_2(x)] = \mathcal{L}[f(x)] + \mathcal{L}\left[u_2(x)\right]\mathcal{L}\left[x^{-\alpha}\right], \tag{2.4}$$

respectively. After we solve for $\mathcal{L}[u_i(x)],\ i=1,2$, we will use the inversion formulas for the Laplace transform presented in [5] to get a numerical approximation for the unknown function $u_i(x),\ i=1,2$.

2.2. Inversion formula for the Laplace transform.

We now give an outline for the inversion formula given in [5]. To see the fundamental structure of the procedure, we consider the shift semigroup

$$T(t)u(s) := u(t+s),$$

which can be shown to be strongly continuous (see [19] or [9]) on the space of continuous functions vanishing at infinity denoted by $(C_0(\mathbb{R}_+, X), ||\cdot||_\infty)$, where X is a Banach space and the generator A of T is given by the derivative $A = \frac{d}{ds}$. One formally thinks of T as the operator e^{tA} acting on the Banach space C_0 for each $t \geq 0$. Since the resolvent operator $R(\lambda, A)$ is known to be the Laplace transform of the semigroup T, one obtains that

$$R(\lambda, A)u = (\lambda I - A)^{-1}u = \int_0^\infty e^{-\lambda t}T(t)u\,dt,$$

for all $\lambda \in \mathbb{C}_+$ and $u \in X$. In particular, if one denotes the Laplace transform of u by $\mathcal{L}[u](\lambda) := \int_0^\infty e^{-\lambda t}u(t)dt$, then

$$R(\lambda, \frac{d}{ds})u(0) = \int_0^\infty e^{-\lambda t}T(t)u(0)dt = \int_0^\infty e^{-\lambda t}u(t)dt = \mathcal{L}[u](\lambda),$$

for $\lambda \in \mathbb{C}_+$. Furthermore, since $R^{n+1}(\lambda, A) = \frac{(-1)^n}{n!}R(\lambda, A)^{(n)}$ for all $n \in \mathbb{N}$ where $R^{(n)}$ denotes the nth-derivative of R with respect to λ, one obtains that

$$R(\lambda, \frac{d}{ds})^{n+1}u(0) = \frac{(-1)^n}{n!}\frac{d^n}{d\lambda^n}R(\lambda, \frac{d}{ds})u(0) = \frac{(-1)^n}{n!}\mathcal{L}[u]^{(n)}(\lambda). \tag{2.5}$$

In the celebrated paper of Benner and Thomée [21], the authors showed that if r is any $\mathcal{A}$-stable rational approximation to the exponential function of order $m \geq 1$ (i.e., $|r(z)| \leq 1$ and $r(z) = e^z + O(|z|^{m+1})$ as $z \to 0$) and T is the generator of a bounded strongly continuous semigroup T with generator A, then there exists $C_m > 0$ such that

$$\sup_{s\in[0,\infty)} \left\| r^n\left(\frac{t}{n}A\right)u(s) - T(t)u(s)\right\|_X \leq C_m\frac{t^{m+1}}{n^m}\|A^{m+1}u\|_\infty.$$

and $\lim_{n\to\infty} r^n\left(\frac{t}{n}A\right)u = T(t)u$ uniformly on compact intervals of $t \in [0, \infty)$ for all u.

In particular, for the shift semigroup on $C_0(\mathbb{R}_+, X)$, one obtains that

$$\left\| r^n\left(\frac{t}{n}\frac{d}{ds}\right)u(0) - T(t)u(0)\right\|_X \leq C_m\frac{t^{m+1}}{n^m}\|u^{(m+1)}\|_\infty, \tag{2.6}$$

and $\lim_{n\to\infty} r^n\left(\frac{t}{n}\frac{d}{ds}\right)u(0) = u(t)$ uniformly on compact intervals.

If one uses partial fraction decomposition on an $\mathcal{A}$-stable rational approximation to the exponential r accurate of order $m \geq 1$, then there exists constants B_0, $B_{n,i,j}$, $b_i \in \mathbb{C}$ with $Re(b_i) > 0$, and $d, \overline{r}_i \in \mathbb{N}$ (where d is the number of poles of r and $\overline{r}_i$ is the multiplicity of the pole b_i) such that

$$r^n(z) = B_0^n + \sum_{i=1}^{d}\sum_{j=1}^{n\cdot\overline{r}_i}\frac{B_{n,i,j}}{(b_i - z)^j}.$$

Upon application of the Hille-Phillips functional calculus together with (2.5), one obtains

$$r^n\left(\frac{t}{n}\frac{d}{ds}\right)u(0) = B_0^n u(0) + \sum_{i=1}^{d}\sum_{j=1}^{n\cdot\overline{r}_i}B_{n,i,j}\left(\frac{n}{j}\right)^j\frac{(-1)^j}{(j-1)!}\mathcal{L}[u]^{(j-1)}\left(\frac{b_i n}{t}\right). \tag{2.7}$$

If additionally one considers the $\mathcal{A}$-stable rational approximations to the exponential given by the subdiagonal Padé approximants, then $B_0 = 0$ and the formula given by (2.7) depends only on the Laplace transform of u and the underlying rational function r. Notice that in the general case, $u(0) = \lim_{\lambda\to\infty} \lambda\mathcal{L}[u](\lambda)$, and one can use any $\mathcal{A}$-stable rational approximation to the exponential of order $m \geq 1$ provided one first calculates these limit. However, for simplicity, we consider here rational functions given by the subdiagonal Padé approximants. Thus, the rational inversion formula for the Laplace transform is given by

$$\mathcal{L}^{-1}_{r,n}[\mathcal{L}[u]](t) := \sum_{i=1}^{d} \sum_{j=1}^{n\cdot\overline{r}_i} B_{n,i,j} \left(\frac{n}{j}\right)^j \frac{(-1)^j}{(j-1)!} \mathcal{L}[u]^{(j-1)} \left(\frac{b_i n}{t}\right). \tag{2.8}$$

Now, as in [9], one defines the error function $E_r(n,t,u) := ||\mathcal{L}^{-1}_{r,n}[\mathcal{L}[u]](t) - u(t)||_X$. It then follows from (2.6) that

$$E_r(n,t,u) \leq C_m \frac{t^{m+1}}{n^m} ||u^{(m+1)}||_\infty,$$

for sufficiently smooth functions u, and $\lim_{n\to\infty} \mathcal{L}^{-1}_{r,n}[\mathcal{L}[u]](t) = u(t)$ uniformly on compact intervals for all $u \in C_0$.

In [20], the author showed that all of the previously described results hold for a class wider than the one of strongly continuous semigroups which includes the shift semigroup on the space of bounded and continuous functions and, more generally, on the space of exponentially bounded functions denoted by

$$C_{b,\omega} := \{u : [0,\infty) \to X : \|u(t)\| \leq Me^{\omega t}\}$$

with norm $\|u\|_{\omega,\infty} := \sup_{t\geq 0} |e^{-\omega t}u(t)|$. However, in the later case, an extra $e^{\omega t}$ term appears on the right side of the error estimate (2.6). In this way, one obtains the following result:

Theorem 2.1. *Let u_1 and u_2 be the solutions of the first and second order Abel's Integral Equation* (1.1) *and* (1.2) *respectively and let r be a subdiagonal Padé approximant of order $m \geq 1$. If u_1, $u_2 \in C^{m+1}_{b,\omega}([0,\infty), X)$, then*

$$\left\| \mathcal{L}^{-1}_{r,m} \left[\frac{(\cdot)^{1-\alpha}\mathcal{L}[f](\dot{)}}{\Gamma(1-\alpha)} \right] - u_1(t) \right\|_X \leq C_m \frac{e^{\omega t} t^{m+1}}{n^m} ||u_1^{(m+1)}||_{\omega,\infty}, \tag{2.9}$$

and

$$\left\| \mathcal{L}^{-1}_{r,m} \left[\frac{(\cdot)^{1-\alpha}\mathcal{L}[f](\cdot)}{(\cdot)^{1-\alpha} - \Gamma(1-\alpha)} \right] - u_2(t) \right\|_X \leq C_m \frac{e^{\omega t} t^{m+1}}{n^m} ||u_2^{(m+1)}||_{\omega,\infty}. \tag{2.10}$$

Furthermore,

$$\lim_{n\to\infty} \mathcal{L}^{-1}_{r,m} \left[\frac{(\cdot)^{1-\alpha}\mathcal{L}[f](\cdot)}{\Gamma(1-\alpha)} \right] = u_1(t) \text{ and } \lim_{n\to\infty} \mathcal{L}^{-1}_{r,m} \left[\frac{(\cdot)^{1-\alpha}\mathcal{L}[f](\cdot)}{(\cdot)^{1-\alpha} - \Gamma(1-\alpha)} \right] = u_2(t) \tag{2.11}$$

uniformly on compact intervals for all $u \in C_{b,\omega}([0,\infty), X)$.

Notice that if $t \in [0,1]$, the error estimate for the approximation of u_i is smaller than $\frac{C_m e^{\omega}}{n^m}\|u_i^{(m+1)}\|_{\omega,\infty}$ for $\omega \geq 0$ or than $\frac{C_m}{n^m}\|u_i^{(m+1)}\|_{\omega,\infty}$ for $\omega < 0$.

3. **Numerical examples**

In this section, we provide examples which are revisited from [6] and [7]. As mentioned in Section 2, we will be using the rational functions given by the subdiagonal Padé approximants for all of the examples that follow. More specifically, we use [9/10] subdiagonal Padé approximants in Examples 1 and 2 and use [2/3] and [9/10] subdiagonal Padé approximants in Example 3. We use a program written in *Mathematica* to calculate the Padé approximants and the values in the examples.

Example 1 The following example is the same as Example 3 in [6]. Consider the Abel equation of the first kind

$$\int_0^x \frac{u(t)}{(x-t)^{1/2}}dt = e^x - 1. \tag{3.1}$$

Taking the Laplace transform of both sides of (3.1), we get from the convolution property of the Laplace Transform,

$$\mathcal{L}[u]\mathcal{L}[x^{1/2}] = \mathcal{L}[e^x - 1].$$

This implies that $\mathcal{L}[u] = \frac{\mathcal{L}[e^x-1]}{\mathcal{L}[x^{1/2}]}$. The exact solution of (3.1) can be computed to be $u(x) = \frac{e^x erf(\sqrt{x})}{\sqrt{\pi}}$. Table 1 gives the exact values for $u(x)$, the approximation using the method in [6] with $n = 3$, and the approximation using the rational inversion formula outlined in section 2. We see that although the method used in [6] is quite accurate, the rational inversion formula method presented in section 2 is more accurate for this example.

X	Exact	Huang et al. ($n = 3$)	Rational Inversion Formula
0.1	0.21529	0.21629	0.21547
0.2	0.32588	0.32727	0.326137
0.3	0.42757	0.42925	0.427876
0.4	0.52933	0.53126	0.529691
0.5	0.63503	0.63715	0.635432
0.6	0.74704	0.74933	0.747478
0.7	0.86719	0.86962	0.86766
0.8	0.99709	0.99963	0.997594
0.9	1.13830	1.14091	1.13883
1	1.29239	1.29503	1.29295
Max Abs. Err.		2.647×10^{-3}	5.632×10^{-4}

TABLE 1. Comparision of our method to Huang et al. in [6].

Example 2 The following example is the same as Example 5.3 in [7]. Consider the Abel equation of the second kind

$$u(x) = 2\sqrt{x} - \int_0^x \frac{u(t)}{(x-t)^{1/2}}dt, \;\; x \in [0,1]. \tag{3.2}$$

Taking the Laplace transform of both sides of (3.2), we get from the convolution property and linearity of the Laplace Transform,

$$\mathcal{L}[u] = \frac{\mathcal{L}[2\sqrt{x}]}{1 + \mathcal{L}[x^{-1/2}]}.$$

The exact solution of (3.2) can be computed to be $u(x) = 1 - e^{\pi x} erfc(\sqrt{\pi x})$. We then apply (2.8) to $\mathcal{L}[u]$. In [7], the author compares his result with two other methods presented in [22], the Chebyshev wavelets method and the block-pulse functions (BPFs) method. Table 2 gives the exact values for $u(x)$, the approximations from three of the methods in Yang's table 1 [7], and the approximation using the rational inversion formula outlined in section 2. We see again that the rational inversion formula method presented in section 2 is extremely accurate. Our method does out perform the wavelet method and the BPF method but is slightly less accurate than the method of Yang in [7] for this example.

x	Yang([4/4])	Wavelets method ($k = 0, M = 16$)	BPFs method ($m = 16$)	Rational Inversion Formula
0.1	4.33846×10^{-9}	1.62983×10^{-3}	1.15872×10^{-2}	5.31211×10^{-6}
0.2	4.87786×10^{-8}	2.82352×10^{-3}	1.13995×10^{-2}	7.40412×10^{-6}
0.3	1.82276×10^{-7}	1.89633×10^{-3}	9.55367×10^{-3}	9.26409×10^{-6}
0.4	4.42272×10^{-7}	1.43922×10^{-3}	1.68378×10^{-3}	1.05841×10^{-5}
0.5	8.53771×10^{-7}	1.32002×10^{-3}	7.61903×10^{-3}	1.17567×10^{-5}
0.6	1.43214×10^{-6}	1.21446×10^{-3}	1.53846×10^{-3}	1.27792×10^{-5}
0.7	2.18575×10^{-6}	9.86938×10^{-4}	3.09894×10^{-3}	1.39448×10^{-5}
0.8	3.11805×10^{-6}	2.45968×10^{-4}	2.98197×10^{-3}	1.49093×10^{-5}
0.9	4.22898×10^{-6}	2.45968×10^{-4}	7.08482×10^{-4}	1.056083×10^{-5}

TABLE 2. Comparision of absolute errors to the results in [7].

Example 3 The following example is the same as Example 5.2 in [7]. Consider the Abel equation of the first kind

$$\int_0^x \frac{u(t)}{(x-t)^{1/2}}dt = e^{-x} - 1. \tag{3.3}$$

Taking the Laplace transform of both sides of (3.3), we get from the convolution property of the Laplace Transform,

$$\mathcal{L}[u]\mathcal{L}[x^{1/2}] = \mathcal{L}[e^{-x} - 1].$$

This implies that $\mathcal{L}[u] = \frac{\mathcal{L}[e^{-x}-1]}{\mathcal{L}[x^{1/2}]}$. The exact solution of (3.3) can be computed to be $u(x) = \frac{ie^{-x}erf(\sqrt{-x})}{\sqrt{\pi}}$. We then apply (2.8) to $\mathcal{L}[u]$. Table 3 gives the exact values for

$u(x)$ and the approximation using the rational inversion formula, which we will signify as RIF, outlined in section 2 with [2/3] and [9/10] subdiagonal Padé approximants. We see again that the rational inversion formula method presented in section 2 is extremely accurate for this example.

x	Exact	RIF (Padé [2/3])	RIF (Padé [9/10])
.1	-0.188418	-0.186286	-0.188597
.2	-0.249615	-0.246571	-0.249869
.3	-0.286649	-0.282883	-0.286961
.4	-0.310643	-0.306249	-0.311003
.5	-0.326265	-0.321300	-0.326668
.6	-0.336073	-0.330573	-0.336514
.7	-0.341666	-0.335657	-0.342144
.8	-0.344130	-0.337628	-0.34641
.9	-0.344235	-0.337252	-0.344777
1	-0.342552	-0.335095	-0.343124
Max Abs. Err.		7.457×10^{-3}	5.719×10^{-4}

TABLE 3

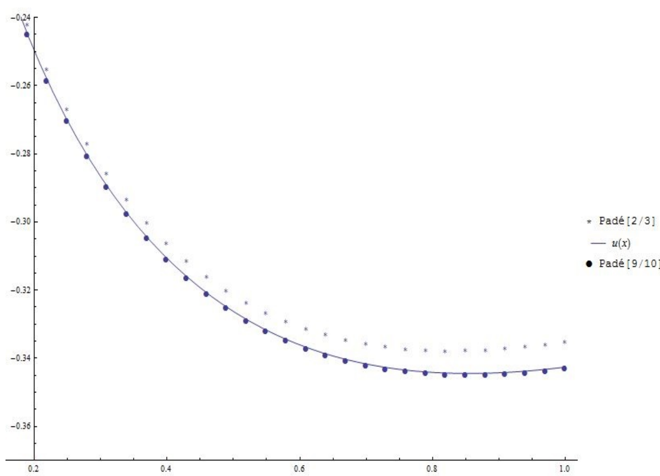

FIGURE 1. Comparison of the exact solution and the subdiagonal Padé approximants.

4. Conclusion

In this paper, we provide a new numerical method for solving Abel's integral equations of the first and second kind. We show that using the ration inversion formula outlined in Section 2 that we were able to get very accurate approximations to the solutions to the integral equations in the examples. We use subdiagonal Padé approximants as the rational functions for the inversion formula. With very minimal increase in computing time, our program can be run with a higher degree Padé approximants to possibly get even more accurate results. We could also possibly use other rational functions in the inversion formula in order to increase accuracy.

Acknowledgments

Patricio Jara acknowledges the support of the National Science Foundation Award DMS-1008101.

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