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CONVERGENCE ANALYSIS OF EXPONENTIAL EXPANDING MESHES COMPACT-FDM FOR POISSON EQUATION IN POLAR COORDINATE SYSTEM

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ABSTRACT. We develop a third (four) order accurate new nine-point compact finite difference scheme for the numerical solution of two-dimensional Poisson equation in polar form. The peculiar character of the exponential expanding mesh parameters help us in resolving interior or boundary layer in the partial differential equations. The proposed scheme takes care of grid singularity and oblique coefficient that accompany the polar form of Poisson equation. A detailed discrete convergence analysis for the difference scheme has been developed based on monotone and irreducible property of the iteration matrix. Numerical accuracy of the solutions has been obtained that shows the applicability of the scheme in the presence of singularity and thin layers. Comparing the proposed third order compact scheme with the corresponding fourth order uniform mesh strategy, the solution accuracy proved to be highly satisfactory.

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1. INTRODUCTION

In this work, we investigate numerical solution of elliptic boundary value problems (EBVPs) of semi-linear type, which is associated with Dirichlet's boundary values. It is common practice to consider approximating solution technique, because the semi-linear problems may not exhibit theoretical solution in general. We are familiar with some of the approximate solution methods such as finite element, boundary element, collocation method, B-spline and Haar wavelets in the context of EBVPs [1, 2, 3]. The second and fourth order finite difference schemes for the numerical estimates of Poisson equation in polar coordinate system has been discussed by Mittal [4]. With these available solution procedures, finite difference method (FDM) is one of an elegant tool, used to discretize the EBVPs so as to obtain system of recurrence relations, that can be easily solved with the help of matrix algebra.

We consider the following two space variables EBVPs in polar coordinate

$$\Delta\phi(r,\theta) \equiv \left(\partial^{rr} + d(r)\partial^{r} + b(r)\partial^{\theta\theta}\right)\phi(r,\theta) = H(r,\theta,\phi), (r,\theta) \in \Omega \tag{1.1}$$

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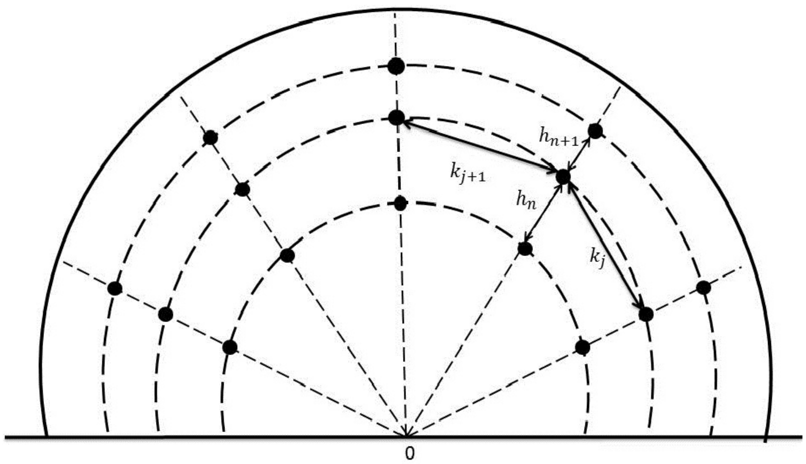

FIGURE 1. Computational molecules of $[0 < r < 1] \times [0 < \theta < \pi]$.

and associated Dirichlet's boundary values

$$\phi(r, \theta) = H^\star(r, \theta), (r, \theta) \in \partial\Omega \tag{1.2}$$

where r, θ are radial and polar coordinates, $b(r) = 1/r^2, d(r) = 1/r$, and $\partial\Omega$ is bounding the finite semi-circular region $\Omega = [0 < r < 1] \times [0 < \theta < \pi]$.

The major difficulty to the solution of above equation arises from the singularity at origin $r = 0$. In the subsequent section, we develop a new method and the procedure so as to retain order and accuracy of the scheme in the vicinity of singularity on a variable mesh step-size network.

2. EXPONENTIAL EXPANDING MESHES AND COMPACT SCHEMES

In order to obtain the discrete form to the equation (1.1), we discretize the region Ω using nine-point meshes as $\Omega = \{(r_l, \theta_m) : (l, m) \in D\}, D = \{n-1, n, n+1\} \otimes \{j-1, j, j+1\}$, where $0 = r_0 < r_1 < \cdots < r_N < r_{N+1} = 1$, $h_n = r_n - r_{n-1}$, $n = 1(1)N+1$, $0 = \theta_0 < \theta_1 < \cdots < \theta_J < \theta_{J+1} = \pi$, $k_j = \theta_j - \theta_{j-1}$, $j = 1(1)J+1$, $h_{n+1} = \alpha h_n$, $k_{j+1} = \beta k_j$, $n = 1(1)N$, $j = 1(1)J$, $\alpha, \beta \in \mathbb{R}$ (see figure 1) and $\alpha, \beta > 0$ are the exponential expanding mesh parameters, whose values are associated with the location of layer(s) and their finite range will be obtained while discussing convergence criterion. Such a mesh structure has been found suitable in the application area of electrochemistry [5] and other one dimensional problems [6, 7]. Let $\gamma_{n,j} = k_j/h_n, n = 1(1)N, j = 1(1)J$ be the spatial mesh ratio parameters. The solution values $\phi(r, \theta)$ at the mesh point (r_n, θ_j) is represented by $\phi_{n,j}$ as an exact solution and $\varphi_{n,j}$ as an approximate solution values.

Let us define the operators

$$\mathcal{A}_r\phi_{n,j} = \alpha^{-1}(\alpha+1)^{-1}\phi_{n+1,j} + \alpha^{-1}(\alpha-1)\phi_{n,j} - \alpha(\alpha+1)^{-1}\phi_{n-1,j}, \tag{2.1}$$

$$\mathcal{A}_\theta\phi_{n,j} = \beta^{-1}(\beta+1)^{-1}\phi_{n,j+1} + \beta^{-1}(\beta-1)\phi_{n,j} - \beta(\beta+1)^{-1}\phi_{n,j-1}, \tag{2.2}$$

$$\mathcal{B}_r\phi_{n,j} = 2\alpha^{-1}(\alpha+1)^{-1}\phi_{n+1,j} - 2\alpha^{-1}\phi_{n,j} + 2(\alpha+1)^{-1}\phi_{n-1,j}, \tag{2.3}$$

$$\mathcal{B}_\theta\phi_{n,j} = 2\beta^{-1}(\beta+1)^{-1}\phi_{n,j+1} - 2\beta^{-1}\phi_{n,j} + 2(\beta+1)^{-1}\phi_{n,j-1}, \tag{2.4}$$

With the help of operators (2.1)–(2.4), it is possible to approximate the first and second order partial derivatives in r- and θ- directions. These operators are defined on three-by-three mesh points D and since three is the minimum number of mesh points needed to discretize the highest order derivative term present in equation (1.1), thus the operators (2.1)–(2.4) and their composites are compact.

Initially, we discretize the two space EBVPs with oblique coefficient

$$\mathcal{L}\phi \equiv (\partial^{rr} + b(r)\partial^{\theta\theta})\phi(r,\theta) = H(r,\theta). \tag{2.5}$$

With the application of finite Taylor's series expansion, the finite difference replacement of equation (2.5) is given by

$$\mathcal{L}_{h_n,k_j}\phi_{n,j} = 2h_n^2k_j^2\sum_{(l,m)\in D} f_{l,m}H_{l,m} + T_{n,j}, n = 1(1)N, j = 1(1)J, \tag{2.6}$$

where

$$\begin{aligned} \mathcal{L}_{h_n,k_j} &= 36k_j^2\mathcal{B}_r + 3h_n^2[12b_n + 4h_n(\alpha-1)b_n^r + h_n^2(\alpha^2-\alpha+1)b_n^{rr}]\mathcal{B}_\theta \\ &\quad + 12(\beta-1)k_j^2\mathcal{B}_r\mathcal{A}_\theta + 6h_n^2[2(\alpha-1)b_r + (\alpha^2-\alpha+1)h_nb_n^r]\mathcal{A}_r\mathcal{B}_\theta \\ &\quad + 3[h_n^2(\alpha^2-\alpha+1)b_n + k_j^2(\beta^2-\beta+1)]\mathcal{B}_r\mathcal{B}_\theta, \end{aligned} \tag{2.7}$$

is the discretization of the local operator $\mathcal{L} = \partial^{rr} + b(r)\partial^{\theta\theta}$, $b_n = 1/r_n^2$, $b_n^r = -2/r_n^3$, $b_n^{rr} = 6/r_n^4$ and

$$T_{n,j} = \begin{cases} O(h_n^2k_j^2(h_n^3 + h_n^2k_j + h_nk_j^2 + k_j^3)), \alpha \neq 1 \vee \beta \neq 1, \\ O(h_n^2k_j^2(h_n^4 + h_n^2k_j^2 + k_j^4)), \alpha = 1 \wedge \beta = 1, \end{cases} \tag{2.8}$$

be the local truncated error (LTE) and

$$\begin{aligned} f_{n,j} &= [(2\beta^2-\beta+2)(\alpha^2+1) - (\beta^2-8\beta+1)\alpha]/[\alpha\beta], \\ f_{n+1,j} &= [(2\beta^2-\beta+2)(\alpha-1) + 3\alpha^2\beta]/[(\alpha+1)\alpha\beta], \\ f_{n-1,j} &= [(2\beta^2-\beta+2)\alpha(1-\alpha) + 3\beta]/[(\alpha+1)\beta], \\ f_{n,j+1} &= [2(\beta-1)(\alpha^2+1) + \alpha(3\beta^2-\beta+1)]/[\alpha\beta(\beta+1)], \\ f_{n,j-1} &= [2\beta(1-\beta)(\alpha^2+1) + \alpha(\beta^2-\beta+3)]/[\alpha(\beta+1)], \\ f_{n+1,j+1} &= 2(\beta-1)(\alpha-1)/[\alpha\beta(\alpha+1)(\beta+1)], \\ f_{n+1,j-1} &= -\beta^2 f_{n+1,j+1}, f_{n-1,j+1} = -\alpha^2 f_{n+1,j+1}, f_{n-1,j-1} = \alpha^2\beta^2 f_{n+1,j+1}. \end{aligned}$$

Thus, the method (2.6) is third order accurate for the arbitrary finite values of $\alpha \neq 1$ or $\beta \neq 1$ and in particular, it is fourth order accurate if $\alpha = \beta = 1$ for the numerical solution of EBVPs (2.5).

Our goal is to obtain compact scheme for (1.1), which involves first order partial derivative in radial direction, therefore we need to define some more approximations as follows:

For $\rho = 0, \pm 1$,

$$\begin{bmatrix} \hat{\phi}^r_{n,j+\rho} \\ \hat{\phi}^r_{n+1,j+\rho} \\ \hat{\phi}^r_{n-1,j+\rho} \end{bmatrix} = \frac{1}{\alpha(\alpha+1)h_n} \begin{bmatrix} \alpha^2 - 1 & 1 & -\alpha^2 \\ -(1+\alpha)^2 & 1+2\alpha & \alpha^2 \\ (1+\alpha)^2 & -1 & -\alpha(\alpha+2) \end{bmatrix} \begin{bmatrix} \phi_{n,j+\rho} \\ \phi_{n+1,j+\rho} \\ \phi_{n-1,j+\rho} \end{bmatrix}. \tag{2.9}$$

Now, we construct the functional for $(l,m) \in D \sim \{(n,j)\}$ as follows

$$\hat{R}_{l,m} = -d_l \hat{\phi}^r_{l,m} + H(r_l, \theta_m, \phi_{l,m}), \quad R_{l,m} = -d_l \phi^r_{l,m} + H(r_l, \theta_m, \phi_{l,m}), \tag{2.10}$$

where $d_l = 1/r_l$ and $d^r_l = -1/r^2_l$, $l = n,\ n \pm 1$.

With the help of series expansion, we obtain

(2.11)

$$\begin{bmatrix} \hat{R}_{n+1,j-1} \\ \hat{R}_{n+1,j} \\ \hat{R}_{n+1,j+1} \end{bmatrix} = \begin{bmatrix} R_{n+1,j-1} \\ R_{n+1,j} \\ R_{n+1,j+1} \end{bmatrix} + \frac{h_n^2}{6}\alpha(\alpha+1)\boldsymbol{M}^{+} \begin{bmatrix} (d_n + \alpha h_n d^r_n)\phi^{rrr}_{n,j} \\ h_n d_n \phi^{rrrr}_{n,j}/4 \\ k_j d_n \phi^{rrr\theta}_{n,j} \end{bmatrix} + \begin{bmatrix} O(h_n^4) \\ O(h_n^4) \\ O(h_n^4) \end{bmatrix},$$

$$\begin{bmatrix} \hat{R}_{n-1,j-1} \\ \hat{R}_{n-1,j} \\ \hat{R}_{n-1,j+1} \end{bmatrix} = \begin{bmatrix} R_{n-1,j-1} \\ R_{n-1,j} \\ R_{n-1,j+1} \end{bmatrix} + \frac{h_n^2}{6}(\alpha+1)\boldsymbol{M}^{-} \begin{bmatrix} (d_n - h_n d^r_n)\phi^{rrr}_{n,j} \\ h_n d_n \phi^{rrrr}_{n,j}/4 \\ k_j d_n \phi^{rrr\theta}_{n,j} \end{bmatrix} + \begin{bmatrix} O(h_n^4) \\ O(h_n^4) \\ O(h_n^4) \end{bmatrix}, \tag{2.12}$$

$$\begin{bmatrix} \hat{R}_{n,j-1} \\ \hat{R}_{n,j+1} \end{bmatrix} = \begin{bmatrix} R_{n,j-1} \\ R_{n,j+1} \end{bmatrix} - \frac{h_n^2}{6}\alpha d_n \boldsymbol{M}^{\mathbf{0}} \begin{bmatrix} \phi^{rrr}_{n,j} \\ h_n \phi^{rrrr}_{n,j}/4 \\ k_j \phi^{rrr\theta}_{n,j} \end{bmatrix} + \begin{bmatrix} O(h_n^4) \\ O(h_n^4) \\ O(h_n^4) \end{bmatrix}, \tag{2.13}$$

where

$$\boldsymbol{M}^{-} = \begin{bmatrix} 1 & \alpha-2 & -1 \\ 1 & \alpha-2 & 0 \\ 1 & \alpha-2 & \beta \end{bmatrix}, \boldsymbol{M}^{+} = \alpha \begin{bmatrix} 1 & 2\alpha-1 & -1 \\ 1 & 2\alpha-1 & 0 \\ 1 & 2\alpha-1 & \beta \end{bmatrix}, \boldsymbol{M}^{\mathbf{0}} = \begin{bmatrix} 1 & \alpha-1 & -1 \\ 1 & \alpha-1 & 1 \\ 1 & \alpha-1 & -1 \end{bmatrix}.$$

Now, we define a linear combination so as to obtain new approximation of first order derivative in radial-direction at the central mesh point (r_n, θ_j) as follows

$$\begin{aligned} \hat{\hat{\phi}}^r_{n,j} = \hat{\phi}^r_{n,j} + h_n[\delta_1(\hat{R}_{n+1,j+1} - \hat{R}_{n-1,j+1}) + \delta_2(\hat{R}_{n+1,j-1} - \hat{R}_{n-1,j-1}) \\ +\delta_3(\hat{\phi}^{\theta\theta}_{n+1,j} - \hat{\phi}^{\theta\theta}_{n,j}) + \delta_4(\hat{\phi}^{\theta\theta}_{n-1,j} - \hat{\phi}^{\theta\theta}_{n,j})] + h_n^2(\delta_5\hat{\phi}^{\theta\theta}_{n+1,j} + \delta_6\hat{\phi}^{\theta\theta}_{n-1,j}) \end{aligned} \tag{2.14}$$

where

$$\hat{\phi}^{\theta\theta}_{n\pm\tau,j} = 2[\phi_{n\pm\tau,j+1} - (\beta+1)\phi_{n\pm\tau,j} + \beta\phi_{n\pm\tau,j-1}]/[\beta(\beta+1)k_j^2], \quad \tau = 0, 1,$$

and δ_i, $i = 1(1)6$, are the undetermined coefficients, to be obtained in such a manner that the resulting difference scheme for the numerical solution of (1.1) gives rise to third and fourth order of truncation error in an exponential mesh and uniform mesh network respectively.

Let us construct the functional at the central mesh (r_n, θ_j) as

$$\hat{R}_{n,j} = -d_n \hat{\hat{\phi}}^{r}_{n,j} + H(r_n, \theta_j, \phi_{n,j}), \quad R_{n,j} = -d_n \phi^{r}_{n,j} + H(r_n, \theta_j, \phi_{n,j}). \tag{2.15}$$

The explicit algebraic expression to the equation (2.10) can be obtained with the help of Taylor's expansion as

(2.16)

$$\begin{aligned}
\hat{R}_{n,j} = R_{n,j} &- (h_n^2/2) d_n \left[(\alpha+1)(h_n(\alpha-1)b_n^{rr} + 2b_n^r)(\delta_1+\delta_2) + 2(\delta_5+\delta_6)\right] \phi^{\theta\theta}_{n,j} \\
&- (h_n^2/6) d_n [(\alpha+1)(h_n(\alpha-1)d_n + 6)(\delta_1+\delta_2) + \alpha] \phi^{rrr}_{n,j} \\
&- h_n^2 d_n [(\alpha+1)(h_n(\alpha-1)b_n^r + b_n)(\delta_1+\delta_2) - \delta_4 + \alpha\delta_3 - h_n(\delta_6 - \alpha\delta_5)] \phi^{r\theta\theta}_{n,j} \\
&- (h_n^3/24) d_n (\alpha-1)[12(\alpha+1)(\delta_1+\delta_2) + \alpha] \phi^{rrrr}_{n,j} - h_n^2 k_j d_n (\alpha+1)(\beta\delta_1 - \delta_2) \phi^{rrr\theta}_{n,j} \\
&- (h_n^2 k_j/3) d_n [3b_n(\alpha+1)(\beta\delta_1 - \delta_2) + (\beta-1)(\alpha\delta_3 - \delta_4)] \phi^{r\theta\theta\theta}_{n,j} \\
&- (h_n^2 k_j/3) d_n [3(\alpha+1)b_n^r(\beta\delta_1 - \delta_2) + (\beta-1)(\delta_5+\delta_6)] \phi^{\theta\theta\theta}_{n,j} \\
&- (h_n^3/2) d_n [b_n(\alpha^2-1)(\delta_1+\delta_2) + \alpha^2\delta_3 + \delta_4] \phi^{rr\theta\theta}_{n,j} + O(h_n^4).
\end{aligned}$$

Incorporating the above functional approximations (2.11)–(2.13) and (2.16) in the compact scheme (2.6), one obtains

$$\mathcal{L}_{h_n,k_j} \phi_{n,j} = 2h_n^2 k_j^2 \sum_{(l,m)\in D} f_{l,m} \hat{R}_{l,m} + T_{n,j}, \quad n = 1(1)N, j = 1(1)J, \tag{2.17}$$

as the finite difference replacement of equation (1.1) and $\delta_4 = b_n(\alpha+1)(\delta_1+\delta_2)+\alpha\delta_3$, $\delta_6 = -b_n^r(\alpha+1)(\delta_1+\delta_2) - \delta_5$, $\delta_1 = -\delta_2 - \alpha\beta(\alpha^2+\alpha+1)/\{2(\alpha+1)[(\alpha^2+1)(2\beta^2-\beta+2) - \alpha(\beta^2-8\beta+1)]\}$, gives $T_{n,j} = O(h_n^2 k_j^2 (h_n^3 + h_n^2 k_j + h_n k_j^2 + k_j^3))$, for the arbitrary values of δ_2, δ_3 and δ_5(may be zero). In addition to these values of undetermined coefficients, if $\alpha = \beta = 1$ and $\delta_2 = -1/32$, $\delta_3 = b_n/16$, $\delta_5 = b_n^r/16$, the local truncation error in equation (2.17) becomes $T_{n,j} = O(h_n^2 k_j^2 (h_n^4 + h_n^2 k_j^2 + k_j^4))$. Thus, we conclude this section with the observation that it is possible to develop a lower order (three) compact scheme on exponential expanding meshes and the scheme achieve an accuracy of order four, if the meshes are uniformly distributed.

3. APPLICATION TO POISSON EQUATION IN POLAR COORDINATES

Consider the Poisson equation in polar coordinates

$$\left(\partial^{rr} + \frac{1}{r}\partial^r + \frac{1}{r^2}\partial^{\theta\theta}\right)\phi(r,\theta) = H(r,\theta), \tag{3.1}$$

defined on the finite domain Ω. The application of method (2.17) to the equation (3.1) gives rise to terms like $\frac{1}{r_{n-1}}$, which fails to compute at $n=1$, as $\frac{1}{r_0}$ leads to zero divisor. A similar observation may occurs with $H_{n\pm1,j\pm1}$. For example, if $H(r,\theta)=\frac{\sin(\theta)}{r-1}$, then $H_{n+1,j\pm1}=\frac{\sin(\theta_{j\pm1})}{r_{n+1}-1}$ and at $n=N$, $r_{n+1}=1$, thus we again land up with zero divisor. Keeping these observation, it is suitable to replace all $\frac{1}{r_{n\pm1}}, H_{n\pm1,j+1}$ terms in its equivalent finite Taylor series up to the accuracy compatible with $T_{n,j}$. Such a replacement gives rise to the difference scheme

$$\begin{aligned}\Delta\phi_{n,j}&+h_n^2k_j^2[36H_{n,j}+12(\beta-1)k_jH^{\theta}_{n,j}+3(\beta^2-\beta+1)k_j^2H^{\theta\theta}_{n,j}\\&+4(\alpha-1)(\beta-1)h_nk_jH^{r\theta}_{n,j}+3(\alpha^2-\alpha+1)h_n^2H^{rr}_{n,j}\\&+(h_n/r_n)((\alpha^2+\alpha+1)h_n\\&+12r_n(\alpha-1))H^{r}_{n,j}]=T_{n,j},\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}\Delta=&\frac{k_j^2}{r_n^2}[(5\alpha^2-7\alpha+5)h_n^2-12(\alpha-1)r_nh_n-36r_n^2]\mathcal{B}_r\\&-\frac{4k_j^2}{r_n}(\beta-1)[(\alpha-1)h_n+3r_n]\mathcal{B}_r\mathcal{A}_\theta\\&+(2h_n^2/r_n^4)[\{(\alpha^2+\alpha+1)-9(\alpha^2-\alpha+1)r_n\}h_n^2+12(\alpha-1)h_nr_n-18r_n^2]\mathcal{B}_\theta\\&+(h_n^3/r_n^3)[(11\alpha^2-13\alpha+11)\mathcal{A}_r\mathcal{B}_\theta-(5\alpha^2-7\alpha+5)k_j^2\mathcal{A}_r]\\&+(h_n^2/r_n^2)[4(\alpha-1)k_j^2(3\mathcal{A}_r+(\beta-1)\mathcal{A}_r\mathcal{A}_\theta)-12(\alpha-1)\mathcal{A}_r\mathcal{B}_\theta-3(\alpha^2-\alpha+1)\mathcal{B}_r\mathcal{B}_\theta]\\&-(h_nk_j^2/r_n)[36\mathcal{A}_r+3(\beta^2-\beta+1)\mathcal{A}_r\mathcal{B}_\theta+12(\beta-1)\mathcal{A}_r\mathcal{B}_\theta]-3k_j^2(\beta^2-\beta+1)\mathcal{B}_r\mathcal{B}_\theta\end{aligned}$$

The scheme (3.2) is compact and free from the terms $H_{n\pm1,j\pm1}$ or $1/r_{n\pm1}$, and thus easily computed inside the domain of integration Ω.

4. CONVERGENCE ANALYSIS

In this section, we prove that the compact scheme (2.17) for the numerical solution of (1.1) converges for sufficiently small mesh spacing. At the mesh point (r_n,θ_j), equation(1.1) can be written as

$$\left(\partial^{rr}+d_n\partial^{r}+b_n\partial^{\theta\theta}\right)\phi_{n,j}=H(r_n,\theta_j,\phi_{n,j}),\tag{4.1}$$

where $b_n=1/r_n^2>0$ and $d_n=1/r_n$.

The discretized compact scheme (2.17) for the equation (1.1) can be expressed as

$$G_{n,j}+O(h_n^5)=0,\quad n=1(1)N, j=1(1)J,\tag{4.2}$$

where $G_{n,j}=-h_n^{-2}\mathcal{L}_{h_nk_j}\phi_{n,j}+2\gamma_{n,j}^2h_n^2\sum_{(l,m)\in D}f_{l,m}\hat{R}_{l,m}$.

In the vector notation

$$\boldsymbol{G}(\boldsymbol{\phi})+\boldsymbol{T}=\boldsymbol{O}_{NJ\times NJ},\tag{4.3}$$

where $\boldsymbol{G}(\boldsymbol{\phi}) = [G_{11}, \ldots, G_{N1}, \ldots, G_{1J}, \ldots, G_{NJ}]^T$, $\boldsymbol{\phi} = [\phi_{11}, \ldots, \phi_{N1}, \ldots, \phi_{1J}, \ldots, \phi_{NJ}]^T$ and $\boldsymbol{T} = [T_{11}, \ldots, T_{N1}, \ldots, T_{1J}, \ldots, T_{NJ}]^T$ be the vector of fifth order of local truncation error.

Let $\epsilon_{n,j} = \varphi_{n,j} - \phi_{n,j}$ be the discretization error of approximate and exact solution values and $\boldsymbol{\epsilon} = [\epsilon_{11}, \ldots, \epsilon_{N1}, \ldots, \epsilon_{1J}, \ldots, \epsilon_{NJ}]^T$ be the error vector.

The approximate solution φ satisfy

$$\boldsymbol{G}(\boldsymbol{\varphi}) = \boldsymbol{O}_{NJ\times NJ}. \tag{4.4}$$

For $(l,m) \in D \sim \{(n,j)\}$, define $\hat{S}_{l,m} = -d(r_l)\hat{\varphi}^r_{l,m} + H(r_l, \theta_m, \varphi_{l,m})$ and $\hat{S}_{n,j} = -d(r_n)\hat{\hat{\varphi}}^r_{n,j} + H(r_n, \theta_j, \varphi_{n,j})$ and let

$$\hat{E}_{l,m} = \hat{S}_{l,m} - \hat{R}_{l,m}, (l,m) \in D. \tag{4.5}$$

With the help of Mean value theorem , it is easy to write

$$\hat{E}_{l,m} = X_{l,m}\epsilon_{l,m} + Y_{l,m}\hat{\epsilon}^r_{l,m}, (l,m) \in D \sim \{(n,j)\}, \quad \hat{E}_{n,j} = X_{n,j}\epsilon_{n,j} + Y_{n,j}\hat{\hat{\epsilon}}^r_{n,j}, \tag{4.6}$$

where $X_{l,m} = X(r_l, \theta_m)$ corresponds to $\frac{\partial H}{\partial \phi}$ and $Y_{l,m} = Y(r_l, \theta_m)$, $(l,m) \in D$ are suitable finite constants and $\hat{\epsilon}^r_{l,m}, (l,m) \in D$ can be obtained from the equation (2.9) simply replacing ϕ by ϵ. In a similar manner $\hat{\hat{\epsilon}}^r_{n,j}$ can be obtained from equation (2.14).

Hence, we can explicitly write the error equation as

$$\boldsymbol{G}(\boldsymbol{\varphi}) - \boldsymbol{G}(\boldsymbol{\phi}) \equiv [-h_n^2 \mathcal{L}_{h_n k_j}\epsilon_{n,j} + 2\gamma^2_{n,j}h_n^2 \sum_{(l,m)\in D} f_{l,m}\hat{E}_{l,m}]_{n=1(1)N, j=1(1)J} \tag{4.7}$$

or equivalently in matrix notation, we have

$$\boldsymbol{G}(\boldsymbol{\varphi}) - \boldsymbol{G}(\boldsymbol{\phi}) = \boldsymbol{M}\boldsymbol{\epsilon} \tag{4.8}$$

where $\boldsymbol{M} = [M_{p,q}], p, q = 1(1)NJ$, is a block tri-diagonal matrix and has only non-zero elements for $\alpha, \beta \neq (\sqrt{5} \pm 1)/2$ at the following locations:

For $j = 2(1)J$

$$M_{(j-1)N+n,(j-2)N+n-1} = \frac{12(\alpha^2 - \alpha - 1)}{(\alpha+1)(\beta+1)r_n^2} + \frac{12\gamma^2_{n,j}(\beta^2 - \beta - 1)}{(\alpha+1)(\beta+1)} + O(h_n), \quad n = 2(1)N,$$

$$M_{(j-1)N+n,(j-2)N+n} = -\frac{12(\alpha^2 + 3\alpha + 1)}{\alpha(\beta+1)r_n^2} - \frac{12\gamma^2_{n,j}(\beta^2 - \beta - 1)}{\alpha(\beta+1)} + O(h_n), \quad n = 1(1)N,$$

$$M_{(j-1)N+n,(j-2)N+n+1} = -\frac{12(\alpha^2+\alpha-1)}{\alpha(\alpha+1)(\beta+1)r_n^2} + \frac{12\gamma_{n,j}^2(\beta^2-\beta-1)}{\alpha(\alpha+1)(\beta+1)} + O(h_n), \quad n = 1(1)N-1,$$

For $j = 1(1)J$

$$M_{(j-1)N+n,(j-1)N+n-1} = -\frac{12(\alpha^2-\alpha-1)}{(\alpha+1)\beta r_n^2} - \frac{12\gamma_{n,j}^2(\beta^2+3\beta+1)}{(\alpha+1)\beta} + O(h_n), \quad n = 2(1)N,$$

$$M_{(j-1)N+n,(j-1)N+n} = \frac{12(\alpha^2+3\alpha+1)}{\alpha\beta r_n^2} + \frac{12\gamma_{n,j}^2(\beta^2+3\beta+1)}{\alpha\beta} + O(h_n), \quad n = 1(1)N,$$

$$M_{(j-1)N+n,(j-1)N+n+1} = \frac{12(\alpha^2+\alpha-1)}{\alpha(\alpha+1)\beta r_n^2} - \frac{12\gamma_{n,j}^2(\beta^2+3\beta+1)}{\alpha(\alpha+1)\beta} + O(h_n), \quad n = 1(1)N-1,$$

For $j = 1(1)J-1$

$$M_{(j-1)N+n,jN+n-1} = \frac{12(\alpha^2-\alpha-1)}{(\alpha+1)\beta(\beta+1)r_n^2} - \frac{12\gamma_{n,j}^2(\beta^2+\beta-1)}{(\alpha+1)\beta(\beta+1)} + O(h_n), \quad n = 2(1)N,$$

$$M_{(j-1)N+n,jN+n} = -\frac{12(\alpha^2+3\alpha+1)}{\alpha\beta(\beta+1)r_n^2} + \frac{12\gamma_{n,j}^2(\beta^2+\beta-1)}{\alpha\beta(\beta+1)} + O(h_n), \quad n = 1(1)N,$$

$$M_{(j-1)N+n,jN+n+1} = -\frac{12(\alpha^2+\alpha-1)}{\alpha(\alpha+1)\beta(\beta+1)r_n^2} - \frac{12\gamma_{n,j}^2(\beta^2+\beta-1)}{\alpha(\alpha+1)\beta(\beta+1)} + O(h_n), \quad n = 1(1)N-1,$$

From the equations (4.3), (4.4) and (4.8), we obtain

$$\boldsymbol{M}\boldsymbol{\epsilon} = \boldsymbol{T} \tag{4.9}$$

The lower, upper and main tri-diagonal blocks of the matrix $\boldsymbol{M}$ have non-zero values at its sub-, sup- and main-diagonal provided $\alpha, \beta \neq (\sqrt{5} \pm 1)/2$. Further, if the arrow $p \to q$ denotes the directed path to each non-zero values $M_{p,q}$ of the matrix $\boldsymbol{M}$, then there exists a directed path $(p \to p_1), (p_1 \to p_2), \ldots, (p_n \to q)$ connecting any two ordered pair of nodes (p, q). Therefore, graph of the matrix $\boldsymbol{M}$ is strongly connected and hence $\boldsymbol{M}$ is irreducible [8, 9].

Let $x = \min X_{n,j}, y = \min Y_{n,j}, \gamma = \min \gamma_{n,j}, 1 \leq n \leq N, 1 \leq j \leq J$. Let W_q denote the weak row elements sum of the matrix $\boldsymbol{M}$. Then, for sufficiently small values of h_n or in the limiting case as $h_n \to 0^+$ and $\mid \alpha - \sqrt{5}/2 \mid \leq 1/2, \mid \beta - \sqrt{5}/2 \mid \leq 1/2$, we find

$$
\begin{aligned}
W_1 &\geq \frac{12(\alpha^2+5\alpha+5)}{(\alpha+1)(\beta+1)} + \frac{12\gamma^2(\beta^2+5\beta+5)}{(\alpha+1)(\beta+1)} > 0,\\
W_q &\geq \frac{72}{(\beta+1)} > 0, \quad q = 2(1)N-1,\\
W_N &\geq \frac{12(5\alpha^2+5\alpha+1)}{\alpha(\alpha+1)(\beta+1)} + \frac{12\gamma^2(\beta^2+5\beta+5)}{\alpha(\alpha+1)(\beta+1)} > 0,\\
W_{(t-1)N+1} &\geq \frac{72\gamma^2}{\alpha+1} > 0, \quad t = 2(1)J-1,\\
W_{(t-1)N+q} &\geq 36x\gamma^2 h_q^2 \geq 0, \quad t = 2(1)J-1, \quad q = 2(1)N-1, \quad x \geq 0,\\
W_{(t-1)N+N} &\geq \frac{72\gamma^2}{\alpha(\alpha+1)} > 0, \quad t = 2(1)J-1,\\
W_{(J-1)N+1} &\geq \frac{12(\alpha^2+5\alpha+5)}{(\alpha+1)\beta(\beta+1)} + \frac{12\gamma^2(5\beta^2+5\beta+1)}{(\alpha+1)\beta(\beta+1)} > 0,\\
W_{(J-1)N+q} &\geq \frac{72}{\beta(\beta+1)} > 0, \quad q = 2(1)N-1,\\
W_{(J-1)N+N} &\geq \frac{12(5\alpha^2+5\alpha+1)}{\alpha(\alpha+1)\beta(\beta+1)} + \frac{12\gamma^2(5\beta^2+5\beta+1)}{\alpha(\alpha+1)\beta(\beta+1)} > 0,
\end{aligned}
$$

since $1/r_n^2 > 1$, $n = 1(1)N$.

As all weak row sum (except corresponding to the main diagonal) are positive, thus the matrix $\boldsymbol{M}$ is monotone [10, 11]. As a consequence, $\boldsymbol{M}$ is monotone and irreducible, if $\mid \alpha - \sqrt{5}/2 \mid < 1/2, \mid \beta - \sqrt{5}/2 \mid < 1$. Hence, $\boldsymbol{M}^{-1}$ exists and $\boldsymbol{M}^{-1} > 0$.

Let $M_{p,q}^{-1}$ be the $(p, q)^{th}$ element of $\boldsymbol{M}^{-1}$ and define the matrix norm

$$
\|\boldsymbol{M}^{-1}\|_\infty = \max_{p=1(1)NJ} \left[\begin{array}{c} |M_{p,1}^{-1}| + \sum_{q=2}^{N-1} |M_{p,q}^{-1}| + |M_{p,N}^{-1}| + |M_{p,(J-1)N+1}^{-1}| + \sum_{q=2}^{N-1} |M_{p,(J-1)N+q}^{-1}| \\ +|M_{p,NJ}^{-1}| + \sum_{t=2}^{J-1} \left(|M_{p,(t-1)N+1}^{-1}| + \sum_{q=2}^{N-1} |M_{p,(t-1)N+q}^{-1}| + |M_{p,tN}^{-1}| \right) \end{array} \right],
$$

and the vector norm

$$
\|\boldsymbol{T}\|_\infty = \max_{n=1(1)N} \sum_{j=1(1)J} T_{n,j}.
$$

The elementary matrix identity $\boldsymbol{M}^{-1}(\boldsymbol{M}I) = I$, where I is the $NJ \times 1$ column matrix with one as all of its elements, gives rise

$$
\sum_{q=1(1)NJ} M_{p,q}^{-1} W_q = 1, p = 1(1)NJ. \tag{4.10}
$$

With the help of equation (4.10), it is possible to determine the bounds on the elements of matrix $\boldsymbol{M}^{-1}$. Applying finite Taylor's expansion, we observed,

For $h = \max_{n=1(1)N} h_n$ and $p = 1(1)NJ$:

$$M^{-1}_{p,1} \leq \frac{1}{W_1} \leq \frac{(\alpha+1)(\beta+1)}{12[\alpha^2+5\alpha+5+\gamma^2(\beta^2+5\beta+5)]} + O(h),$$

$$\sum_{q=2}^{N-1} M^{-1}_{p,q} \leq \frac{1}{\min_{q=2(1)N-1} W_q} \leq \frac{1}{72}(\beta+1) + O(h),$$

$$M^{-1}_{p,N} \leq \frac{1}{W_N} \leq \frac{(\alpha+1)(\beta+1)}{12[5\alpha^2+5\alpha+1+\gamma(\beta^2+5\beta+5)]} + O(h),$$

$$\sum_{t=2(1)J-1} M^{-1}_{p,(t-1)N+1} \leq \frac{1}{\min_{t=2(1)J-1} W_{(t-1)N+1}} \leq \frac{\alpha+1}{72\gamma^2} + O(h),$$

$$\sum_{t=2}^{J-1}\sum_{q=2}^{N-1} M^{-1}_{p,(r-1)N+q} \leq \begin{cases} \sum_{q=1}^{NJ} M^{-1}_{p,q} W_q = 1, x = 0, \\ \frac{1}{\min_{\substack{q=2(1)N-1 \\ t=2(1)J-1}} W_{(t-1)N+q}} \leq \frac{1}{36x\gamma^2 O(h^2)}, x > 0, \end{cases}$$

$$\sum_{t=2}^{J-1} M^{-1}_{p,tN} \leq \frac{1}{\min_{t=2(1)J-1} W_{tN}} \leq \frac{\alpha(\alpha+1)}{72\gamma^2} + O(h),$$

$$M^{-1}_{p,(J-1)N+1} \leq \frac{1}{W_{(J-1)N+1}} \leq \frac{(\alpha+1)\beta(\beta+1)}{12[\alpha^2+5\alpha+5+\gamma^2(5\beta^2+5\beta+1)]} + O(h),$$

$$\sum_{q=2}^{N-1} M^{-1}_{p,(J-1)N+q} \leq \frac{1}{\min_{q=2(1)N-1} W_{(J-1)N+q}} \leq \frac{1}{72}\beta(\beta+1) + O(h),$$

$$M^{-1}_{p,NJ} \leq \frac{1}{W_{NJ}} \leq \frac{\alpha(\alpha+1)\beta(\beta+1)}{12[5\alpha^2+5\alpha+1+\gamma(5\beta^2+5\beta+1)]} + O(h),$$

Incorporating the above inequalities in equation (4.9), the bounds of error are given by

$$\|\epsilon\|_\infty \leq \|\boldsymbol{M}^{-1}\|_\infty . \|\boldsymbol{T}\|_\infty \leq \begin{cases} \frac{h^3}{36\gamma^2 x} + O(h^4), x > 0 \\ O(h^5), x = 0 \end{cases} \tag{4.11}$$

Thus, $\epsilon \to 0$ as $h \to 0^+$. Hence, the compact method (2.17) for the numerical solution of semi-linear EBVPs (1.1) converges, if $x \geq 0$ i.e. $\frac{\partial H}{\partial \phi} \geq 0$.

5. NUMERICAL VALIDATION

The presented nine-point compact scheme will be applied on Poisson equation, Helmholtz equation, Schrödinger equation and on a semi-linear equation. The solution values for the linear equations are computed using Gauss-Seidel method. For semi-linear equation, Newton-Raphson method is applied. In all the problems, zero vector is taken as an initial guess with 10^{-14} as a tolerance of the iterative method and the minimum number of converging iterations is denoted by (I). All the algebraic calculations are performed using symbolic tool of Maple software and numerical calculations have been performed with C programming using Mac OS.

Example 5.1 ([12])**.** Consider the Poisson equation $\Delta\phi = (4+\pi^2)\cosh(\pi\theta)$, having equilibrium temperature as $\phi(r,\theta) = r^2\cosh(\pi\theta)$.

Example 5.2 ([13])**.** We consider the Helmholtz equation in two space dimensions $\Delta\phi = \phi$. The theoretical solution is given by $\phi(r,\theta) = e^{\frac{r}{2}(\cos\theta+\sqrt{3}\sin\theta)}$.

Example 5.3 ([12])**.** Consider the stationary Schrödinger equation representing nuclear motion $\Delta\phi = \omega r^2\phi$. The theoretical solution is given by $\phi(r,\theta) = [r^2\cos(2\theta) + 2][\cosh(\sqrt{\omega}r^2\cos\theta\sin\theta) + \sinh(\sqrt{\omega}r^2\cos\theta\sin\theta)]/2$.

Example 5.4 ([14])**.** We solve semi-linear Poisson equation $\Delta\phi = 4\phi^3$. The theoretical solution is given by $\phi(r,\theta) = 1/[4 + r(\cos\theta + \sin\theta)]$.

All the above examples are solved using proposed third and fourth order compact scheme by taking $\alpha \neq 1, \beta \neq 1$ and $\alpha = \beta = 1$ respectively. The boundary values are obtained from the theoretical solution as a test procedure. The accuracy of approximate and theoretical solution values in terms of maximum absolute errors (MAE) $\mathcal{E}^{\infty}_{N,J} = \max_{\substack{n=1(1)N\\ j=1(1)J}} |\phi_{n,j} - \varphi_{n,j}|$ and convergence order $\Theta = \log_2\left(\mathcal{E}^{\infty}_{N,J}/\mathcal{E}^{\infty}_{2N+1,2J+1}\right)$ are presented in Table 1-4 for various mesh spacing $(N = J)$. The numerical results show that the exponential expanding meshes third order compact scheme is superior as compared with uniform meshes fourth order compact scheme both in terms of accuracy and convergence iteration number.

6. CONCLUSION

A new non-uniform mesh nine-point compact scheme for the solution of two space dimensional singular EBVPs have been presented with third and fourth order of accuracy. The application of proposed method to Cartesian coordinate is straightforward by taking $b(r) = 1$.

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TABLE 1. MAE of solution values for example 1

N	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ
5	1.00	1.00	116	3.45e+01	- -	0.632	0.894	89	5.68e-01	- -
11	1.00	1.00	412	2.45e-00	3.8	0.640	0.946	254	4.34e-02	3.7
23	1.00	1.00	1368	1.63e-01	3.9	0.910	0.970	1059	8.84e-03	2.3
47	1.00	1.00	4558	1.03e-02	4.0	0.910	0.990	3329	1.56e-03	2.5

TABLE 2. MAE of solution values for example 2

N	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ
7	1.00	1.00	117	1.95e-04	- -	1.220	1.010	103	9.10e-05	- -
15	1.00	1.00	376	1.65e-05	3.6	1.100	1.010	332	6.00e-06	3.9
31	1.00	1.00	1129	1.97e-06	3.1	1.070	1.010	829	3.68e-07	4.0
63	1.00	1.00	3038	2.43e-07	3.0	1.040	1.000	2086	2.97e-08	3.6

TABLE 3. MAE of solution values for example 3 at $\omega = 25$

N	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ	α	β	I	$\mathcal{E}^{\infty}_{N,J}$	Θ
5	1.00	1.00	70	1.37e-01	- -	1.110	1.180	59	7.54e-02	- -
11	1.00	1.00	222	9.93e-03	3.8	1.110	1.050	204	5.71e-03	3.7
23	1.00	1.00	692	6.16e-04	4.0	1.040	1.020	620	3.64e-04	4.0
47	1.00	1.00	1950	4.18e-05	3.9	1.020	1.010	1744	2.33e-05	4.0

Table 4. MAE of solution values for example 4

N	α	β	I	$\mathcal{E}_{N,J}^{\infty}$	Θ	α	β	I	$\mathcal{E}_{N,J}^{\infty}$	Θ
5	1.00	1.00	61	1.67e-05	- -	1.120	0.930	53	6.03e-06	- -
11	1.00	1.00	195	2.33e-06	2.8	1.110	0.990	150	5.54e-07	3.4
23	1.00	1.00	568	3.00e-07	3.0	1.070	1.000	379	6.58e-08	3.1
47	1.00	1.00	1439	3.79e-08	3.0	1.050	1.000	980	5.61e-09	3.6

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