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NUMERICAL SOLUTION OF FOURTH ORDER FRACTIONAL INTEGRO-DIFFERENTIAL EQUATION BY USING LEGENDRE WAVELETS.

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ABSTRACT. A numerical method is proposed to solve fourth order linear or nonlinear fractional integro-differential equation. The Legendre wavelets have been used as basis in approximation of a function. It converts the integro-differential equation into a linear or nonlinear system of algebraic equations which can be solved easily. The application of method is illustrated with the help of test examples. The main advantage of the proposed numerical method is that after discretization, the coefficient matrix of algebraic equation becomes sparse. The wavelet method is computer friendly, thus solving higher order fractional integro-differential equation becomes a matter of dimension increasing.

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1. INTRODUCTION

In last few years, considerable attention has been given to fractional differential equations due to their numerous applications in the areas of science and engineering. Finding an analytical solution for fractional differential equation is not an easy task as no general method is available for the solution of fractional differential equation [1, 2]. Numerical methods for approximating fractional integrals and derivatives have been studied by many authors. Few among the recent methods involves fractional differential transform method [3], Homotopy analysis method [4], Variational iteration method, Homotopy perturbation method [5] and collocation method [6, 7].

Recently wavelets based methods has been getting considerable interest to solve ordinary differential equations as well as integral equations [8, 9, 10]. And, this development of wavelets method have been led to the solution of fractional differential equations and fractional integro-differential equations by using Haar wavelets [11], CAS wavelets [12], Chebyshev wavelets [13, 14], Legendre wavelets [15] etc. The construction and application of wavelet numerical method has typically focused on the selection of different wavelets and the derivation of wavelet-based discrete forms.

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In this paper, the Legendre wavelets [16] is implemented to derive an approximate solutions to linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations [17]. The standard form of the fourth-order fractional integro-differential equation (1.1) can be written in terms of operator forms as

(1.1)
$$D^{\alpha}y(x) = f(x) + \gamma y(x) + \int_0^x [g(t)y(t) + h(t)F(y(t))]dt; \quad 0 < x < 1, \quad 3 < \alpha \leq 4$$

with the boundary conditions $y(0) = \gamma_0$, $y''(0) = \gamma_2$, $y(1) = \beta_0$, $y''(1) = \beta_2$, where D^{α} indicates the Caputo fractional derivative of order α, and $F(y(x))$ is a linear or nonlinear continuous function, γ, γ_0, γ_2, β_0 and β_2 are real constants, f , g and h are known functions.

2. **WAVELETS**

Wavelets form a family of functions constructed from dilation and translation of a single function which is called as the mother wavelet [18]. When the dilation parameter 'ν' and the translation parameter 'ω' vary continuously, we have the following family of continuous wavelets: $\Psi_{\nu,\omega}(x) = |\nu|^{-1/2}\Psi(\frac{x-\omega}{\nu})$, $\nu, \omega \in \mathbb{R}$, $\nu \neq 0$.

If we restrict the parameters 'ν' and 'ω' to discrete values as $\nu = a_0^{-k}$, $\omega = nb_0 a_0^{-k}$, $a_0 > 1, b_0 > 0$ and n, k are positive integers, we have the following family of discrete wavelets $\Psi_{k,n}(x) = |a_0|^{k/2}\Psi(a_0^k x - nb_0)$, where $\Psi_{k,n}(x)$ form an orthonormal basis.

2.1. **LEGENDRE WAVELETS.**

Legendre wavelets $\Psi_{n,m}(x) = \Psi(k, \hat{n}, m, x)$ have four arguements: $\hat{n} = 2n - 1$, $n = 1, 2, 3, \ldots, 2^{k-1}$, k can assume any positive integer, m is the order of Legendre polynomials and t is the normalized time [19]. They are defined on the interval $[0, 1)$ as

(2.1)
$$\Psi_{n,m}(x) = \begin{cases} (2m+1)^{\frac{1}{2}} 2^{\frac{k}{2}} L_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{elsewhere.} \end{cases}$$

where $k = 2, 3, \ldots$, $n = 1, 2, 3, \ldots 2^{k-1}$, $m = 0, 1, 2, 3, \ldots M - 1$, M is fixed positive integer. $L_m(x)$ are Legendre polynomials of degree m such that

$$L_0(x) = 1,$$

$$L_1(x) = x,$$

$$L_{m+1}(x) = \frac{2m+1}{m+1} x L_m(x) - \frac{m}{m+1} L_{m-1}(x),$$

where $m = 1, 2, \ldots$.

2.2. **FUNCTION EXPANSION WITH WAVELETS.** A function $f(x) \in L^2[0,1)$ may be expanded by using legendre wavelets [19] as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}(x) \tag{2.2}$$

where $c_{n,m} = \langle f(x), \Psi_{n,m}(x)\rangle = \int_0^1 f(x)\Psi_{n,m}(x)dx$ in which $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2[0,1)$.

If the infinite series is truncated, then it can be written as

$$f(x) \approx \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\Psi_{nm}(x) = C^T\Psi(x) \tag{2.3}$$

where $C = [c_{1,0}, \dots, c_{1,(M-1)}, c_{2,0}, \dots, c_{2,(M-1)}, \dots, c_{2^{k-1},0}, \dots, c_{2^{k-1},(M-1)}]$ and $\Psi(t) = [\Psi_{1,0}, \dots, \Psi_{1,(M-1)}, \Psi_{2,0}, \dots, \Psi_{2,(M-1)}, \dots, \Psi_{2^{k-1},0}, \dots, \Psi_{2^{k-1},(M-1)}]$.

3. PRELIMINARIES OF FRACTIONAL CALCULUS

There are various definitions of a fractional derivative of order $\alpha > 0$. The two commonly used definitions are Riemann-Liouville fractional derivative and Caputo fractional derivative. Each definition [1, 2] uses Riemann-Liouville fractional integration and derivatives of whole order. The Riemann-Liouville fractional integration of order α is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt, x > 0, J^0 f(x) = f(x), \tag{3.1}$$

and the Caputo fractional derivatives of order α is defined as

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x), \tag{3.2}$$

where D^m is the usual integer differential operator of order m and $J^{m-\alpha}$ is the Riemann-Liouville integral operator of order $m-\alpha$ and $m-1 < \alpha \leq m$.

The relation between the Riemann-Liouville operator and Caputo operator is given by the following lemma [1, 2]:

Lemma: If $m-1 < \alpha \leq m$; $m \in N$, then $D^\alpha J^\alpha f(x) = f(x)$ and

$$J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ x > 0.$$

4. METHODOLOGY FOR THE SOLUTION

Consider the approximation of fractional derivative $D^\alpha y(x)$ by using Legendre wavelet as

$$D^\alpha y(x) = C^T\Psi(x), \quad 3 < \alpha \leq 4, \tag{4.1}$$

where C^T is as defined in equation (2.3).

Operating J^α on both sides, we get

$$J^\alpha(D^\alpha y)(x) = J^\alpha(C^T\Psi(x)), \quad 3 < \alpha \leq 4. \tag{4.2}$$

And, using the lemma defined in last section, we obtain

$$y(x) - \sum_{k=0}^{3} y^{(k)}(0)\frac{(x)^k}{k!} = C^T P^\alpha \Psi(x),$$

which implies

$$y(x) = \sum_{k=0}^{3} y^{(k)}(0)\frac{(x)^k}{k!} + C^T P^\alpha \Psi(x), \tag{4.3}$$

where P^α is the fractional operational matrix of Legendre wavelets [19]. Legendre wavelets can be also expanded into an m'-term of Block Pulse function [20] as:

$$\Psi_{n,m}(x) \cong \sum_{k=1}^{m'} h_k \Phi_k(x), \tag{4.4}$$

hence we get

$$\Psi(x) \cong \psi_{m'\times m'}\varphi_{m'}(x), \tag{4.5}$$

where $\psi_{m'\times m'}$ is the coeffient matrix and $\varphi_{m'} = [\phi_1, \phi_2, \ldots, \phi_{m'}]$.

Kilicman and Zhour have given the block pulse operational matrix of fractional integration F^α as follows [20]:

$$J^\alpha \varphi_{m'}(x) \cong F^\alpha \varphi_{m'}(x), \tag{4.6}$$

where

$$F^\alpha = \frac{1}{m'^\alpha \Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m'-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m'-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m'-3} \\ \cdot & \cdot & \cdot & \cdot & \cdots & \vdots \\ 0 & 0 & \cdot & \cdot & 1 & \vdots \\ 0 & 0 & 0 & \cdot & \cdots & 1 \end{bmatrix}, \tag{4.7}$$

and $\xi_k = (k+1)^{(\alpha+1)} - 2k^{(\alpha+1)} + (k-1)^{(\alpha+1)}$.

Now on operating fractional integral operator of order α on Legendre wavelets, we get

$$J^\alpha \Psi(x) \cong J^\alpha \psi_{m'\times m'}\varphi_{m'} \cong \psi_{m'\times m'} F^\alpha \varphi_{m'}(x). \tag{4.8}$$

Also, P^α denotes the fractional operational matrix of Legendre wavelets which implies

$$P^\alpha \Psi(x) \cong \psi_{m'\times m'} F^\alpha \varphi_{m'}(x) \tag{4.9}$$

hence we obtained the fractional operational matrix of Legendre wavelets by using block-pulse functions as

$$P^{\alpha} \cong \psi_{m' \times m'} F^{\alpha} \psi^{-1}_{m' \times m'}. \tag{4.10}$$

First we have used the equations (4.1), (4.3) in equation (1.1) and the boundary conditions are also incorporated. Further, on substituting m collocation points between 0 and 1 given by $x_i = \frac{2i-1}{2(m-2)}$, $i = 1, 2, \ldots, m = 2^{k-1}M$, finally we get $m+2$ number of linear or nonlinear equations. On solving these, the values of unknown coefficients C^T are found which in turn are used to find the approximate solution of the fractional integro-differential equation (1.1).

5. ILLUSTRATION WITH EXAMPLES

5.1. **Example 1.** Consider the following linear fourth-order fractional integro-differential equation :

$$D^{\alpha} y(x) = x(1+e^{x}) + 3e^{x} + y(x) - \int_0^x y(t)dt, \quad 0 < x < 1, 3 < \alpha \leq 4, \tag{5.1}$$

subject to the following boundary conditions:
$y(0) = 1$, $y''(0) = 2$,
$y(1) = 1 + e$, $y''(1) = 3e$.

For $\alpha = 4$, the exact solution in this example is known and is given by

$$y(x) = 1 + xe^{x}. \tag{5.2}$$

Figure 1 represents the exact solution for Example 1 while Figure 2 represents the corresponding approximate solutions with the help of Legendre wavelets by using different number of basis functions. Figure 3 shows the error in the approximation by using Legendre wavelets with different number of basis functions.

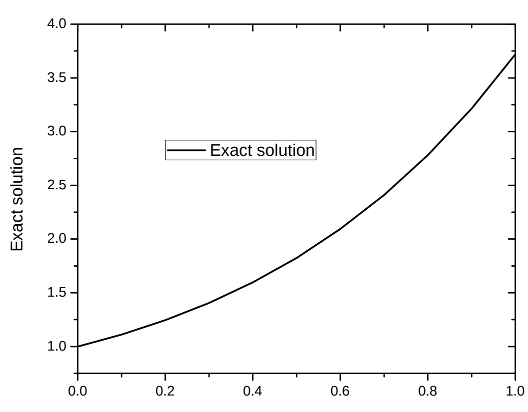

FIGURE 1. Exact solution for Example 1.

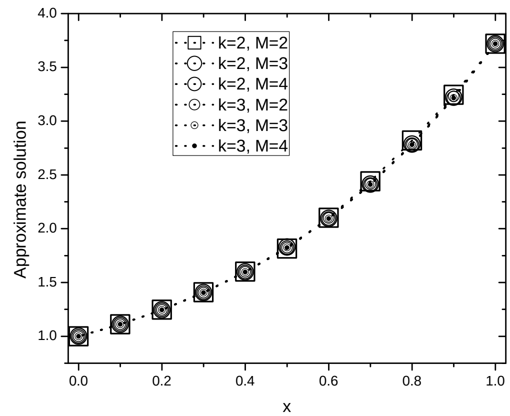

FIGURE 2. Approximate solutions for Example 1 for different values of k and M

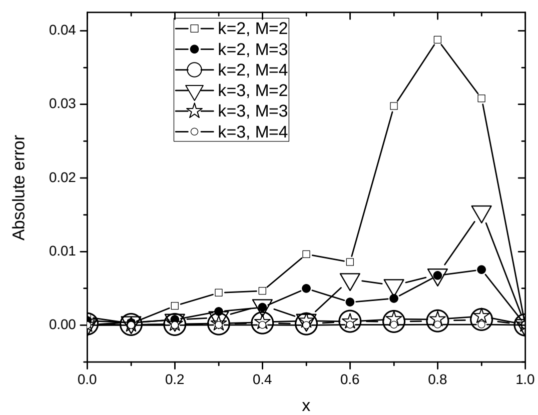

FIGURE 3. Absolute errors for Example 1 for different values of k and M

5.2. **Example 2.** Consider the following nonlinear fourth-order fractional integro-differential equation:

$$D^{\alpha}y(x) = 1 + \int_0^x e^{-t}y^2(t)dt, \quad 0 < x < 1, 3 < \alpha \leq 4, \tag{5.3}$$

subject to the following boundary conditions:
$y(0) = 1$, $y''(0) = 1$,
$y(1) = e$, $y''(1) = e$.

For $\alpha = 4$, the exact solution in this example is known and is given by

$$y(x) = e^{x}. \tag{5.4}$$

Figure 4 shows the exact solution for Example 2 while Figure 5 represents the corresponding approximate solutions with the help of Legendre wavelets by using

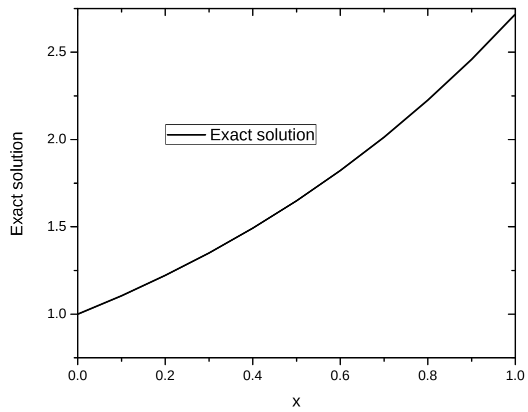

FIGURE 4. Exact solution for Example 2.

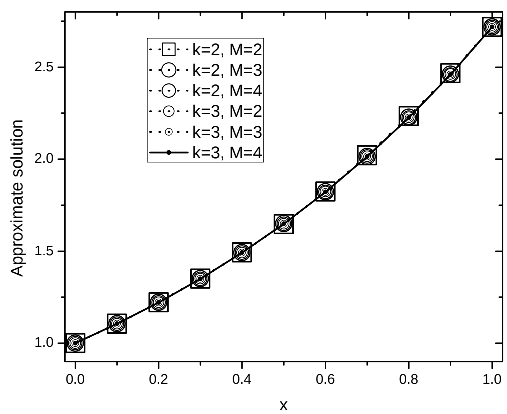

FIGURE 5. Approximate solutions for Example 2 for different values of k and M

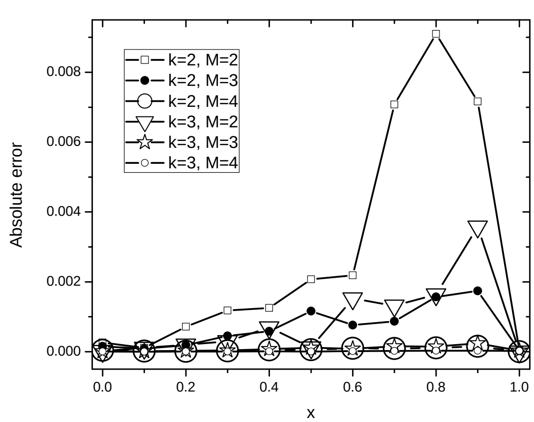

FIGURE 6. Absolute errors for Example 2 for different values of k and M

different number of basis functions. Figure 6 shows the error in the approximation by using Legendre wavelets with different number of basis functions.

6. CONCLUSIONS

The proposed method is easy to implement. One of advantage of Legendre wavelets over similar kind of wavelet like Chebyshev wavelet is that Legendre polynomial involve less computational cost than Chebyshev polynomial. Since the weight function for Legendre polynomial over the interval $[-1, 1]$ is 1 while for Chebyshev polynomial is $\sqrt{1-t^2}$. Further, it is noted from Figure 3 and Figure 6 that as the number of basis functions are increased, the error is decreased for both linear as well as nonlinear problem. So any desired accuracy can be easily obtained by considering more number of Legendre wavelet basis functions in the approximation.

REFERENCES

[1] K. B. Oldham and J. Spanier, *Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order*, Academic Press, Inc., New York-London, 1974.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., 1993.

[3] A. Arikoglu, and I. Ozkol, Solution of fractional integro-differential equations by using fractional differential transform method, *Chaos Solit. Fract.*, 40:521–529, 2009.

[4] X. Zhang, B. Tang and Y. He, Homotopy analysis method for higher-order fractional integro-differential equations, *Comput. Math. Appl.*, 62:3194–3203, 2011.

[5] Y. Nawaz, Variational iteration method and homotopy perturbation method for fourth-order fractional integro-differential equations, *Comput. Math. Appl.*, 61:2330–2341, 2011.

[6] E. A. Rawashdeh, Numerical solution of fractional integro-differential equations by collocation method, *Applied Mathematics and Computation*, 176:1–6, 2006.

[7] Y. Mahmoudi, Wavelet Galerkin method for numerical solution of nonlinear integral equation, *Appl. Math. Comput.*, 167:1119–1129, 2005.

[8] S. Yousefi and M. Razzaghi, Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations, *Math. Comput. Simulat.*, 70:1–8, 2005.

[9] X. Zheng and X. Yang, Techniques for solving integral and differential equations by Legendre wavelets, *Int. J. Syst. Sci.*, 40:1127–1137, 2009.

[10] S. G. Venkatesh, S. K. Ayyaswamy and S. Raja Balachandar, Convergence Analysis of Legendre Wavelets Method for Solving Fredholm Integral Equations, *Appl. Math. Sci.*,6:2289–2296, 2012.

[11] Y. Chen, M. Yi, and C. Yu, Error analysis for numerical solution of fractional differential equation by Haar wavelets method, *J. Comput. Sci.*, 3:367–373, 2012.

[12] H. Saeedi, M. M. Moghadam, N. Mollahasani and G. N. Chuev, A CAS wavelet method for solving nonlinear Fredholm integro-differential equations of fractional order, *Commun. Nonlinear Sci.*, 16:1154–1163, 2011.

[13] M. H. Heydari, M. R. Hooshmandasl, F. Mohammadi and C. Cattani, Wavelets method for solving systems of nonlinear singular fractional Volterra integro-differential equations, *Commun. Nonlinear Sci.*, 19:37–48, 2014.

[14] Y. Wang and Q. Fan, The second kind Chebyshev wavelet method for solving fractional differential equations, *Appl. Math. Comput.*, 218:8592–8601, 2012.

[15] F. Mohammadi, M. M. Hosseini and S. T. Mohyud-Din, Legendre wavelet Galerkin method for solving ordinary differential equations with non-analytic solution, *Int. J. Syst. Sci.*, 42:579–585, 2011.

[16] M. ur Rehman and R. A. Khan, The Legendre wavelet method for solving fractional differential equations, *Commun. Nonlinear Sci.*, 16:4163–4173,2011.

[17] S. Momani and M. A. Noor, Numerical methods for fourth-order fractional integro-differential equations, *Appl. Math. Comput.*, 182:754–760, 2006.

[18] J. C. Goswami and A. K. Chan, *Fundamentals of Wavelets: Theory, Algorithms, and Applications*, *John Wiley and Sons, Inc.*, 2011.

[19] M. Razzaghi and S. Yousefi, The Legendre wavelets operational matrix of integration, *Int. J. Syst. Sci.*, 32:49–502, 2001.

[20] A. Kilicman and Z. A. A. A. Zhour, Kronecker operational matrices for fractional calculus and some applications, *Appl. Math. Comput.*, 187:250–265, 2007.