Neural, Parallel, and Scientific Computations 23 (2015) 421-432

THE STUDY OF OPTIMAL STRATEGY FOR SELLING TICKETS FOR A SPORTING EVENT

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ABSTRACT. How to sell tickets optimally for a sporting event is a challenging problem in sports management. At Claflin University, we have regular sport events on campus. In our sport management, the goals are to maximize the income from selling tickets, and to attract more people to attend the sporting event, but we may not be able to realize these two goals at the same level. We are able to affect the tickets sale by increasing or decreasing its price. Furthermore, we have several constraints, like the capacity of a gymnasium or stadium, the acceptable price of tickets to sellers and buyers, etc. Based on the properties of this problem, the optimal control model will be adopted to help design selling strategy for some sport event. The typical solution method for optimal control model is based on solving Pontryagin Maximum Principle. Because of its nonlinearity, we analyze the Principle for our model, and develop a numerical algorithm to solve it. And at last, based on the numerical results, applicable strategy for selling tickets has been given.

AMS (MOS) Subject Classification. 35K60, 35K57

1. INTRODUCTION

In sports management, one of common and challenging problems is how to sell tickets optimally. As to different sport events, the managers may have different goals, which will be decided by the characteristics of sport events. In this paper, we studied the sport events happened on campus of Claflin University. It is basketball games between Claflin school team and other school teams. As to selling tickets for this type of sport event, the specific goals are to maximize the income from selling tickets, and to attract more people to attend the sporting event. Since we may not be able to realize these two goals at the same level, there will be different focuses between these two goals, which will be considered in our investigation. In the process of selling tickets, the sale of tickets is dynamic. It is driven by inherent economic and marketing law. It can also be affected by specific marketing strategies, which is called controls in our investigation. Thus, we are facing to optimize some goals in a controlled dynamic system, which implies optimal control model is a proper mathematics model in our investigation. Furthermore, in the modeling process we will consider other constraints, like the capacity of a gymnasium or stadium, the

Received September 15, 2015

acceptable price of tickets to sellers and buyers, etc. The typical solution method for optimal control model is based on solving Pontryagin Maximum Principle. However, in reality because of the nonlinearity of optimal control models, it is difficult to solve it analytically. In our investigation, we will develop numerical methods to solve it. And at last, based on the numerical results, applicable strategy for selling tickets has been given. Following is the key theoretic parts within optimal control models we used in the investigation.

Dynamical system is a concept in mathematics where a fixed rule describes the time dependence of a point in a geometrical space. Dynamics are used to describe the 'changing' of states, which are mathematically modeled as differential equations representing various components and the interactions between them. we can apply 'control' to affect the evolution of the dynamical system, which is called controlled dynamics:

$$\begin{cases} \dot{x}(t) & = f(x(t), u(t), t), \quad t > 0 \\ x(0) & = x^0 \end{cases}$$

where controls $u(\cdot)$ are measurable functions, but it will be approximated by piecewised functions, which is general practice in numerical experiments. As to any admissible controls, and its associated state trajectories, we can define the payroll functional:

$$J[u(t)] = \int_{t_0}^{t_f} r(x(t), u(t), t)dt$$

without loss of generality, the decision maker's goal is to find a control $u^*(t)$, which maximizes the payoff:

$$J[u(t)] \leq J[u^*(t)]$$

for all admissible controls. Optimizing above functional subject to controlled dynamics are Optimal Control models.

$$\begin{aligned} & Max_{u \in C} J(u) = \textstyle\int_{t_0}^{t_f} f(x(t), u(t), t)dt \\ subject\ to & \\ & \begin{cases} \dot{x} = & g(x(t), u(t), t) \\ x(t_0) & = \alpha \\ C = & \{u : [t_0, t_1] \to U \subset R^k\} \end{cases} \end{aligned} \tag{1.1}$$

After an optimal control model has been constructed, we should investigate the major mathematical issues of optimal control theory: (a) The existence of optimal control. (b) How to characterize optimal control mathematically? (c) How to construct an optimal control.

Optimal control models (1.1) are usually solved by Pontryagin Maximum Principle [6], which is necessary optimality condition. We define following function:

$$(\lambda_0, \lambda) = (\lambda_0, \lambda_1, \dots, \lambda_n) : [t_0, t_1] \to \mathbb{R}^{n+1},$$

with λ_0 constant, which is called costate variable. We define the Hamiltonian function $\mathcal{H} : [t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by

$$\mathcal{H}(t, x, u, \lambda_0, \lambda) = \lambda_0 f(t, x, u) + \lambda g(t, x, u)$$

The following result is fundamental necessary condition for optimal control:
Theorem 1 (Pontryagin). Let us consider the problem (1) with $f \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$ and $g \in C^1([t_0, t_1] \times \mathbb{R}^{n+k})$. Let u^* be an optimal control and x^* be the associated trajectory. Then there exists a multiplier (λ_0^*, λ^*) with

1. λ_0^* constant
2. $\lambda^* : [t_0, t_1] \to \mathbb{R}^n$ continuous

such that $(\lambda_0^*, \lambda^*) \neq (0, 0)$ and

1. Pontryagin Maximum Principle(PMP) for all$\tau \in [t_0, t_1]$ we have

$$u^*(\tau) \in argmax_{v \in U} \mathcal{H}(\tau, x^*(\tau), v, \lambda_0^*, \lambda^*(\tau))$$

$$\mathcal{H}(\tau, x^*(\tau), u^*(\tau), \lambda_0^*, \lambda^*(\tau)) = \max_{v \in U} \mathcal{H}(\tau, x^*(\tau), v, (\tau), \lambda_0^*, \lambda^*(\tau))$$

2. Adjoint equation(AE) in $\in [t_0, t_1]$ we have

$$\lambda^* = -\nabla_x \mathcal{H}$$

3. Transversality condition(TC)

$$\lambda^*(t_1) = 0$$

4. $\lambda_0^* = 1$

One of the main result about the sufficient conditions for a control to be optimal is due to Mangasarian (O. L. Mangasarian [12]).
Theorem 2 (Mangasarian). As to the maximum problem (1) with $f \in \mathcal{C}^1$ and $g \in \mathcal{C}^1$. Let the control set U be convex. Let u^* be a normal extremal control, x^* the associated trajectory and$\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ the associated multiplier. Consider the Hamiltonian function $\mathcal{H}$ and let us suppose that the function $(x, u) \mapsto \mathcal{H}(t, x, u, \lambda^*)$ is, for every $t \in [t_0, t_1]$, concave. Then u^* is optimal.

A further sufficient condition for the following particular situation of the problem is due to Arrow:

$$Max_{u \in C} J(u) = \int_{t_0}^{t_f} f(x(t), u(t), t) dt$$

subject to

$$\left\{ \begin{array}{rl} \dot{x} = & g(x(t), u(t), t) \\ x(t_0) & = \alpha \\ C = & \{u : [t_0, t_1] \to U \subset KC\} \end{array} \right. \tag{1.2}$$

with $U \subset \mathbb{R}^k$. Let the function $\mathcal{U}(t, x, \lambda) = argmax_{u \in \mathcal{U}} \mathcal{H}(t, x, u, \lambda)$, where $\mathcal{H}(t, x, u, \lambda) = f(t, x, u) + \lambda g(t, x, u)$ is the Hamiltonian. Define the maximized Hamiltonian function

$\mathcal{H}^0(t,x,\lambda) = \mathcal{H}(t,x,\mathcal{U}(t,x,\lambda),\lambda)$. The following result is from by Arrow (K. J. Arrow [1]).

Theorem 3 (Arrow). As to the maximum problem (2) with $f \in \mathcal{C}^1$ and $g \in \mathcal{C}^1$. Let u^* be a normal extremal control, x^* the associated trajectory and $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$ the associated multiplier. Consider the Hamiltonian function $\mathcal{H}^0$ and let us suppose that, for every $t \in [t_0, t_1] \times \mathbb{R}^n$, the function $x \mapsto \mathcal{H}^0(t, x, \lambda^*)$ is concave. Moreover, we suppose that the function $\mathcal{U}$ along the curve $t \mapsto (t, x^*(t), \lambda^*(t))$ is equal to u^*, i.e. $u^*(t) = \mathcal{U}(t, x^*(t), \lambda^*(t))$ $\forall t \in [t_0, t_1]$. Then u^* is optimal.

2. **Modeling**

2.1. **Analysis of Modeling background.** As to the sporting event manager, his problem is (1) to maximize the income from selling tickets, and (2) to attract more people to attend the sporting event, which implies he wants to keep the sale to a certain level. We are assuming that the population is kept unchanged in current market. And the population could be divided into three subgroups:

- Group 1: The people who have bought the tickets; the rate of change of the population of this group is $x_1(t)$. Thus, $x_1(t)$ is equivalent to sales rate of tickets.
- Group 2: The people who have not bought the tickets, but they may or may not buy. The rate of change of the population of this group is $x_2(t)$.
- Group 3: The people who will never buy the tickets. They have "immunization" to sport event. The rate of change of the population of this group is $x_3(t)$.

From Figure 1, we exhibit the immigration of population between these three groups. The rationality of migration is as follow. Group 2 could buy tickets and migrate to Group 1. Group 2 could also refuse to join this sport event and migrate to Group 3. People of Group 1 could return tickets and migrate to Group 3. The rule

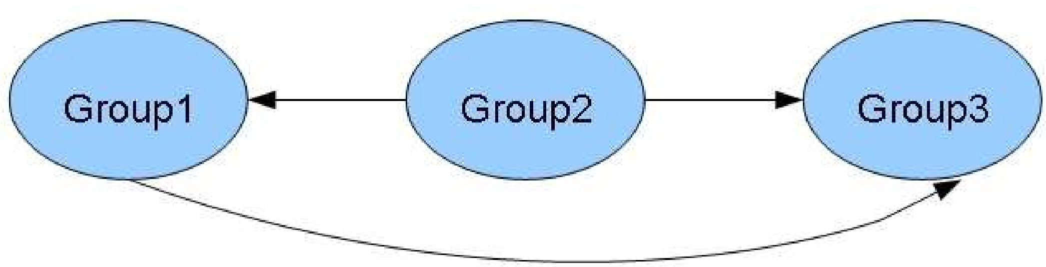

Figure 1. Immigration of population

and driving force for this immigration are as follows:

- Rule 1: The growing rate of Group 1 is proportional to the product of Group 1 and Group 2, and Group 1 also keeps same rate of migrating to Group 3, which is proportional to the population of Group 1.

$$\dot{x}_1(t) = r_{12}x_1(t)x_2(t) - r_{13}x_1(t)$$

- Rule 2: Group 2 will migrate to Group 1. Group 2 will also migrate to Group 3, and the rate is proportional to the population of Group 2.

$$\dot{x}_2(t) = -r_{12}x_1(t)x_2(t) - r_{23}x_2(t)$$

Thus, the rate of change of Group 3 is:

$$\dot{x}_3(t) = r_{13}x_1(t) - r_{23}x_2(t)$$

- Rule 3: The sporting event manager could use their control (advertisement and promotion) to affect the interaction between Group 1 and Group 2. Thus,

$$r_{12} = \alpha + \beta u(t)$$

Furthermore, based on above state variables the objective function is:

$$Max_u J(u) = \int_{t_0}^{t_1} px_1(t) - cu(t)^2 dt - w(x_1(t_f) - v)^2$$

where

p is the price of the ticket.
c is the cost of controls
w is the weight between two objectives.
v is the ideal ticket sale rate the manager wants

The rationality of construction of objective function is:

- Net profits are given by *income from selling tickets - cost of advertisement*, i.e. $px_1(t) - cu(t)^2$.
- The sports manager wants to keep the sale $x_1(t)$ to a certain level v in the end of event, so he needs to minimize $(x_1(t_f) - v)^2$.
- Parameter w is the key to express the manager's preference between these two objectives.

2.2. **Models.** Based on above analysis of modeling background, we set up optimal control model as follows:

$$Max_{u\in C} J(u) = \textstyle\int_{t_0}^{t_1} px_1(t) - cu(t)^2 dt - w(x_1(t) - v)^2$$

subject to

$$\left\{ \begin{array}{ll} \dot{x}_1(t) & = r_{12}x_1(t)x_2(t) - r_{13}x_1(t) \\ \dot{x}_2(t) & = -r_{12}x_1(t)x_2(t) - r_{23}x_2(t) \\ \dot{x}_3(t) & = r_{13}x_1(t) - r_{23}x_2(t) \\ x_1(t_0) & = 1,\ x_2(t_0) = 10,\ x_3(t_0) = 3 \end{array} \right. \tag{2.1}$$

Taken into account of computing, above model can be reduced to following optimal control model since $\dot{x}_1(t)+\dot{x}_2(t)+\dot{x}_3(t)=0$:

$$Max_{u\in C}J(u)=\int_{t_0}^{t_1}px_1(t)-cu(t)^2dt-w(x_1(t)-v)^2$$

subject to

$$\begin{cases} \dot{x}_1(t) & =r_{12}x_1(t)x_2(t)-r_{13}x_1(t) \\ \dot{x}_2(t) & =-r_{12}x_1(t)x_2(t)-r_{23}x_2(t) \\ x_1(t_0) & =1,\ x_2(t_0)=10,\ x_3(t_0)=3 \end{cases} \tag{2.2}$$

3. **Solution Methods**

The existence of optimal control is guaranteed by the convexity of objective function. We would like to solve above model and construct optimal control by Pontryagin Maximum Principle in Theorem 1. We first write out the Hamiltonian for Optimal Control model (2.2) as follows:

$$\mathcal{H}=cu^2-px_1+p_1[(\alpha+\beta u)x_1(t)x_2(t)-r_{13}x_1(t)]+p_2[-(\alpha+\beta u)x_1(t)x_2(t)-r_{23}x_2(t)]$$

Pontryagin Maximum Principle will be written explicitly as follows. State equations:

$$\begin{cases} \dot{x}_1(t) & =r_{12}x_1(t)x_2(t)-r_{13}x_1(t) \\ \dot{x}_2(t) & =-r_{12}x_1(t)x_2(t)-r_{23}x_2(t) \end{cases}$$

Co-State equations:

$$\begin{cases} \dot{p}_1(t) & =p-p_1(t)((\alpha+\beta u(t))x_1(t)-r_{13})+p_2(t)(\alpha+\beta u(t))x_2(t) \\ \dot{p}_2(t) & =-p_1(t)((\alpha+\beta u(t))x_1(t)+p_2(t)(\alpha+\beta u(t))x_1(t)+r_{23} \end{cases}$$

Optimal control:

$$2cu(t)+(p_1(t)-p_2(t))\beta x_1(t)x_2(t)=0$$

$$\Rightarrow$$

$$u(t)=\frac{1}{2c}(p_2(t)-p_1(t))\beta x_1(t)x_2(t)$$

Boundary conditions:

$$\begin{cases} x_1(t_0) & =1 \\ x_2(t_0) & =10 \\ p_1(t_f) & =2w(x_1(t_f)-v) \\ p_2(t_f) & =0 \end{cases}$$

Because above Hamiltonian is convex function of control, we obtain optimal control using $\frac{\partial\mathcal{H}}{\partial u}=0$. In above system there are 2 equations with 2 given initial conditions and 2 co-state equations with 2 terminal conditions and 1 equation for the control. We develop and revise the iterative algorithm ([3], [11]) to solve above system. The basic steps are expressed in the following process:

- Step 1 Generate initial controls from admissible controls randomly ;

- Step 2 Use control from step 1 to solve the state equations forward by Runge-Kutta;
- Step 3 Use $x_i(t_f)$ solved in second step we proceed to get $p_i(t_f)$. Then we find the co-state equations by solving the state and co-state equations backwards by Runge-Kutta methods.
- Step 4 Use the values of state and co-state variables we check if $\frac{\partial \mathcal{H}}{\partial u} = 0$ is satisfied. If yes, we have the optimal control strategies; if not, using steepest descent algorithm to get new control trajectories, repeat the above steps starting with the first step.

Based on above process, we developed the following revised steepest descent algorithm to solve optimal control model:

Step 1: Generate randomly a discrete approximation to the controls $u(t), \quad t \in [t_0, t_f]$, that is:

$$u_1(t) = u_1(t_k), \quad t \in [t_k, t_{k+1}], \quad k = 1, 2, \ldots, N$$

Step 2: use $u(t)$ to integrate the state equation forward with initial condition of state variables. The resulting state trajectory is stored as piecewise-constant vector.

Step 3: calculate $p_i(t_f)$ using $x_i(t_f)$ from $p_i(t_f) = \frac{\partial}{\partial x} h(x(t_f))$ and integrate the co-state equations backward.

Step 4: Use the discrete value of state and co-state variables $x_i(t), p_i(t)$ to evaluate $\frac{\partial \mathcal{H}}{\partial u}$.

Step 5: If $\| \frac{\partial \mathcal{H}}{\partial u} \| \leq \epsilon$, where $\| \frac{\partial \mathcal{H}}{\partial u} \|^2 = \int_{t_0}^{t_f} (\frac{\partial \mathcal{H}}{\partial u})^T (\frac{\partial \mathcal{H}}{\partial u})$, then terminate the iterative procedure and output the optimal state and control. If the stopping criterion is not satisfied, generate a new pair of piecewise constant controls given by line search:

$$u(t_{k+1}) = u(t_k) - \Delta \frac{\partial \mathcal{H}}{\partial u}(t_k), \quad k = 1, 2, \ldots, N,$$

where step length Δ will be chosen by Armijo Rule to decrease $\mathcal{H}$. Then go back to Step 2.

4. **Results**

We realized above algorithm with Matlab. In our experiment, We are interested in studying the changes of objective function values, state trajectories, and controls of sport event manager, which were brought by different preference between those two goals. We realized the different preference by choosing different values for weight w. Different values of w implies the decision maker' preference between total incoming and final sales rate. In numerical experience, we solve the model for

$$w = 0, 2, 4, 6, 8$$

And other parameters' values are as follows:

$$c = 1.2; v = 7.5; p = 2; \alpha = 1; \beta = 0.5; r_{13} = 1; r_{23} = 0.4$$

From numerical results (Table 1), we can see that the value of state variable (sale rate) at the end is increasing as w increases, which are expected since he wants the sale rate closer to what he planned. From the graph, and numerical results, we can also see that the control becomes bigger and bigger as w increases. This simply means the he needs to invest more controls to push up sales rate. Furthermore, in the Table 1 we can see the when w increases, the objective function value J decreases. This is what we expected, since bigger w implies the decision maker places more importance on final sales rate than on net profits, and he has to invest more in controls/advertisement.

Table 1. Objective Values and Terminal State Values

	$w=0$	$w=2$	$w=4$	$w=6$	$w=8$
J	5.76	5.72	5.64	5.53	5.42
$x_1(t_f)$	7.092	7.145	7.187	7.2231	7.25

In following graphs, Figure 2 are comparison of convergence trajectories of objective functions. Figure 3 are comparison of optimal control trajectories. Figure 4 are comparison of the trajectories of state variables.

5. Acknowledgement

This research was supported by Title III Undergraduate Research (SEED Project Spring 2015). I give my special thanks to my advisor Dr. Negash Medhin, from North Carolina State University, who provided insight and expertise that greatly assisted the research.

We thank Dr. Peters W. Angela, Vice Provost for Academic Programs for support and assistance with funding our research, and Mr. Saleem Sajid for fiscal assistance.

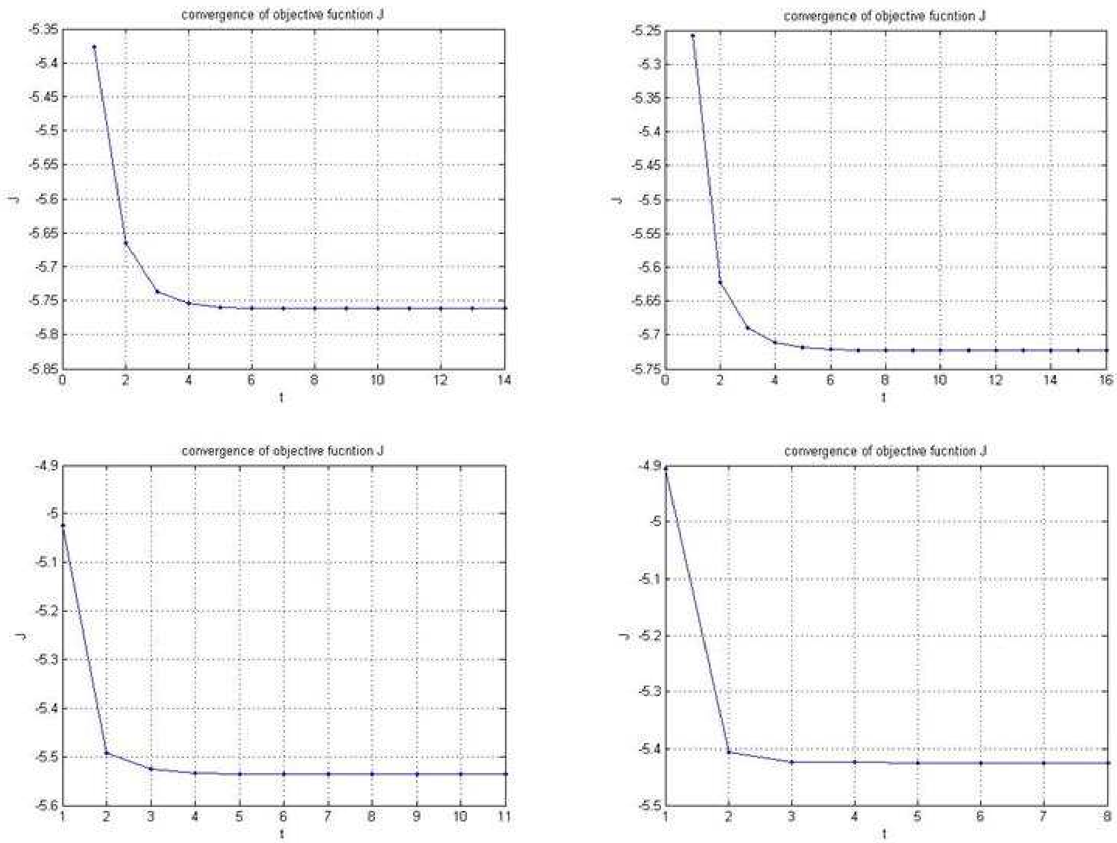

Figure 2. Convergence Trajectories of Objective Function

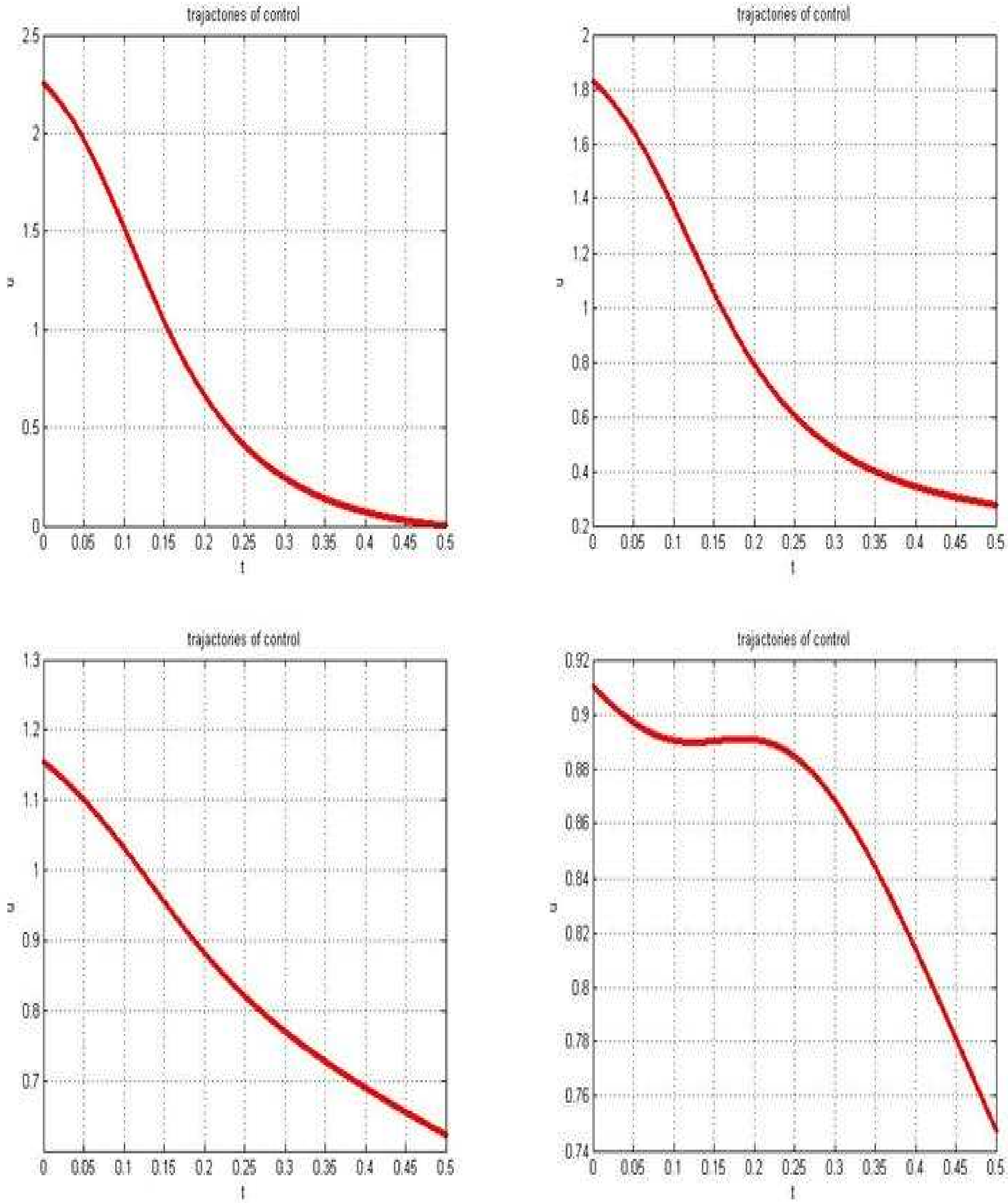

FIGURE 3. Optimal Control Trajectories

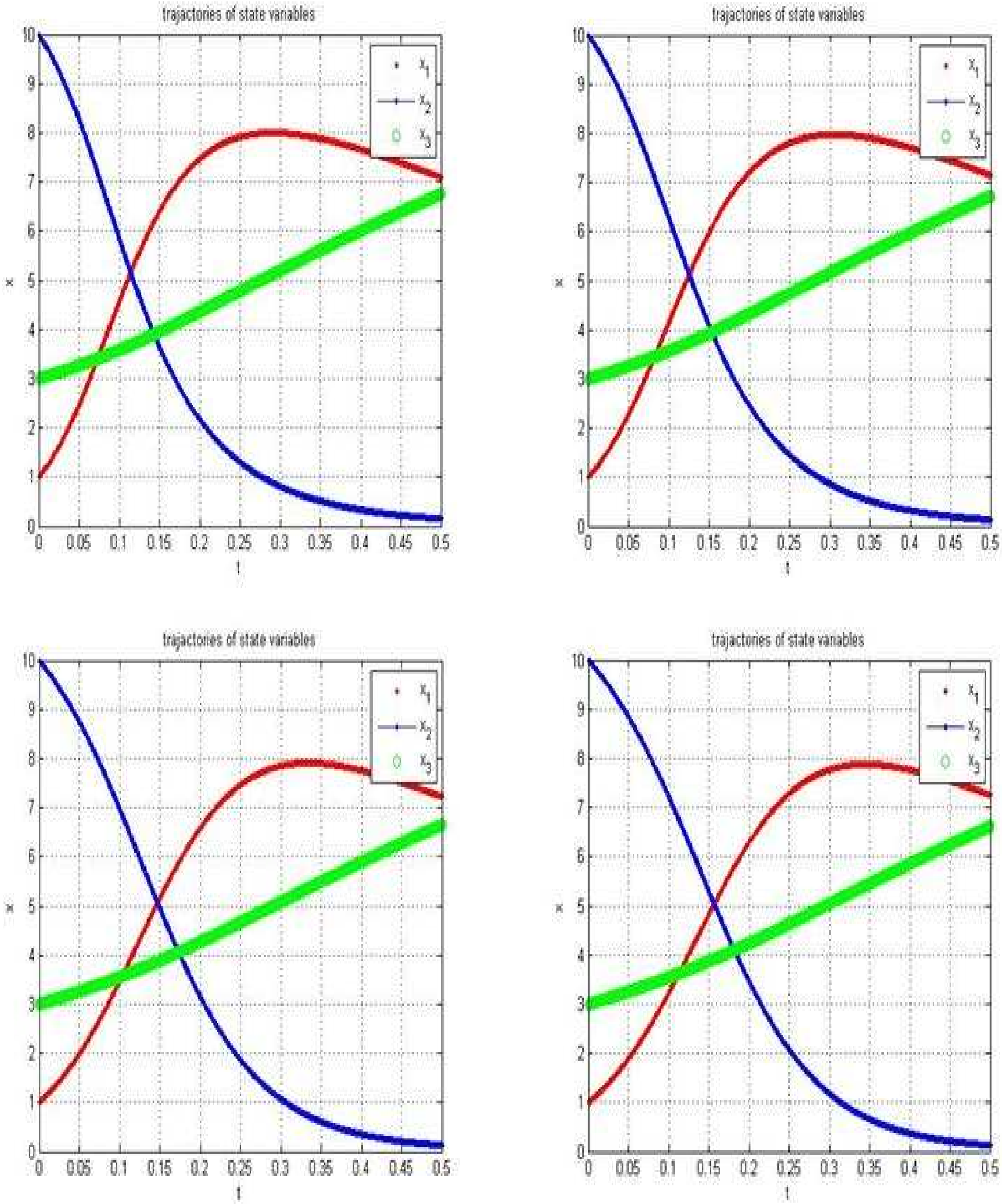

FIGURE 4. State Variables Trajectories

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