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DQM SOLUTION OF NATURAL CONVECTION FLOW OF WATER-BASED NANOFLUIDS

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ABSTRACT. In this study, unsteady natural convection heat transfer of water-based nanofluid in a square cavity with heat source at the left vertical wall is studied by solving the equation of conservation of mass, momentum and energy. Stream function-vorticity form of the governing equations are solved by using the differential quadrature method (DQM). Vorticity transport and energy equations are transformed to the form of modified Helmholtz equations by discretizing the time derivative terms first. This procedure eliminates the need of another time integration scheme in vorticity transport and energy equations, and has the advantage of using large time increments. The computational results are obtained for Rayleigh number values between 10^3 and 10^6, volume fraction of nanoparticles changing from 0 to 0.2 and the length of the heater varying from 0.25 to 1.0. Also, two types of nanoparticles (Al_2O_3 and Cu) are tested. The results are show that the type of the nanoparticles and the length of the heat source affect the flow and temperature flow.

Key Words DQM, Natural convection flow, Nanofluid.

1. INTRODUCTION

In many engineering applications such as building heating and cooling systems, heat exchangers, cooling of electronic components etc. natural convection heat transfer has an importance and it has been analyzed using many different numerical methods. Heat exchanger devices which are used in many different industries should be small and light. Beside this, they must provide higher performance. Conventional heat transfer fluids such as water and engine oil are used as a base fluid but they have low thermal conductivity. There are different kinds of techniques to enhance their heat transfer capacities. One of them is the use of mixture of base fluid and nanoparticles which is introduced by Choi [5]. They proposed that heat transfer fluids with high heat transfer properties can be obtained by adding metallic nanoparticles in conventional heat transfer fluids with low heat transfer properties. Xuan and Li [16] developed a method about preparation of nanofluid which contains nanophase powders and a base fluid. Eastman et al. [6] showed that ethylene glycol with copper nanometer-sized particles has a higher thermal conductivity than nanoparticle-containing fluids or nanofluids containing oxide particle.

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A numerical study on buoyancy-driven heat transfer enhancement in a two-dimensional enclosure utilizing nanofluids is presented by Khanafer et al. [11]. Finite volume method along with the alternating direct method is used as a numerical method for transport equations. Another numerical study for the natural convection heat transfer enhancement filled with nanofluids is done by Jou and Tzeng in [10]. In the computations, they used finite difference method for stream function-vorticity formulation of transport equations. The effect of Rayleigh number and aspect ration are examined and obtained results show that when the Rayleigh number and volume fraction of nanoparticles increase the average heat transfer coefficient increase. Tiwari and Das [15] numerically studied a two-sided lid-driven differentially heated square cavity by investigating the behavior of nanofluids and computational results are obtained using finite volume method. The study of Oztop and Abu-Nada [13] represented a numerical study for natural convection in partially heated rectangular enclosure using finite volume method. In this study, they obtained nanofluids using different kind of nanoparticles. Aminossadati and Ghasemi numerically studied natural convection of water-based nanofluids in an enclosure localised heat source at the bottom and in an inclined enclosure in [3] and [8], respectively. Finite volume approach using SIMPLE algorithm is used as a numerical method and results are given to show that the influence of pertinent parameters. Another numerical study for steady-state natural convection of water-based nanofluids in an inclined enclosure is given by Büyük Öğüt in [4]. In the study, DQM solutions are given by considering five types of nanoparticles. Dual reciprocity boundary element method (DRBEM) procedure is used in the solution of unsteady natural convection of water-based nanofluids in [9] and they used implicit Euler scheme for the time integration. A mathematical model of natural convection boundary layer flow along an inverted cone is analyzed in [7].

In this paper, we consider DQM solution of two-dimensional unsteady natural convection heat transfer of water-based nanofluid. In the governing equations, stream function, vorticity and temperature variables are used. The need of a time integration scheme is eliminated by converting the vorticity transport and temperature equations to the modified Helmholtz equations. The same idea was used by Alsoy-Akgün and Tezer-Sezgin in [1] and [2], and in these studies, natural convection and natural convection under a magnetic field problems were solved by using DRBEM and DQM, respectively. Also, unknown vorticity boundary condition are approximated by using DQM. In the computations, polynomial based DQM is used with Gauss-Chebyshev-Lobatto (GCL) mesh points which cluster through to end points and lead to more stable results compared with uniform mesh points. All the results are given for varying values of Rayleigh number, volume fraction, heater length and for two type of nanoparticles. The results are compared with previous studies in the literature.

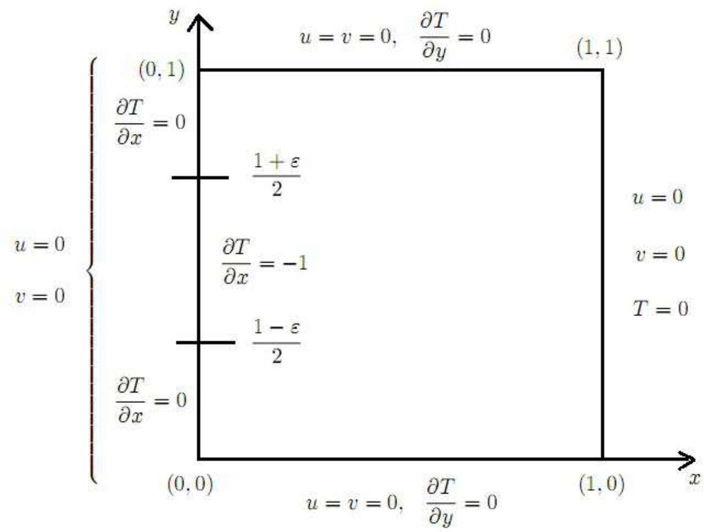

FIGURE 1. Boundary conditions for the problem.

2. GOVERNING EQUATIONS

The physical system considered in the present study is described in Figure 1. Two dimensional square enclosure filled with a nanofluid which is Newtonian, incompressible and laminar. The left wall is heated with a thermally insulated heat source which is placed at the center of the wall and the length of the heater is changed by using a parameter. The right wall is cooled and the adiabatic boundary conditions are imposed on the top and bottom walls. The water-based nanofluid contains different type of nanoparticles (Al_2O_3 and Cu) and they are assumed to have a uniform shape and size. Also, the fluid and the nanoparticles are in thermal equilibrium and no slip occur between them. The thermophysical properties of the nanoparticles and base fluid (water) are given in Table 1. The thermophysical properties of the nanofluid are assumed to be constant except for the density variation in the buoyancy forced which is determined by using the Boussinesq approximation.

The governing equations for non-dimensional natural convection flow equations in two-dimensional Cartesian coordinates can be written in terms of stream function-vorticity-temperature ($\psi - w - T$) [3, 9] as follows,

$$\begin{aligned}
&\nabla^2\psi = -w, \\
&\frac{\mu_{nf}}{\rho_{nf}\alpha_f}\nabla^2 w = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} - RaPr\frac{(\rho\beta)_{nf}}{\rho_{nf}\beta_f}\frac{\partial T}{\partial x}, \qquad (2.1)\\
&\frac{\alpha_{nf}}{\alpha_f}\nabla^2 T = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y},
\end{aligned}$$

TABLE 1. Thermophysical properties of base fluid nanoparticles [4, 13].

Physical properties	Fluid phase (Water)	Cu	Al_2O_3
$C_p(J/kgK)$	4179	385	765
$\rho(kg/m^3)$	997.1	8933	3970
$k(W/mK)$	0.613	400	40
$\beta \times 10^{-5}(1/K)$	21	1.67	2.4
$\alpha \times 10^{7}(m^2/s)$	1.47	1163.1	131.7

where velocity components u, v and vorticity w are defined as

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \qquad w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \tag{2.2}$$

where Pr and Ra are Prandtl number and Rayleigh number, respectively. The parameters α_{nf} and ρ_{nf} are thermal diffusivity and effective density of the nanofluid, respectively, and they are defined as [13]

$$\alpha_{nf} = \frac{k_{eff}}{(\rho C_p)_{nf}}, \qquad \rho_{nf} = (1-\varphi)\rho_f + \varphi\rho_s,$$

where k_{eff} is the effective thermal conductivity, C_p is the specific heat at constant pressure and φ is nanoparticle volume fraction. The subscripts 'nf', 'f' and 's' refer to nanofluid, fluid and solid, respectively. For spherical nanoparticles, the Maxwell-Garnett's model of the effective thermal conductivity of nanofluid is approximated as [13]

$$\frac{k_{eff}}{k_f} = \frac{k_s + 2k_f - 2\varphi(k_f - k_s)}{k_s + 2k_f + \varphi(k_f - k_s)}.$$

The heat capacitance of nanofluid and the thermal expansion coefficient of the nanofluid which is the part of the Boussinesq term can be given as [3, 4]

$$(\rho C_p)_{nf} = (1-\varphi)(\rho C_p)_f + \varphi(\rho C_p)_s, \qquad (\rho\beta)_{nf} = (1-\varphi)(\rho\beta)_f + \varphi(\rho\beta)_s.$$

Here, ρ, k and α represent density, thermal conductivity and thermal diffusivity of fluid or solid, respectively. The viscosity of nanofluid is obtained by using the approximation [13]

$$\mu_{nf} = \frac{\mu_f}{(1-\varphi)^{2.5}},$$

where μ_f is the dynamic viscosity of the fluid.

The average Nusselt number (Nu_{av})is determined by integrated local Nusselt number (Nu) along the heat source and they are defined as [4]

$$Nu = \frac{k_{eff}}{k_f}\frac{1}{T_s(y)}, \qquad Nu_{av} = \frac{1}{\varepsilon}\int_0^{\varepsilon} Nu dy,$$

where $T_s(y)$ is local dimensionless temperature.

3. SOLUTION PROCEDURE

Natural convection flow of water-based nanofluid equations are given in a two-dimensional square region and proper wall conditions are specified. No-slip conditions on the walls imply the zero value for stream function on the boundary ($\psi^{(0)} = \psi(x, y, t_0) = 0$). Vorticity boundary conditions are not known and obtained from vorticity definition (2.2), and for the temperature either Dirichlet type (cooled wall) or adiabatic boundary conditions are assigned on the boundary.

The time derivatives in the vorticity transport and energy equations are approximated at the beginning of the solution procedure using the forward finite difference approximation

$$\frac{\partial w}{\partial t} = \frac{w^{(n+1)} - w^{(n)}}{\Delta t} \quad \texttt{and} \quad \frac{\partial T}{\partial t} = \frac{T^{(n+1)} - T^{(n)}}{\Delta t}, \tag{3.1}$$

where $w^{(n)} = w(x, y, t_n)$, $T^{(n)} = T(x, y, t_n)$, $t_n = n\Delta t$ and Δt is the time step. Vorticity and temperature in the Laplace terms are also approximated at the two time levels by using relaxation parameters θ_w and θ_T as

$$w^{(n+1)} = \theta_w w^{(n+1)} + (1 - \theta_w)w^{(n)} \quad \text{and} \quad T^{(n+1)} = \theta_T T^{(n+1)} + (1 - \theta_T)T^{(n)},$$

in order to smooth the values between two consecutive time levels. These give two modified Helmholtz equations for the vorticity transport and energy equations. Thus, the following equations are used iteratively for solving natural convection flow of water-based nanofluid

$$\begin{aligned}
\nabla^2\psi^{(n+1)} &= -w^{(n)}, \\
\nabla^2 w^{(n+1)} - \lambda_w^2 w^{(n+1)} &= \frac{(\theta_w - 1)}{\theta_w}\nabla^2 w^{(n)} - \lambda_w^2 w^{(n)} \\
&\quad + \frac{\rho_{nf}\alpha_f}{\mu_{nf}\theta_w}\left(\frac{\partial\psi^{(n+1)}}{\partial y}\frac{\partial w^{(n)}}{\partial x} - \frac{\partial\psi^{(n+1)}}{\partial x}\frac{\partial w^{(n)}}{\partial y}\right) \\
&\quad - RaPr\frac{(\rho\beta)_{nf}\alpha_f}{\beta_f\mu_{nf}\theta_w}\frac{\partial T^{(n)}}{\partial x}, \\
\nabla^2 T^{(n+1)} - \lambda_T^2 T^{(n+1)} &= \frac{(\theta_T - 1)}{\theta_T}\nabla^2 T^{(n)} - \lambda_T^2 T^{(n)} \\
&\quad + \frac{\alpha_f}{\alpha_{nf}\theta_T}\left(\frac{\partial\psi^{(n+1)}}{\partial y}\frac{\partial T^{(n)}}{\partial x} - \frac{\partial\psi^{(n+1)}}{\partial x}\frac{\partial T^{(n)}}{\partial y}\right),
\end{aligned} \tag{3.2}$$

where $\lambda_w^2 = \frac{\rho_{nf}\alpha_f}{\mu_{nf}\Delta t\theta_w}$ and $\lambda_T^2 = \frac{\alpha_f}{\alpha_{nf}\Delta t\theta_T}$, and n indicates iteration number. Then, DQM can be used for the solution of the governing equations.

4. DQM FORMULATION OF THE PROBLEM

The main idea of the DQM, the derivative of the function $f(x)$, which is sufficiently smooth in the domain, can be approximated by using linear sum of all the functional values in the whole domain such as [14]

$$f(x_i)=\sum_{k=1}^{N} l_k(x)f(x_k), \qquad f^{(1)}(x_i)=\sum_{k=1}^{N} a_{ik}f(x_k), \qquad f^{(2)}(x_i)=\sum_{k=1}^{N} b_{ik}f(x_k),$$

where x_i's are a grid point, N is the number of the grid points, $l_k(x)$'s are weighting coefficients, $a_{ik}=l_k^{(1)}(x_i)$ and $b_{ik}=l_k^{(2)}(x_i)$. These coefficients are computed by employing some explicit formulations in [14]. If the Lagrange interpolation polynomials are taken as set of weighting coefficients, then it is called polynomial base of differential quadrature (PDQ) method.

The discretized equations employing the DQM corresponding to stream function, vorticity and temperature equations (3.2) are

$$\sum_{k=1}^{N} b_{ik}\psi_{kj}^{(n+1)}+\sum_{k=1}^{M} \bar{b}_{jk}\psi_{ik}^{(n+1)}=-w_{ij}^{(n)}, \tag{4.1a}$$

$$\sum_{k=1}^{N} b_{ik}w_{kj}^{(n+1)}+\sum_{k=1}^{M} \bar{b}_{jk}w_{ik}^{(n+1)}-\lambda_w^2 w_{ij}^{(n+1)}=b_1, \tag{4.1b}$$

$$\sum_{k=1}^{N} b_{ik}T_{kj}^{(n+1)}+\sum_{k=1}^{M} \bar{b}_{jk}T_{ik}^{(n+1)}-\lambda_T^2 T_{ij}^{(n+1)}=b_2, \tag{4.1c}$$

where

$$\begin{aligned}
b_1 &= \left(\frac{\theta_w-1}{\theta_w}\right)\left(\sum_{k=1}^{N} b_{ik}w_{kj}^{(n)}+\sum_{k=1}^{M} \bar{b}_{jk}w_{ik}^{(n)}\right)-\lambda_w^2 w_{ij}^{(n)}\\
&\quad+\frac{\rho_{nf}\alpha_f}{\mu_{nf}\theta_w}\left(\sum_{k=1}^{M} \bar{a}_{jk}\psi_{ik}^{(n+1)}\sum_{k=1}^{N} a_{ik}w_{kj}^{(n)}-\sum_{k=1}^{N} a_{ik}\psi_{kj}^{(n+1)}\sum_{k=1}^{M} \bar{a}_{jk}w_{ik}^{(n)}\right)\\
&\quad-RaPr\frac{(\rho\beta)_{nf}\alpha_f}{\beta_f\mu_{nf}\theta_w}\sum_{k=1}^{N} a_{ik}T_{kj}^{(n)},\\
b_2 &= \left(\frac{\theta_T-1}{\theta_T}\right)\left(\sum_{k=1}^{N} b_{ik}T_{kj}^{(n)}+\sum_{k=1}^{M} \bar{b}_{jk}T_{ik}^{(n)}\right)-\lambda_T^2 T_{ij}^{(n)}\\
&\quad+\frac{\alpha_f}{\alpha_{nf}\theta_T}\left(\sum_{k=1}^{M} \bar{a}_{jk}\psi_{ik}^{(n+1)}\sum_{k=1}^{N} a_{ik}T_{kj}^{(n)}-\sum_{k=1}^{N} a_{ik}\psi_{kj}^{(n+1)}\sum_{k=1}^{M} \bar{a}_{jk}T_{ik}^{(n)}\right).
\end{aligned}$$

Here, $i=1,\dots,N$, $j=1,\dots,M$, and N and M represent the total number of grid points in x- and y-directions, respectively. For the DQ method, a non-uniform grid point distribution is used which is expressed in [12] and [2].

Due to the no-slip boundary condition for velocity, the Dirichlet type boundary conditions for the stream function which is zero are inserted to the equation (4.1a) directly as $\psi_{1j}=0$, $\psi_{Nj}=0$, $\psi_{i1}=0$, and $\psi_{iM}=0$, for $i=1,\dots,N$ and $j=1,\dots,M$. The boundary conditions for the vorticity can be obtained from (2.2) and can also be approximated by the DQ method as follows

$$w_{1j}^{(n+1)} = \sum_{k=1}^{N} a_{1k} v_{kj}^{(n+1)} - \sum_{k=1}^{M} \bar{a}_{jk} u_{1k}^{(n+1)}, \quad w_{Nj}^{(n+1)} = \sum_{k=1}^{N} a_{Nk} v_{kj}^{(n+1)} - \sum_{k=1}^{M} \bar{a}_{jk} u_{Nk}^{(n+1)},$$

$$w_{i1}^{(n+1)} = \sum_{k=1}^{N} a_{ik} v_{k1}^{(n+1)} - \sum_{k=1}^{M} \bar{a}_{1k} u_{ik}^{(n+1)}, \quad w_{iM}^{(n+1)} = \sum_{k=1}^{N} a_{ik} v_{kM}^{(n+1)} - \sum_{k=1}^{M} \bar{a}_{Mk} u_{ik}^{(n+1)},$$

where $j = 1, \ldots, M$, $i = 2, \ldots, N-1$ and the velocity components are obtained from (2.2), and their DQ approximations are

$$u_{ik}^{(n+1)} = \sum_{k=1}^{M} \bar{a}_{jk} \psi_{ik}^{(n+1)}, \qquad v_{ik}^{(n+1)} = -\sum_{k=1}^{N} a_{ik} \psi_{kj}^{(n+1)}.$$

These equations are added to the equation (4.1b) resulting in an over-determined system. The Dirichlet temperature boundary conditions can be inserted directly to the equation (4.1c), but the Neumann boundary conditions (adiabatic conditions) are also expanded bu using DQM and added to the equation (4.1c) resulting in an over-determined system. Over-determined equations for vorticity and energy equations (4.1b, 4.1c) are going to be solved by using least squares method, whereas stream function equation (4.1a) is a square linear system of equations. These three equations are solved iteratively.

5. NUMERICAL RESULTS

In this work, we simulated unsteady natural convection heat transfer of water based nanofluid in a partially heated square cavity. The computational results of the problem are given for Rayleigh number values between 10^3 and 10^6, volume fraction of nanoparticles changing from 0 to 0.2, the length of the heater varying from 0.25 to 1.0 and two different type of nanoparticles (Cu and Al_2O_3). Due to the no-slip boundary conditions for the velocity, the boundary conditions of the stream function are taken zero. The vorticity boundary conditions are obtained from the vorticity definition in (2.2) by using the DQM approximations. The right wall is cooled ($T = 0$) whereas adiabatic boundary conditions are imposed on the top and bottom walls. The value of heat flux at the left wall is changing with the parameter ε. The stopping criteria to obtain the steady-state results is taken as $\epsilon = 10^{-5}$ for all variables.

At the beginning of the study, the results are obtained by changing the number of mesh points and the time increments (Δt) to obtain minimal acceptable numbers of the grid points. The computations are done by taking $Ra = 10^4$, $\varphi = 0.0$ and $\varepsilon = 0.25$ for Cu-based nanofluid. Necessary iteration numbers are given in Table 2 and the streamlines, isotherms and vorticity contours are drawn for each cases in Figure 2. From the table and figure one can see that when the number of the grid point increases we get better results but we need small time increment and more

TABLE 2. Iteration numbers and time increment (Δt) of Cu-based nanofluid when $Ra = 10^4$, $\varphi = 0.0$ and $\varepsilon = 0.25$.

Grid distribution	Δt	Iteration number
16×16	0.01	139
19×19	0.005	173
20×20	0.005	262
22×22	0.003	416

TABLE 3. Validation of the method

	$Ra = 10^4$	$Ra = 10^5$	$Ra = 10^6$
Nu_{av}	$2,224$	$4,501$	8,792

iteration to reach steady-state solutions. After comparing all results, we can say that the results for 20×20 grid points are better than the results for 19×19 grid points and close the results for 22×22 grid points. Since there is no significant difference between the solutions 20×20 grid points and 22×22 grid points, we continue to the rest of the computations for $Ra = 10^4$ with 20×20 grid points. At the rest of the study, we used minimal acceptable numbers of grid points and time increment for all other cases. Also obtained numerical result is validated by comparing the average Nusselt numbers for the left wall with the study of Öğüt [4]. From the Table 3 that DQM results are good agreement with the ones given in this study.

To show the effect of the Rayleigh number on nanofluids, numerical experiment were performed, as given in Figure 3. The results are given at steady-state for $Ra = 10^3 - 10^6$ with $\varphi = 0.2$ and $\varepsilon = 0.25$. From the figure one can say that, for low Rayleigh number there is not enough convection in the system, so the viscous forces are dominating the system. When $Ra = 10^3$, the vortex of the streamlines takes the shape of a circle and the minimum value is obtained at the center of the cavity. From the vorticity contour lines it can be seen that a circular vortex occur at the center of the cavity. At the right side of the cavity, the isotherms are perpendicular to the top and the bottom walls due to the adiabatic boundary conditions but near the heater they tend to develop loops. When Rayleigh number increases, the circular vortex of streamlines becomes an ellipse in the clockwise direction and boundary layer occur near the vertical walls. The center vortex of the vorticity starts divide two parts and these two new vortices move towards the right bottom and left upper corners. Also, boundary layer occur with the increase of the value of vorticity near the vertical walls. Due to the high value of Rayleigh number the isotherms become horizontal and form boundary layer take place near the heater and right wall. Therefore, these behavior indicating that the convection forces starts to dominate the system over the viscous forces. These results are physically expected behaviors and they are good agreement with the ones in [4].

The effect of volume fraction and the heater length on streamline, vorticity and temperature for different values of Rayleigh number are given in Figure 4 and Figure 5. In the analysis Cu is used as nanoparticle. The presence of the nanoparticle has an important effect on the flow and temperature patterns. When the volume fraction increases, the energy change increasing. Thus, flow and isotherm strength increases but vorticity is not effected as much as the others. It is clear that streamlines, vorticity and temperature contourlines are influenced by the changing of the heater length. When the length of the heater increases, because there is more heat transfer, the temperature of the system increases.

Also, in order to show that the effects of type of nanoparticle, the same analysis is done by using Al_2O_3 as a nanoparticle and results are given in Figure 6. When the heat transfer rates of Cu-based nanofluid and Al_2O_3-based nanofluid are compared one can see that greater heat transfer rate is obtained by using Cu-based nanofluid. This is expected behavior because Cu has higher thermal conductivity than Al_2O_3. All these behaviors are observed in [4] and [13].

6. **CONCLUSION**

In this paper, unsteady natural convection of water-based nanofluid in a partially heated square enclosure from the left wall is numerically studied using DQM. In the DQM procedure, solutions and their derivatives are interpolated by using polynomials. Therefore, it is quite simple in terms of computation and enables us to use considerably small number of the grid points. At the beginning of the solution procedure, the forward finite difference approximations are used for the time derivatives. Vorticity and temperature variables which are located in the Laplace terms are approximated with relaxation parameters. Thus, two modified Helmholtz equations are obtained for vorticity transport and temperature equations. By using this procedure, the need of a time integration scheme is eliminated. Computational results are obtained to show the effects of Rayleigh number, volume fraction, heater length and type of nanoparticle. It is observed that the heat rate transfer can be increased by using nanoparticles. In particular, the heat transfer rate can be increased by using Cu as a nanoparticle. Also, heat transfer can be increased using higher values of Rayleigh number and increasing the heater length.

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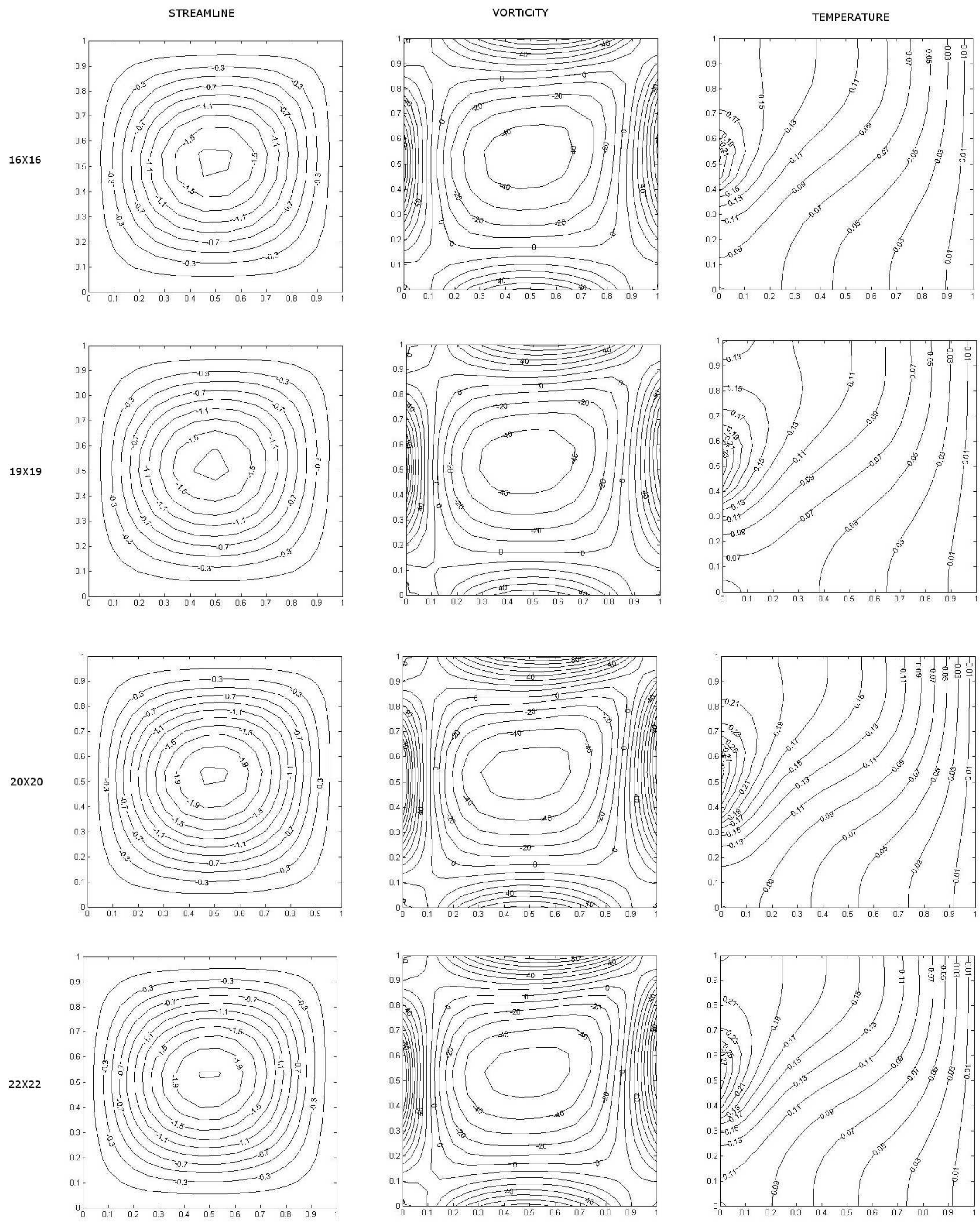

FIGURE 2. Cu-based nanofluid for grid distribution: 16×16, 19×19, 20×20 and 22×22 when $Ra = 10^4$, $\varphi = 0.0$ and $\varepsilon = 0.25$.

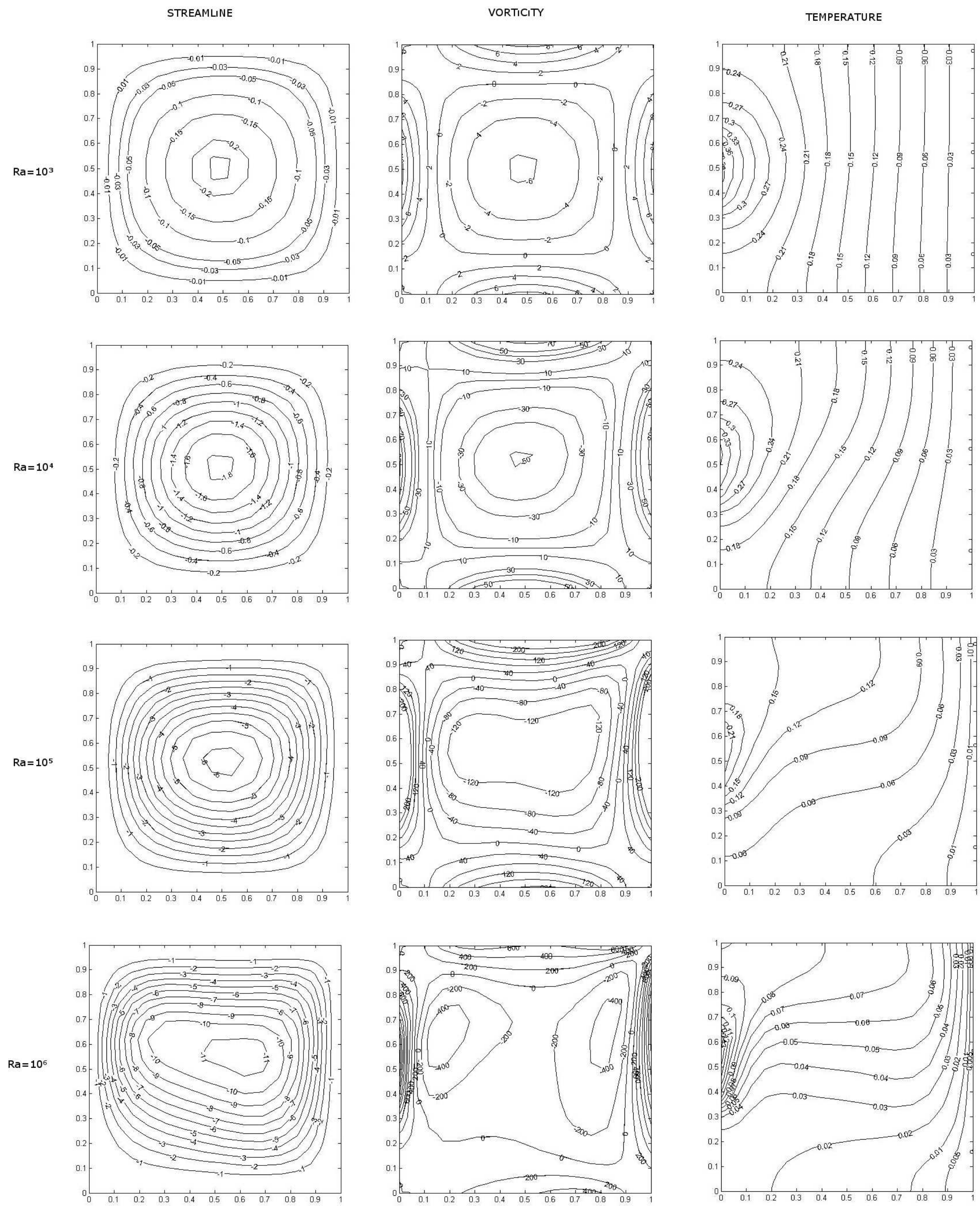

FIGURE 3. Cu-based nanofluid for several Rayleigh numbers when $\varphi = 0.2$ and $\varepsilon = 0.25$.

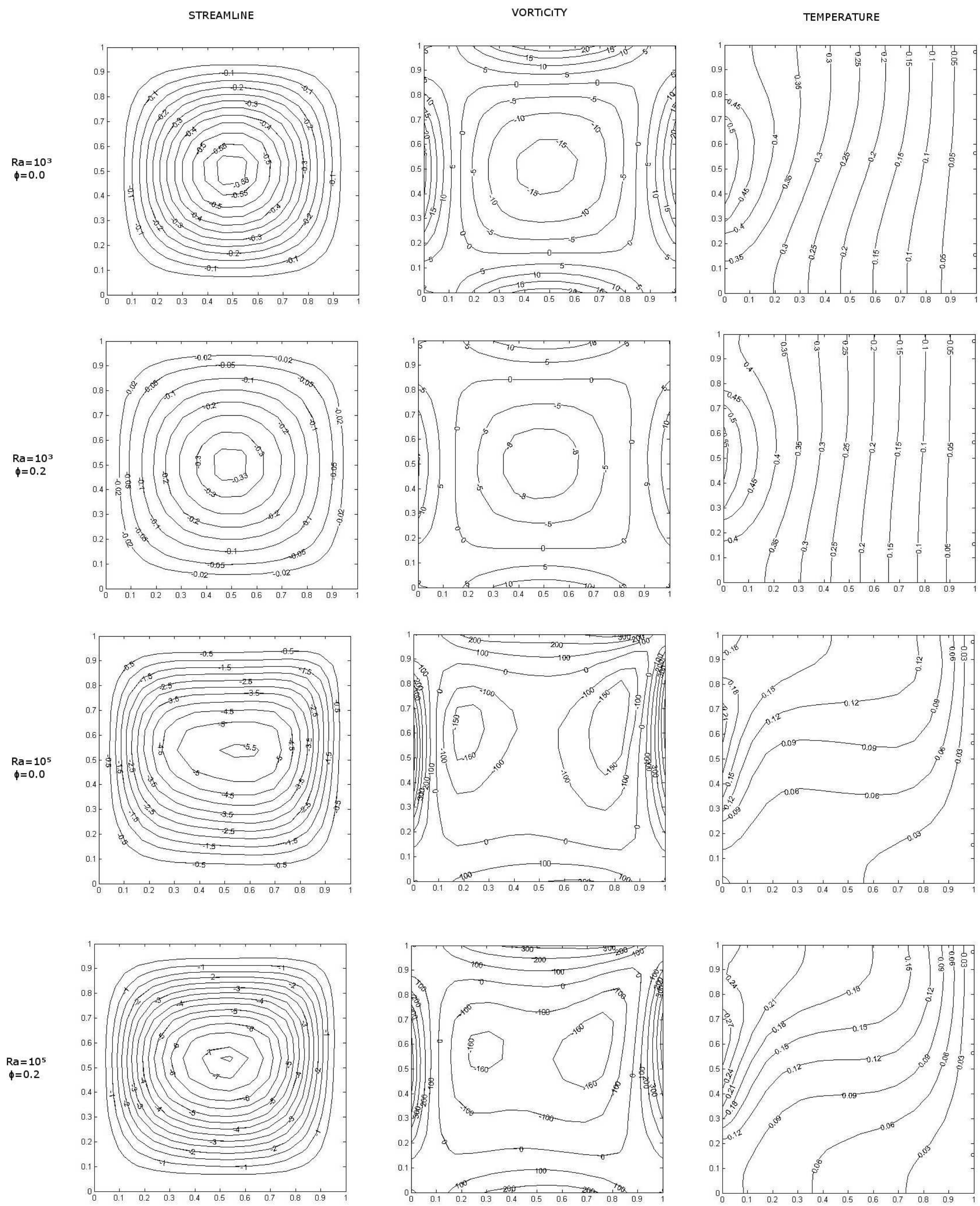

FIGURE 4. *Cu*-based nanofluid for several Rayleigh numbers and φ when $\varepsilon = 0.5$.

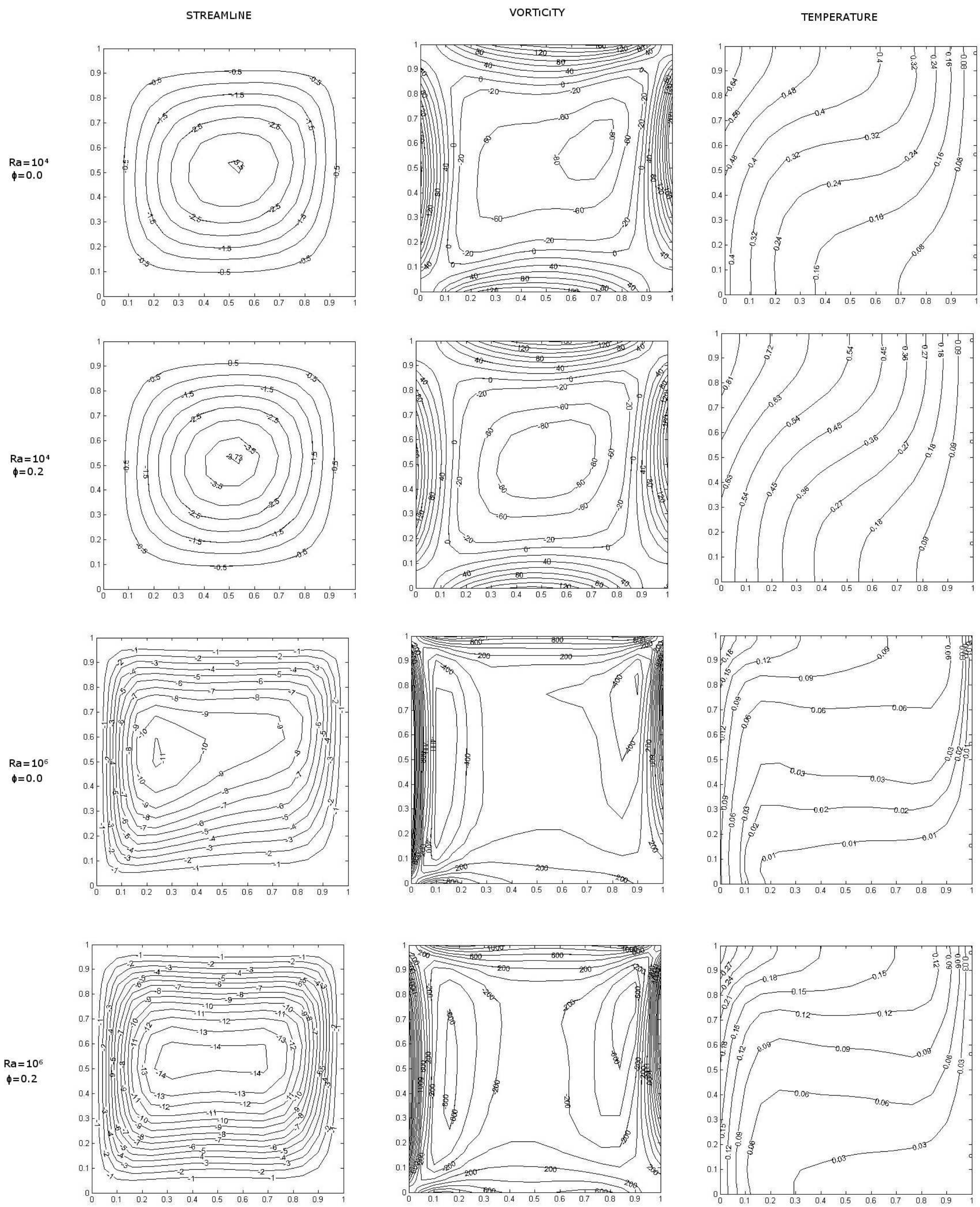

FIGURE 5. Cu-based nanofluid for several Rayleigh numbers and φ when $\varepsilon = 1$.

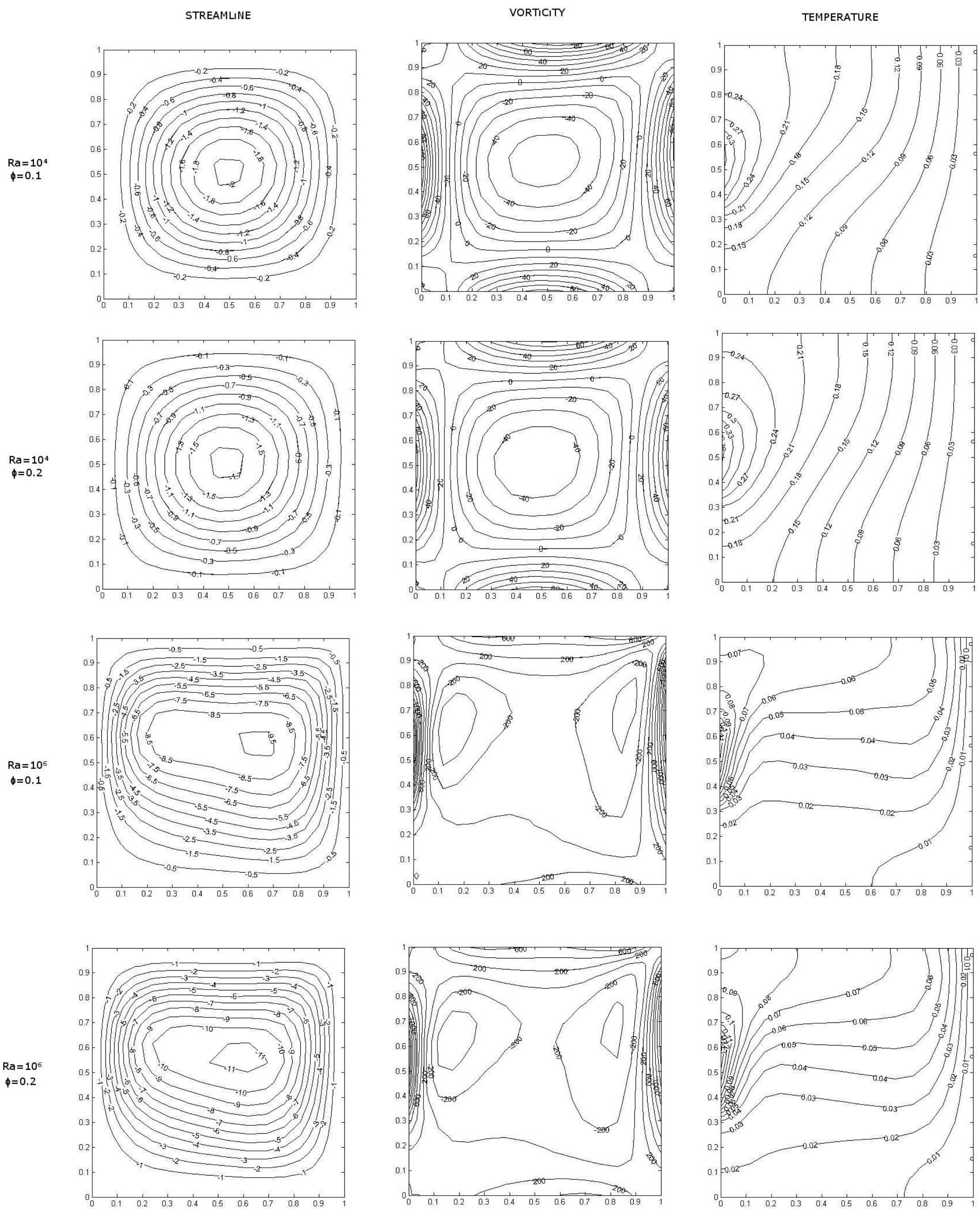

FIGURE 6. Al_2O_3-based nanofluid for several Rayleigh numbers and φ when $\varepsilon = 0.25$.