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AN ASYMPTOTIC NUMERICAL METHOD FOR SINGULARLY PERTURBED FOURTH ORDER ODE OF CONVECTION-DIFFUSION TYPE TURNING POINT PROBLEM

M. CHANDRU AND V. SHANTHI

Department of Mathematics, National Institute of Technology,
Tiruchirappalli-620 015, Tamil Nadu, India

ABSTRACT. In this paper, we consider a singularly perturbed turning point problem of convection-diffusion of fourth-order ordinary differential equations with a small positive parameter(ε). The given fourth-order boundary value problem is transformed into a system of weakly coupled systems of two second-order ordinary differential equations, one without parameter and other with parameter ε multiplying highest derivatives with suitable boundary conditions. A computational method is presented for solving the system of both linear and non-linear problems. In the linear case, we first find the zero-order asymptotic approximation expansion of the solution in the second equation. Then, the second equation of the system is solved by the numerical method which is constructed for this problem which involves Shishkin mesh. As in the case of non-linear problem, Newton's method of quasilinearization is applied for the second equation of the system. Numerical results are presented which support the theoretical results.

Key Words. Singularly Perturbed Turning Point Problem, Fourth-Order Differential Equation, Asymptotic Expansion Approximation, Convection-Diffusion, Finite Difference Scheme, Linear and Non-Linear.

AMS Subject Classification. 65L10

1. INTRODUCTION

For the past three decades, numerical treatment of Singularly Perturbed Ordinary Differential Equations (SPODEs) has been pursued. This type of problems occurs in applied mathematics that arise in diverse areas, including linearized Navier-Stokes equation at high Reynolds number, heat transport problems with large peclet numbers, magneto-hydrodynamic duct problems at Hartman numbers and the drift-diffusion equation of semi conductor device modeling. Several research articles have been published in the past two decades on non-classical methods, which cover mostly second-order equations. The research paper [1] presents the detailed review on Singularly Perturbed Turning Point Problem (SPTPP). The author reviewed some existing literatures on asymptotic and numerical analysis of SPTPP and interior layer problem. In specific, the authors investigated interior TPP, boundary TPP (Geophysics)

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and considered many real time applications related to one-dimensional second order singularly perturbed two point Boundary Value Problem (BVP) with TPP (both interior and boundary) on linear, semi-linear, quasi-linear and non-linear types. This paper uses a positive variable ε, which will be multiplied the highest derivative of the differential equation.

The classification of singularly perturbed higher-order problems depends on how the order of the original equation is affected if one set $\varepsilon = 0$ [2]. The problem is said to be convection-diffusion type, if the order is reduced by one and it is reaction-diffusion type, when the order is reduced by two. According to the knowledge of author on literature survey, there is no research work available on singularly perturbed higher-order BVP with TPP. However, only a few authors have developed numerical methods for singularly perturbed higher-order differential equations without TPP [3, 4, 5] and asymptotic numerical technique has been used to solve the problem. Boundary and interior layer are usually present in the solutions of Singular Perturbation Problems (SPPs). The solution varies rapidly when the layer is present, whereas it behaves regular and varies gradually when it is away from the layer. Therefore many complications may be faced in solving singularly perturbed boundary value problem using standard numerical methods.

In the recent years, large number of special purpose methods have been proposed to provide accurate results. To solve these problems, either additional information about the solution could be used to produce accurate efficient methods, or an attempt may be made to produce a posteriori adaptive method [6]. The former method may involve a priori modification of the mesh or the operator. Some recent works have been done in singularly perturbed non TPPs with interior layers. The authors of [7] examined a singularly perturbed BVP of reaction-diffusion type with discontinuous source term. Further, the authors have used Hybrid Difference Scheme (HDS) [8] in order to improve the order of accuracy obtained in [7]. Two parameter problem has been proposed using HDS in [9] for getting an almost second order convergence. In [10], the authors discussed a singularly perturbed convection-diffusion problem with Robin type boundary condition. The detailed information on analytical and numerical treatment of SPPs is available in [2, 11, 12, 13, 14, 15].

In general, the numerical treatment of TPP is more difficult than SPPs without turning points, because the co-efficient of the convection term vanishes inside the domain of interest. Natesan proposed a numerical method for a second order SPTPP [6] and suggested some computational techniques. Geng et al. have proposed reproducing kernel method for solving SPTPPs with interior layer [16] and the authors mentioned few recent articles on SPTPP. Linß considered singularly perturbed semi-linear boundary value problem in [17] and [18]. Liu introduced geometric approach to solve singular boundary value problem with turning points [19]. Both linear and nonlinear singularly perturbed two point boundary turning point convection diffusion type are discussed in [20]. In [21], Rai and Sharma described a numerical method based on fitted operator finite difference scheme for the boundary value problems for singularly perturbed delay differential equations with turning point and mixed shifts. Ghoul et al. [22] studied singularly perturbed quasilinear boundary value problem with interior shock layer. The authors introduced an appropriate stretching transformation and constructing interior layer corrective terms to solve this problem. Recently, Riordan and Quinn deal with linear singularly perturbed interior turning point problem with a continuous convection coefficient in [23], which motivates the author of this paper to solve the class of singularly perturbed boundary value problems of the following type.

$$-\varepsilon y^{iv}(x)+a(x)y'''(x)+b(x)y''(x)+c(x)y(x)=f(x), \quad x\in\Omega:=(-1,1), \tag{1.1}$$

$$y(-1)=p, \quad y(1)=q, \quad y''(-1)=-r, \quad y''(1)=-s, \tag{1.2}$$

with $y\in C^4(\Omega)\cap C^2(\overline{\Omega})$. The functions $a(x)$, $b(x)$, $c(x)$ and $f(x)$ belong to $C^4(\bar{\Omega})$ and they satisfy the following conditions:

$$a(0)=0, a'(0)\leq 0, |a(x)|\geq\alpha>0, \text{ for } 0<|x|\leq 1, \tag{1.3}$$

$$|a'(x)|\geq \tfrac{|a'(0)|}{2} \quad \forall\, x\in\bar{\Omega} \tag{1.4}$$

$$b(x)\geq\beta>0, \quad \forall\, x\in\bar{\Omega} \tag{1.5}$$

$$0\geq c(x)\geq-\gamma, \gamma>0, \tag{1.6}$$

$$\alpha-\theta\gamma\geq\lambda>0, \quad \theta>1 \text{ is arbitrarily close to 1, for some } \lambda. \tag{1.7}$$

With the above assumptions, the TPP (1.1–1.2) possess an unique solution exhibiting two boundary layer of exponential type at both end points $x=[-1,1]$ [24].

The rest of the paper is organized as follows. Section 2 provides analytic behavior of the solution of SPP (1.1–1.2). Some analytical and numerical results for TPP for second order ODEs are given in Section 3. Necessary computational method is described in Section 4. Section 5 discusses the estimate of error in detail. Section 6 gives an alternative procedure to problem (1.1–1.2) when condition (1.6) is not met. Non-linear problem is considered in Section 7. Numerical examples are given in Section 8. The paper ends with a conclusion in Section 9. Throughout this paper, C denotes a generic positive constant that is independent of nodal point (i), mesh size (h) and singular perturbation parameter ε.

Remark 1.1. The solution of the problem (1.1–1.2) exhibits a boundary layer at $x=-1$ which is less severe because the boundary conditions of Neumann type are presented in [2] for the derivative of the solution. The condition (1.3) says that the problem (1.1–1.2) is a turning point problem. The condition (1.6) is imposed to ensure that the below system (2.1–2.2), which is equivalent to (1.1–1.2) is known as the quasi monotonicity condition [2].

2. SOME ANALYTICAL RESULTS

In this section we states some analytical results for the solution $\overline{y}$ and its derivatives. Problem (1.1–1.2) can be transformed into an equivalent problem of the form

$$A\overline{y} = F \Leftrightarrow \begin{cases} P_1\overline{y} := -y_1''(x) - y_2(x) = 0, \ x \in \Omega \\ P_2\overline{y} := \varepsilon y_2''(x) - a(x)y_2'(x) - b(x)y_2(x) \\ +c(x)y_1(x) = f(x), \ x \in \Omega, \end{cases} \tag{2.1}$$

$$\begin{cases} R_1\overline{y} := y_1(-1) = p, \quad R_2\overline{y} := y_1(1) = q, \\ R_3\overline{y} := y_2(-1) = r, \quad R_4\overline{y} := y_2(1) = s, \end{cases} \tag{2.2}$$

where $\overline{y} = (y_1, y_2)$. Provided the conditions (1.3–1.7) hold good to the problem (2.1–2.2).

This section presents the maximum principle and stability result on the solution for problem (2.1–2.2) and obtains asymptotic expansion approximation for the problem (2.1–2.2). Further, estimates for the derivatives of the solutions are given.

2.1. Maximum Principle and Stability Result.

Theorem 2.1 (Maximum Principle)**.** *Consider the BVP* (2.1–2.2)*. If $y_1(-1) \geq 0$, $y_1(1) \geq 0$, $y_2(-1) \geq 0$, $y_2(1) \geq 0$, $P_1\overline{y}(x) \geq 0$ and $P_2(x)\overline{y} \geq 0$ for $x \in \Omega$ implies that $\overline{y}(x) \geq 0$ for $x \in \overline{\Omega}$.*

Proof. Define a test function $\bar{s}(x) = (s_1(x), s_2(x)) > 0$ by,

$$\begin{aligned} s_1(x) &= 2(1+\eta)\left(1 - \frac{x^2}{2}\right), \quad \text{where } 0 < \eta \ll 1, \ x \in (-1,1) \\ s_2(x) &= \begin{cases} 1 + \left(\dfrac{\eta}{2} - x\right), & x \in (-1,0) \\ 1 + \left(\dfrac{\eta}{2} + x\right), & x \in (0,1) \end{cases} \end{aligned}$$

where $0 < 2\eta < 1$, $x \in \overline{\Omega}$. Then it is easy to check that $\overline{y}(x) \geq 0$ for $x \in \overline{\Omega}$, $P_1\overline{s}(x) > 0$, and $P_2\overline{s}(x) > 0$, $x \in \Omega$. Now assume that the theorem is not true and define

$$\xi = \max\left\{\max_{x\in\overline{\Omega}}\left(\frac{-y_1}{s_1}\right)(x), \max_{x\in\overline{\Omega}}\left(\frac{-y_2}{s_2}\right)(x)\right\}.$$

Then $\xi > 0$. Also $(y_1 + \xi s_1)(x) \geq 0$ and $(y_2 + \xi s_2)(x) \geq 0$ for $x \in \overline{\Omega}$. Furthermore, there exists a point $x_0 \in \overline{\Omega}$ such that $(y_1 + \xi s_1)(x_0) = 0$ or $(y_2 + \xi s_2)(x_0) = 0$ for $x_0 \in \overline{\Omega}$.

Case 1: $(y_1 + \xi s_1)(x_0) = 0$ for $x_0 \in \Omega$. This implies that $(y_1 + \xi s_1)$ attains its minimum at $x = x_0$. Then $0 < P_1(\overline{y} + \xi\overline{s}) = -(y_1 + \xi s_1)''(x_0) - (y_2 + \xi s_2)(x_0) \leq 0$, a contradiction.

Case 2: $(y_2 + \xi s_2)(x_0) = 0$ for $x_0 \in \Omega$. This implies that $(y_2 + \xi s_2)$ attains its minimum at $x = x_0$. Then $0 < P_2(\overline{y} + \xi\overline{s}) = \varepsilon(y_2 + \xi s_2)''(x_0) - a(x_0)(y_2 + \xi s_2)'(x_0) - b(x_0)(y_2 + \xi s_2)(x_0) + c(x_0)(y_1 + \xi s_1)(x_0) \leq 0$, a contradiction. Hence it can be concluded that $\overline{y}(x) \geq 0$ for $x \in \overline{\Omega}$. $\square$

Lemma 2.2. *If $\overline{y}(x)$ is a smooth function then*

$$|\overline{y}(x)| \leq C \max\left\{ |y_1(-1)|, |y_1(1)|, |y_2(-1)|, |y_2(1)|, \max_{x\in\Omega} |P_1\overline{y}(x)|, \max_{x\in\Omega} |P_2\overline{y}(x)|, \right\} \quad \forall\ x \in \overline{\Omega}.$$

Proof. Using the barrier function, $\overline{\omega}^{\pm} = (\omega_1^{\pm}(x), \omega_2^{\pm}(x))$ by,

$$\begin{aligned} \omega_1^{\pm}(x) &= 2(1+\eta)\left(1 - \frac{x^2}{2}\right) A \pm y_1(x), \quad \text{where } 0 < \eta \ll 1 \ x \in (-1,1) \\ \omega_2^{\pm}(x) &= \begin{cases} 1 + \left(\dfrac{\eta}{2} - x\right) A \pm y_2(x), & x \in (-1,0) \\ 1 + \left(\dfrac{\eta}{2} - x\right) A \pm y_2(x), & x \in (0,1) \end{cases} \end{aligned}$$

where,

$$A = C \max\left\{ |y_1(-1)|, |y_1(1)|, |y_2(-1)|, |y_2(1)|, \max_{x\in\Omega} P_1\overline{y}(x), \max_{x\in\Omega} P_2\overline{y}(x) \right\},$$

it is easy to prove the above stability result. $\square$

2.2. **Asymptotic Expansion Approximation.** One can look for an asymptotic expansion solution of the BVP (2.1–2.2) in the form.

$$y(x,\varepsilon) = (\overline{u}_0(x) + \overline{v}_0(x)) + \varepsilon(\overline{u}_1(x) + \overline{v}_1(x)) + O(\varepsilon^2).$$

By the method of stretching variable [12] one can obtain a zero order asymptotic expansion approximation $\overline{y}_{as} = (\overline{u}_0, \overline{v}_0)$ where $\overline{u}_0$ is the solution of the reduced problem of (2.1–2.2) given by

$$\begin{cases} -u_{01}''(x) - u_{02}(x) = 0, \\ -a(x)u_{02}'(x) - b(x)u_{02}(x) + c(x)u_{01}(x) = f(x), \end{cases} \tag{2.3}$$

$$u_{01}(-1) = p, u_{01}(1) = q, u_{02}(1) = s \tag{2.4}$$

and $\overline{v}_0$ is a layer correction term given by $\overline{v}_0 = (v_{01}, v_{02})$ with

$$\begin{cases} v_{01}(x) = \dfrac{-\varepsilon^2}{a(-1)^2}[r - u_{02}(-1)]e^{a(-1)x/\varepsilon}, \\ v_{02}(x) = [r - u_{02}(-1)]e^{a(-1)x/\varepsilon}, \end{cases} \tag{2.5}$$

$$\begin{cases} -v_{01}''(\eta_1) - v_{02}(\eta_1) = 0, \\ -v_{02}''(\eta_1) + a(-1)v_{02}'(\eta_1) = 0, \text{ and } \eta_1 = x/\varepsilon, \end{cases} \tag{2.6}$$

$$\begin{cases} v_{01}(-1) = \dfrac{-\varepsilon^2}{a(-1)^2} v_{02}(-1), v_{02}(1) = \dfrac{-\varepsilon^2}{a(-1)^2} v_{02}(1), \\ v_{02}(-1) = r - u_{02}(-1) = 0 \text{ and } v_{02}(1) = v_{02} e^{a(-1)x/\varepsilon} \end{cases} \tag{2.7}$$

The following theorem gives the bound for the difference between the solution of the BVP (2.1–2.2) and its zero order asymptotic expansion approximation.

Remark 2.3. If (u_{01}, u_{02}) is the solution (2.3–2.4), then u_{01} is the solution of the BVP

$$-u_{01}''' - (b(x)/a(x))u_{01}''(x) + (c(x)/a(x))u_{01}(x) = f(x)/a(x), \tag{2.8}$$

$$u_{01}(-1) = p, u_{01}(1) = q, u_{01}''(1) = -s. \tag{2.9}$$

In the following it is assumed that BVP (2.8–2.9) can be solved exactly and closed form solution is available. This problem has a unique solution $u_{01} \in C^0(\overline{\Omega}) \cap C^3(\Omega) \cap C^2(-1, 1]$ [6].

Theorem 2.4. *The zero order asymptotic expansion approximation* $\overline{y}_{as} = \overline{u}_0 + \overline{v}_0$ *of the solution* $\overline{y}$ *of the BVP* (2.1–2.2)*, defined by* (2.3)–(2.6)*, satisfies the inequality*

$$|\overline{y}(x) - \overline{y}_{as}(x)| \leq C\varepsilon \quad \forall\ x \in \overline{\Omega}$$

Proof. Defining barrier functions $\overline{\psi}^{\pm}(x) = (\overline{\psi}_1^{\pm}(x), \overline{\psi}_2^{\pm}(x))$ for $x \in \overline{\Omega}$ by

$$\overline{\psi}_1^{\pm}(x) = C_1 \left[2(1+\eta)\left(1 - \frac{x^2}{2}\right)\right]\varepsilon + C_2\varepsilon^2 \left[1 - (1/2)e^{-\alpha x/2\varepsilon}\right] \pm (y_1 - y_{1,as})(x),$$

$$\overline{\psi}_2^{\pm}(x) = \begin{cases} C_1 \left[1 + \dfrac{\eta}{2} - x\right]\varepsilon + C_2\varepsilon^2 \left[e^{-\alpha x/2\varepsilon}\right] \pm (y_2 - y_{2,as})(x), \ x \in (-1, 0) \\ C_1 \left[1 + \dfrac{\eta}{2} + x\right]\varepsilon + C_2\varepsilon^2 \left[e^{-\alpha x/2\varepsilon}\right] \pm (y_2 - y_{2,as})(x), \ x \in (0, 1) \end{cases}$$

for a suitable choice of C and by Theorem 2.1 we have the required result. □

3. SOME ANALYTICAL AND NUMERICAL RESULTS FOR TPP OF SECOND ORDER ODEs

In this section we presents results for the following singularly perturbed TPP of second order in the interval $[-1, 1]$, their forms are not changed when the equation is solved on any arbitrary closed and bounded intervals.

Consider the SPBVP

$$Ly_2^* \equiv \varepsilon y_2^{*''}(x) - a(x)y_2^{*'}(x) - b(x)y_2^*(x) = f(x) - c(x)u_{01}(x), \ x \in (-1, 1), \tag{3.1}$$

$$y_2^*(-1) = r, \ y_2^*(1) = s, \tag{3.2}$$

where $u_{01}(x)$ is the solution of the BVP (2.8–2.9).

Remark 3.1. The BVP (3.1–3.2) has a unique solution $y_2^* \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ [6].

3.1. **Analytical results.**

Theorem 3.2. *If* (y_1, y_2) *and* y_2^* *are solutions of the BVPs* (2.1–2.2) *and* (3.1–3.2), *respectively, then*

$$|y_2(x) - y_2^*(x)| \leq C\varepsilon, \quad x \in \overline{\Omega}$$

Proof. Since (y_1, y_2) is the solution of (2.1–2.2), then y_2 satisfies the BVP

$$\varepsilon y_2''(x) - a(x)y_2'(x) - b(x)y_2(x) = f(x) - c(x)y_1(x), \quad x \in (-1, 1),$$
$$y_2(-1) = r, \quad y_2(1) = s.$$

Further the function $w = y_2 - y_2^*$ satisfies the BVP

$$\varepsilon w''(x) - a(x)w'(x) - b(x)w(x) = -c(x)[y_1(x) - u_{01}(x)],\ x \in (-1, 1),$$
$$w(-1) = 0, \quad w(1) = 0.$$

From the stability result [11] we have

$$|w(x)| \leq C\varepsilon,$$

that is,

$$|y_2(x) - y_2^*(x)| \leq C\varepsilon.$$

□

3.2. **Numerical Results.**

In this section a second order singularly perturbed TPP (3.1–3.2) is discretized using classical finite difference scheme on piecewise uniform mesh. Considering the SPBVP (3.1–3.2), we show that one can obtain ε-uniform convergence for the classical scheme, when it is applied on piecewise uniform meshes known as Shishkin meshes. Consider the classical upwind scheme on a piecewise uniform mesh $\Omega_\varepsilon^N = \{x_i\}_i^{N-1}$, $N \geq 4$ which is constructed by dividing the domain $\overline{\Omega}$ into three subintervals $\Omega_L = [-1, -1+\tau]$, $\Omega_C = [-1+\tau, 1-\tau]$ and $\Omega_R = [1-\tau, 1]$, such that $\overline{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$. The transition parameter τ is chosen to be

$$\tau = \min\left\{\frac{1}{4}, K\varepsilon \ln N\right\}, \quad K \geq \frac{1}{\min\{\alpha, \beta\}} \quad \text{in } \overline{\Omega}. \tag{3.3}$$

Then Ω_ε^N is obtained by putting a uniform mesh with $N/4$ elements in both Ω_L and Ω_R a uniform mesh with $N/2$ in Ω_C. The resulting fitted finite difference scheme for the TPP (3.1–3.2) is given below:

$$L^h y_{2,i}^* = -\varepsilon\delta^2 y_{2i}^* + a(x_i)D^* y_{2i}^* + b(x_i)y_{2i}^* f(x_i) - c(x_i)u_{01i},\ x_i \in (-1, 1), \tag{3.4}$$

$$y_{20}^* = r, \quad y_{2N}^* = s, \tag{3.5}$$

where

$$\begin{aligned} D^+Z_i &= \frac{Z_{i+1}-Z_i}{x_{i+1}-x_i}, \quad D^-Z_i = \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}, \\ \delta^2 Z_i &= \frac{2(D^+Z_i - D^-Z_i)}{x_{i+1}-x_{i-1}}, \quad D^*Z_i = \begin{cases} D^+Z_i, & \text{if } a(x_i) > 0 \\ D^-Z_i, & \text{if } a(x_i) < 0. \end{cases} \end{aligned}$$

Theorem 3.3. *The error in using scheme* (3.4–3.5) *to solve problem* ((3.1–3.2)) *at the inner grid points x_i, $i = 1, 2, \ldots, N-1$ satisfies*

$$|y_2^*(x_i) - y_{2,i}^*| \leq CN^{-1}(\ln N)^2.$$

Proof. Define,

$$y_{2,i}^* = v_{2,i}^* + w_{2,i}^*$$

where, $v_{2,i}^*$ is a regular solution and $w_{2,i}^*$ is a singular solution respectively. Define,

$$y_2^*(x_i) = v_2^*(x_i) + w_2^*(x_i)$$

following from the result given in [6, 15]. It is easy to obtain,

$$|L^N(v_2^*(x_i) - v_{2,i}^*)| \leq CN^{-1} \tag{3.6}$$

$$|L^N(w_2^*(x_i) - w_{2,i}^*)| \leq CN^{-1}(\ln N)^2. \tag{3.7}$$

Based on the procedure [6, 15] and combining equations (3.6–3.7), we get

$$L^N|y_2^*(x_i) - y_{2,i}^*| \leq CN^{-1}(\ln N)^2.$$

□

4. COMPUTATIONAL METHOD

Consider the BVP (2.1–2.2). Let $u_{01}(x)$ be the solution of the BVP (2.8–2.9). The first step in the computation method is to replace y_1 by u_{01} (as we have said earlier it is assumed that the closed form solution is available for $u_{01}(x)$). Hence, system (2.1) gets decoupled. In the second step, we find a numerical solution for y_2 by applying the scheme (3.4–3.5). Then using this result, an improved value for y_1 is calculated from the first equation of the system (2.1).

5. ERROR ESTIMATE

This section presents the main contribution of the article which gives error estimate between the continuous solution and the corresponding numerical solution in the entire region.

Theorem 5.1. *Let* (y_1, y_2) *be the solution of* (2.1–2.2). *Further, let* $y^*_{2,i}$ *be its numerical solution obtained by scheme* (3.4–3.5). *Then*

$$|y_2(x_i) - y^*_{2,i}| \leq C[N^{-1}(\ln N)^2 + \varepsilon] \quad x \in \Omega$$

Proof. Thus using Theorems 3.2 and 3.3. We conclude that,

$$\begin{aligned} |y_2(x_i) - y^*_{2,i}| &\leq |y_2(x_i) - y^*_2(x_i)| + |y^*_2(x_i) - y^*_{2,i}| \\ &\leq CN^{-1}(\ln N)^2 + C\varepsilon \\ |y_2(x_i) - y^*_{2,i}| &\leq C[N^{-1}(\ln N)^2 + \varepsilon]. \quad x \in \Omega \end{aligned}$$

□

Remark 5.2. So far, it has been assumed that the exact solution u_{01} of the BVP (2.8–2.9) is available. When it is not available, one has to obtain a numerical solution for u_{01} by a suitable finite difference method with a piecewise uniform mesh of N mesh intervals described in Section 3.2. As done earlier, in the second equations the values of y_1 at the above grid points will be taken as $u_{01,i}$, then the resulting equations are solved for $y_{2,i}$.

6. ADJOINT SYSTEM

Consider the BVP (2.1–2.2) and suppose that condition (1.6) is not met. Then we adjoint the following system to (2.1–2.2).

$$(6.1) \quad A^*y(x) = F^* \Leftrightarrow \begin{cases} -\hat{y}_1''(x) - \hat{y}_2(x) = 0, \\ -\varepsilon\hat{y}_2''(x) + a(x)\hat{y}_2' + b(x)\hat{y}_2(x) - c^+(x)\hat{y}_3(x) \\ +c^-(x)\hat{y}_1(x) = -f(x), \quad x \in \Omega, \\ -\hat{y}_3''(x) - \hat{y}_4(x) = 0, \\ -\varepsilon\hat{y}_4''(x) + a(x)\hat{y}_2'(x) + b(x)\hat{y}_4(x) - c^+(x)\hat{y}_1(x) \\ +c^+(x)\hat{y}_3(x) = f(x), \quad x \in \Omega, \end{cases}$$

$$(6.2) \qquad \begin{cases} \hat{y}_1(-1) = -p, \quad \hat{y}_1(1) = -q, \quad \hat{y}_2(-1) = -r, \quad \hat{y}_2(1) = -s, \\ \hat{y}_3(-1) = p, \quad \hat{y}_3(1) = q, \quad \hat{y}_4(-1) = r, \quad \hat{y}_4(1) = s, \end{cases}$$

where

$$c^+(x) = \begin{cases} c(x) & \text{if } c(x) \geq 0, \\ 0 & \text{otherwise}, \end{cases}$$

$$c^-(x) = c(x) - c^+(x) \quad \text{and} \quad \hat{y} = (\hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{y}_4).$$

The results derived in the earlier sections can be extended to the above system. Because of the fact that if $\hat{y} = (y_1, y_2)$ is a solution of (2.1–2.2) then $\hat{y} = (-\hat{y}_1, -\hat{y}_2, \hat{y}_1, \hat{y}_2)$

is a solution of the above problem (6.1–6.2), the results derived earlier for the BVP (2.1–2.2) still hold good even if condition (1.6) is not met. It may be observed that if condition (1.6) is not met to the BVP (1.1–1.2), one has to solve the adjoint system and consequently the number of equation is doubled and hence occupies more memory spaces.

7. NON-LINEAR PROBLEMS

Let us consider the BVP

$$\varepsilon y^{iv}(x) = F(x, y, y'', y'''), \quad x \in \Omega, \tag{7.1}$$

$$y(-1) = p, \quad y(1) = q, \quad y''(-1) = -r, \quad y''(1) = -s, \tag{7.2}$$

where $F(x, y, y'', y''')$ is a smooth function such that

$$\begin{cases} |F_{y'''}(x, y, y'', y''')| \geq \alpha > 0, & 0 < |x| \leq 1, \\ F_{y''}(x, y, y'', y''') \geq \beta > 0, & x \in \Omega,\ y \in \mathbb{R} \text{ (set of reals)} \\ 0 \geq F_y(x, y, y'', y''') \geq -\gamma, & \gamma > 0, \alpha - \gamma(1+\delta) \geq \eta > 0 \end{cases} \tag{7.3}$$

for some η and $\delta > 0$. Assume that the reduced problem

$$F(x, y, y'', y''') = 0, \quad y(-1) = p, \quad y(1) = q, \quad y''(1) = -s, \tag{7.4}$$

has a solution $y_0 \in C^4(\Omega)$. Then (7.1–7.2) has a unique solution and has boundary layer of width $O(\varepsilon)$ near $x = -1$ [4]. Analytical results such as existence, uniqueness and asymptotic behavior of the solution of (7.1–7.2) can be found in [4]. In order to obtain a numerical solution for the BVP (7.1–7.2) Newton's method of quasilinearisation [11] is applied to generate a sequence $\{y^{[m]}\}_0^{\infty}$ of successive approximations with a proper choice of initial guess $y^{[0]}$. In fact, we define $y^{[m+1]}$ for each fixed non-negative integer m, to be the solution of the following linear problem:

$$-\varepsilon y^{iv[m+1]}(x) - a^m(x) y^{'''[m+1]}(x) + b^m(x) y^{''[m+1]}(x) - c^m(x) y^{[m+1]}(x) = -f^m(x), \tag{7.5}$$

$$y^{[m+1]}(-1) = p, \quad y^{[m+1]}(1) = q, \quad y^{''[m+1]}(-1) = -r, \quad y^{''[m+1]}(1) = -s, \tag{7.6}$$

where

$$a^m(x) = F_{y'''}(x, y^{[m]}, y^{''[m]}, y^{'''[m]}), \quad b^m(x) = F_{y''}(x, y^{[m]}, y^{''[m]}, y^{'''[m]}),$$
$$c^m(x) = F_y(x, y^{[m]}, y^{''[m]}, y^{'''[m]}),$$

$$\begin{aligned} -f^m(x) &= F_y(x, y^{[m]}, y^{''[m]}, y^{'''[m]}) - y^{[m]} F_y(x, y^{[m]}, y^{''[m]}, y^{'''[m]}) \\ &- y^{''[m]} F_{y''}(x, y^{[m]}, y^{''[m]}, y^{'''[m]}) + y^{'''[m]} F_{y'''}(x, y^{[m]}, y^{''[m]}, y^{'''[m]}). \end{aligned}$$

we make the following observations:

1. From (7.3) it follows that, for each fixed m,

$$\begin{cases} a^m(x) = |F_{y'''}(x, y^{[m]}, y''^{[m]}, y'''^{[m]})| \geq \alpha > 0, & 0 < |x| \leq 1, \\ b^m(x) = F_{y''}(x, y^{[m]}, y''^{[m]}, y'''^{[m]}) \geq \beta > 0, & x \in \Omega \\ 0 \geq c^m(x) = F_y(x, y^{[m]}, y''^{[m]}, y'''^{[m]}) \geq -\gamma, & \gamma > 0. \end{cases} \tag{7.7}$$

2. If the initial guess $y^{[0]}$ is sufficiently close to the solution $y(x)$ of (7.1–7.2), then one can prove, following the method of proof given in [11],that the sequence $\{y^{[m]}\}_0^{\propto}$ converges to $y(x)$.
3. Problem (7.5–7.6), for each fixed m, is a linear BVP. Hence it can be solved by the method described in Section 4.
4. The following convergence criterion is used to terminate the iteration:

$$\|y^{[m+1]}(x_j) - y^{[m]}(x_j)\| \leq \lambda, \quad x_j \in \Omega, \quad \lambda, m \geq 0. \tag{7.8}$$

8. NUMERICAL EXAMPLES

This section presents a numerical example to show the applicability and efficiency of the method. The numerical results are given in the form of a table. The maximum nodal errors and order of convergence are estimated by using double mesh principle [15]. The following example have a turning point at $x = 1/2$.

Example 8.1. Consider the fourth order linear singularly perturbed turning point problem

$$-\varepsilon y^{iv}(x) + 2(2x-1)y'''(x) + 4y'' + y(x) = 0, \quad x \in (0,1),$$
$$y(0) = 1, \quad y(1) = 1, \quad y''(0) = 1, \quad y''(1) = 1.$$

Its equivalent system is a weakly coupled system is given by

$$y_1''(x) + y_2 = 0, \quad x \in (0,1),$$
$$\varepsilon y_2''(x) - 2(2x-1)y_2'(x) - 4y_2(x) + y_1(x) = 0,$$
$$y_1(0) = y_1(1) = y_2(0) = y_2(1) = 1.$$

Example 8.2. Consider the fourth order non-linear singularly perturbed turning point problem

$$-\varepsilon y^{iv}(x) + 2(2x-1)y'''(x) - y''^2 + y(x) = 0, \quad x \in (0,1),$$
$$y(0) = 1, \quad y(1) = 1, \quad y''(0) = 1, \quad y''(1) = 1.$$

Its equivalent system is a weakly coupled system is given by

$$y_1''(x) + y_2 = 0, \quad x \in (0,1),$$
$$\varepsilon y_2''(x) - 2(2x-1)y_2'(x) + y_2^2(x) + y_1(x) = 0,$$
$$y_1(0) = y_1(1) = y_2(0) = y_2(1) = 1.$$

which validate the theoretical results established in the previous section. As in the case of non-linear the tolerance to be $\lambda = 10^{-5}$ and the initial guess to be $y_{init} = y(0) - x(y(1) - y(0)),\ 0 \le x \le 1$. Initial guess chosen in this paper are comparatively good with initial guess found in [11] and satisfies the tolerance. It also requires less iteration to convergence, which shows the computational time is reduced much.

Here, linear interplant method is used to find the maximum pointwise error E_ε^N for linear and non-linear problems and assumed U_ε^{8192} contains the nodal values of the approximate solution with $N = 8192$. The difference between the numerical solutions for various N and the numerical solution $N = 8192$, maximum pointwise error E_ε^N. The corresponding approximate maximum pointwise error is taken to be

$$E_\varepsilon^N = \max_{x_i \in \overline{\Omega}_\varepsilon^N} |U_\varepsilon^N(x_i) - U_\varepsilon^{8192}(x_i)| \quad \text{and} \quad E^N = \max_\varepsilon E_\varepsilon^N$$

and

$$\rho_{intp}^N = \log_2\left(\frac{E^N}{E^{2N}}\right).$$

The computed maximum point wise error E^N, E_ε^N and the computed orders of convergence ρ_{intp}^N for the Problem 8.1 and Problem 8.2 are given in Table 1 and Table 2 respectively.

TABLE 1. *Maximum point-wise errors E_ε^N for various N and ε for the Problem* 8.1.

ε	Number of mesh points					
	2^6	2^7	2^8	2^9	2^{10}	2^{11}
2^0	9.37935E-04	4.49525E-04	2.17253E-04	1.04161E-04	4.83834E-05	2.06875E-05
2^{-1}	2.45536E-03	1.20544E-03	5.89987E-04	2.84706E-04	1.32681E-04	5.68244E-05
2^{-2}	4.34746E-03	2.17469E-03	1.07493E-03	5.21325E-04	2.43568E-04	1.04447E-05
2^{-3}	3.74498E-03	1.89478E-04	9.42012E-04	4.58232E-04	2.14417E-04	9.20175E-05
2^{-4}	1.61157E-03	8.34141E-04	4.19550E-04	2.05385E-04	9.64097E-05	4.14401E-05
2^{-5}	3.91853E-03	1.72672E-03	7.44526E-04	3.14114E-04	1.27723E-04	4.75507E-05
2^{-6}	5.03639E-03	2.43066E-03	1.15434E-03	5.37844E-04	2.41751E-04	9.99900E-05
2^{-7}	5.20421E-03	2.58943E-03	1.26457E-03	6.05101E-04	2.79006E-04	1.18183E-04
2^{-8}	5.16610E-03	2.60747E-03	1.28755E-03	6.22056E-04	2.89279E-04	1.23482E-04
2^{-9}	5.10955E-03	2.59862E-03	1.29019E-03	6.25944E-04	2.92126E-04	1.25065E-04
2^{-10}	5.06870E-03	2.58827E-03	1.28878E-03	6.26574E-04	2.92904E-04	1.25577E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	5.01528E-03	2.56944E-03	1.28337E-03	6.25472E-04	2.92919E-04	1.25757E-04
E^N	5.20421E-03	2.60747E-03	1.29019E-03	6.26574E-04	2.93084E-04	1.25791E-04
ρ_{intp}^N	0.997029231	1.015065222	1.042028840	1.096171635	1.220280016	

TABLE 2. *Maximum point-wise errors E_ε^N for various N and ε for the Problem* 8.2.

ε	Number of mesh points					
	2^6	2^7	2^8	2^9	2^{10}	2^{11}
2^0	1.87254E-04	9.22903E-05	4.52580E-05	2.18607E-05	1.01926E-05	4.366324E-06
2^{-1}	7.15256E-03	3.55908E-04	1.75387E-04	8.49253E-05	3.96461E-05	1.69942E-05
2^{-2}	2.37975E-03	1.20300E-03	5.97858E-04	2.90720E-04	1.36009E-04	5.83624E-05
2^{-3}	5.95282E-03	3.08867E-03	1.55378E-03	7.60579E-04	3.57005E-04	1.53450E-04
2^{-4}	1.16464E-02	6.28219E-03	3.23776E-03	1.60554E-04	7.58269E-04	3.26972E-04
2^{-5}	1.23235E-02	7.58184E-03	4.46974E-03	2.49075E-03	1.309642E-03	6.26765E-04
2^{-6}	1.24684E-02	7.64329E-03	4.50103E-03	2.49568E-03	1.29963E-03	6.07327E-04
2^{-7}	1.25502E-02	7.6833E-03	4.52792E-03	2.51043E-03	1.30771E-03	6.11206E-04
2^{-8}	1.25898E-02	7.70209E-03	4.54051E-03	2.51722E-03	1.31142E-03	6.12979E-04
2^{-9}	1.26092E-02	7.71122E-03	4.54661E-03	2.52047E-03	1.31319E-03	6.13821E-04
2^{-10}	1.26189E-02	7.71574E-03	4.54963E-03	2.52207E-03	1.31405E-03	6.14231E-04
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-30}	1.26308E-02	7.72218E-03	4.55583E-03	2.52626E-03	1.31738E-03	6.17868E-04
E^N	1.26308E-02	7.72218E-03	4.55583E-03	2.52626E-03	1.31738E-03	6.17868E-04
ρ_{intp}^N	0.709866921	0.761293136	0.850709206	0.939332801	1.071675371	

9. CONCLUSION

We examined a numerical method for solving singularly perturbed fourth-order convection-diffusion type turning point problem. The fourth-order SPPBVPs converted into two system of weakly coupled second-order singularly perturbed differential equation. The proposed numerical method uses the classical upwind difference scheme on a piecewise-uniform mesh (Shishkin mesh). In general, the numerical treatment of TPPs is much more complicated than singular perturbation problems without turning points. This is mainly because of the convection coefficient $a(x)$ vanishes inside the domain of interest. To preserve the stability of the difference scheme we use both the forward and backward difference schemes depending on the sign of $a(x)$. The error values calculated are presented in the Table 1 and Table 2 for linear and non-linear type of problems respectively. Numerical results are significant with the theoretical results.

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