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ASYMPTOTIC BEHAVIOR OF STOCHASTIC HEPATITIS B MODEL WITH DIFFERENT CONTROLS

ANWAR ZEB[a], SULTAN HUSSAIN[a], SYED INAYAT ALI SHAH[b], AND GUL ZAMAN[c†]

[a]Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad Pakistan

[b]Department of Mathematics, University of Malakand, Chakdara Pakistan

[c]Department of Mathematics, Islamia College University, Peshawar, Pakistan

ABSTRACT. In this paper, we introduce a hepatitis B virus stochastic transition model with vaccination and awareness campaign. The purpose of this work is to show the effect of vaccine and awareness in the affected area of hepatitis B. For this purpose first, we formulate the stochastic model for hepatitis B virus, then investigate the asymptotic behavior of vaccination and awareness campaign, through the unique positive solution of our proposed model. We study the different scenarios in order to identify the best control of this virus. Graphical justification is also presented.

Key Words: Hepatitis B virus, stochastic modeling, unique positive solution, stability analysis

Subject Classification: 92D25, 49J15, 93D20, 91G20.

1. INTRODUCATION

A hepatitis B virus (HBV) infection is a global health disease which transmits through unhealthy food as well as by their carriers. According to the analysis of World Health Organization about 2000 millions people have been infected with this virus and about 350 millions, among them, are its carriers [1]. There appear about 4 million acute clinical cases and 25 percent of its carriers each year. Moreover about 1 million die, due to this virus, each year. For the control of HBV mathematicians also play an important rule in the sense of making different models to study the dynamical behavior of this virus [2].

Pang et al. [1] and Zou et al. [3] considered a deterministic HBV transmission model with vaccination, where vaccination took place in China to all newborns. They have shown that, with regular vaccination to newborns, the number of children having HBV infection as well as the number of carriers decreased dramatically. Anderson and May 1991 [6] have discussed the effects of carriers on the transmission of this virus by using a simple mathematical model. Zhao et al. 2000 [4] presented an age structure model to study the long-term effect of vaccination on the HBV transmission in China. Wang et al. 2008 [5] studied HBV infection in a diffusion model restricted

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to a finite domain. Similarly Xu and Ma 2009 [7] proposed an HBV model with spatial diffusion respond in the infection rate. Different authors studied the models describing the HBV as well as stochastic models [8, 9, 10, 11, 12, 13, 14].

In this work, we extend the deterministic model of Pang et al. [1] to a stochastic HBV transmission model with addition of migrated class and vaccination plus awareness campaign to all latent individuals. First, we study the stability of the corresponding deterministic model. Then the asymptotic behavior of acute, carriers and immunity classes are presented. It is shown that the number of susceptible and latent's individuals increase/decrease as linear, through the unique positive solution of our stochastic model under different scenarios in order to identify the expected optimal control of HBV.

The rest of the paper is organized as: in Section 2, we give the formulation of model and discuss the stability of stochastic model. In Section 3, we apply different controls and discuss the different scenarios with the obtained figures through Matlab. Finally, we give conclusion in Section 4.

2. NOTATIONS AND FORMULATION OF THE MODEL

We propose a stochastic transition HBV model with vaccination to acute as well as latent ones. The total population is divided in five classes: the Susceptible $S = S(t)$, Latent $L = L(t)$, Acute ones $A = A(t)$, HBV Carriers $C = C(t)$ and temporary protective Immunity $I = I(t)$ at time $t \geq 0$.

We assume that all the parameters of the model are non-negative and the recruitment rate denotes by λ such that λm, $0 \leq m \leq 1$, number of individuals adds to suspectable while $\lambda(1-m)$ move to latent class, d_i, $i = 1, 2, \ldots, 5$, denotes the death rate of S, L, A, C and I, respectively. The child birth rate is denoted by μ, ω is the rate of failure immunization, β is the transmission coefficient, σ is the rate at which latent individuals becoming infections, n ratio of latent moves to immunity class due to vaccination, δ is the rate of loss of immunity class, r_1 is the rate at which an individual leaves class A, carriers recover with the rate r_2, the infectiousness rate of carriers relative to acute infection is α, q $(0 \leq q \leq 1)$ is the rate at which acute infection individuals become carrier while $1-q$ is the rate at which acute infection individuals clear HBV and move to immunity class. Moreover new born are vaccinated immediately after his/her birth and $\mu(1-w)$ part of them move to immunity class.

We introduce stochastic components on the transition rates, that is, on the transition from S to L with drift term $\beta(A + \alpha C)$ and variance σ_1, L to A with drift σ and volatility σ_2, A to C with drift qr_1 with volatility σ_3, C to I with drift r_2 and volatility σ_4 and I to S with drift δ and volatility σ_5.

We consider a probability space $(\Omega, \mathcal{F}, P)$ and a five dimensional Wiener process $W = (W_1(t), W_2(t), \ldots, W_5(t))$ on it, where each $W_i(t)$, $i = 1, 2, \ldots, 5$ is a standard Wiener process, independent of each other, with mean zero and variance t at time t. The augmentation of the natural filtration of the Wiener processes by P-null sets of $\mathcal{F}$ is denote by $\mathcal{F}^W = (\mathcal{F}^{W(t)})_{t\geq 0}$. On the filtered probability space $(W, \mathcal{F}, \mathcal{F}^{W(t)}, P)$ we will consider five random functions $S(t)$, $L(t)$, $A(t)$, $C(t)$ and $I(t)$.

Evolution of these functions obey the following system of stochastic differential equations

(2.1)

$$\begin{aligned}
dS(t) &= (\lambda m + \mu\omega(1 - \epsilon C(t)) - (\beta(A(t) + \alpha C(t)) + p(1 - \omega) + d_1 - \delta I(t))\, S(t))dt \\
&\quad - \sigma_1(A(t) + \alpha C(t))S(t)dW_1(t) + \sigma_5 S(t)I(t)dW_5(t), \\
dL(t) &= (\lambda(1 - m) + \mu\omega\epsilon C(t) + \beta(A(t) + \alpha C(t))S(t) - (n + \sigma + d_2)L(t))dt \\
&\quad + \sigma_1(A(t) + \alpha C(t))S(t)dW_1(t) - \sigma_2 L(t)dW_2(t), \\
dA(t) &= (\sigma L(t) - (d_3 + r_1)A(t))dt + \sigma_2 L(t)dW_2(t) - \sigma_3 qA(t)dW_3(t), \\
dC(t) &= (qr_1 A(t) - (d_4 + r_2)C(t))dt + \sigma_3 qA(t)dW_3(t) - \sigma_4 C(t)dW_4(t), \\
dI(t) &= (r_2 C(t) + \mu(1 - \omega) + (1 - q)r_1 A(t) + nL(t) + p(1 - \omega)S(t) \\
&\quad - (d_5 + \delta)I(t))dt + \sigma_4 C(t)dW_4(t) - \sigma_5 I(t)dW_5(t),
\end{aligned}$$

where we have assumed that all the coefficients in the model are Lipschitz continuous.

From the above system, the dynamics of $S(t)$ can be described by:

(2.2)

$$\begin{aligned}
S(t) &= e^{-(p(1-\omega)+d_1)t - \int_0^t (\beta(A(u)+\alpha C(u))+\alpha C(u)-\delta I(u))du - \frac{\sigma_1^2}{2}\int_0^t (A(u)+\alpha C(u))^2 du - \frac{\sigma_1^5}{2}\int_0^t I(u)^2 du} \\
&\quad \times e^{-\sigma_1 \int_0^t (A(u)+\alpha C(u))dW_1(u) + \sigma_5 \int_0^t I(u)dW_5(u)} \Bigg[S(0) - \mu\omega\epsilon \int_0^t C(u) e^{(p(1-\omega)+d_1)u} \\
&\quad \times e^{\int_0^u (\beta(A(v)+\alpha C(v))+\alpha C(v)-\delta I(v))dv + \frac{\sigma_1^2}{2}\int_0^u (A(v)+\alpha C(v))^2 dv + \frac{\sigma_1^5}{2}\int_0^u I(v)^2 dv} \\
&\quad \times e^{\sigma_1 \int_0^u (A(v)+\alpha C(v))dW_1(v) - \sigma_5 \int_0^u I(v)dW_5(v)} du \Bigg] \\
&\quad + (\lambda m + \mu\omega) \int_0^t e^{-(p(1-\omega)+d_1)(t-u) - \int_u^t (\beta(A(v)+\alpha C(v))+\alpha C(v)-\delta I(u))du - \frac{\sigma_1^2}{2}\int_u^t (A(v)+\alpha C(v))^2 du} \\
&\quad \times e^{-\frac{\sigma_1^5}{2}\int_u^t I(v)^2 dv - \sigma_1 \int_u^t (A(v)+\alpha C(v))dW_1(v) + \sigma_5 \int_u^t I(v)dW_5(v)} du,
\end{aligned}$$

with expected value

(2.3)

$$\begin{aligned}
E(S(t)) &= Ee^{-(p(1-\omega)+d_1)t - \int_0^t (\beta(A(u)+\alpha C(u))+\alpha C(u)-\delta I(u))du} \Bigg[S(0) - \mu\omega\epsilon E \int_0^t C(u) e^{(p(1-\omega)+d_1)u} \\
&\qquad \times e^{\int_0^u (\beta(A(v)+\alpha C(v))+\alpha C(v)-\delta I(v))dv} \Bigg]
\end{aligned}$$

$$+ (\lambda m + \mu\omega) E \int_0^t e^{-(p(1-\omega)+d_1)(t-u) - \int_u^t (\beta(A(u)+\alpha C(u))+\alpha C(u)-\delta I(u))du} du,$$

where we have used the property of exponential martingales

$$E e^{\int_s^t a(u)dW_u - \frac{1}{2}\int_s^t a^2(u)du} = 1, \tag{2.4}$$

where $a(t)$ is any $\mathcal{F}^W(t)$-measurable function and $W(t)$, $t \geq 0$, is any standard Wiener process.

Dynamics of $L(t)$ is given as

(2.5)

$$L(t) = e^{-\left(n+\sigma+d_2+\frac{\sigma_2^2}{2}\right)t - \sigma_2 W_2(t)} \Bigg[L(0) + \int_0^t (\mu\omega\epsilon C(u) + \beta(A(u) + \alpha C(u))S(u))$$
$$\times e^{\left(n+\sigma+d_2+\frac{\sigma_2^2}{2}\right)u + \sigma_2 W_2(u)} du + \sigma_1 \int_0^t e^{\left(n+\sigma+d_2+\frac{\sigma_2^2}{2}\right)u + \sigma_2 W_2(u)} (A(u)$$
$$+ \alpha C(u))S(u) dW_1(u) \Bigg] + \lambda(1-m) \int_0^t e^{-\left(n+\sigma+d_2+\frac{\sigma_2^2}{2}\right)(t-u) - \sigma_2(W_2(t)-W_2(u))} du.$$

Using properties like (2.4) and of the Itô's integral

$$E \int_0^t b(u) dW_u = 0, \tag{2.6}$$

where $b(t)$ is some $\mathcal{F}^W(t)$-measurable function, we obtain the expected value of $L(t)$ as

$$E(L(t)) = e^{-(n+\sigma+d_2)t} \left[L(0) + E \int_0^t (\mu\omega\epsilon C(u) + \beta(A(u) + \alpha C(u))S(u))\, e^{(n+\sigma+d_2)u} du \right]$$
$$+ \frac{\lambda(1-m)}{n+\sigma+d_2} \left(1 - e^{-(n+\sigma+d_2)t}\right). \tag{2.7}$$

Similarly

(2.8)

$$A(t) = e^{-\left(d_3+r_1+\frac{\sigma_3^2 q^2}{2}\right)t - \sigma_3 q W_3(t)} \left[A(0) + \int_0^t L(u) e^{\left(d_3+r_1+\frac{\sigma_3^2 q^2}{2}\right)u + \sigma_3 q W_3(u)} (\sigma du + \sigma_2) dW_2(u) \right],$$

with expectation

$$E(A(t)) = e^{-(d_3+r_1)t} \left[A(0) + \sigma E \int_0^t L(u) e^{(d_3+r_1)u} du \right]. \tag{2.9}$$

While

(2.10)

$$C(t) = e^{-\left(d_4+r_2+\frac{\sigma_4^2}{2}\right)t - \sigma_4 W_4(t)} \left[C(0) + q \int_0^t A(u) e^{\left(d_4+r_2+\frac{\sigma_4^2}{2}\right)u + \sigma_4 W_4(u)} (r_1 du + \sigma_3 dW_3(u)) \right],$$

with expected value

$$E(C(t)) \;=\; e^{-(d_4+r_2)t} \left[C(0) + q r_1 E \int_0^t A(u) e^{(d_4+r_2)u} du \right]. \tag{2.11}$$

And

$$I(t) = e^{-\left(d_5+\delta+\frac{\sigma_5^2}{2}\right)t-\sigma_5 W_5(t)}\left[I(0) + \int_0^t e^{\left(d_5+\delta+\frac{\sigma_5^2}{2}\right)u+\sigma_5 W_5(u)} \times (\mu(1-\omega) + r_2 C(u) + (1-q)r_1 A(u) + p(1-\omega)S(u) + nL(u))du + \sigma_4 C(u)dW_4(u)\right], \tag{2.12}$$

with

$$E(I(t)) = e^{-(d_5+\delta)t}\left[I(0) + \mu(1-\omega)\int_0^t e^{(d_5+\delta)u}du + \int_0^t e^{(d_5+\delta)u}(r_2 C(u) + (1-q)r_1 A(u) + p(1-\omega)S(u) + nL(u))du\right]. \tag{2.13}$$

Under the assumption $E(S(t+)|S(t)) = k_1$ and $E(L(t+)|L(t)) = k_2$, where $k_1, k_2 \in R$, of linear growth/decay, we have

$$E(A(t)) = A(0)e^{-(d_3+r_1)t} + \frac{\sigma k_2}{d_3+r_1}\left(1 - e^{-(d_3+r_1)t}\right), \tag{2.14}$$

$$\begin{aligned} E(C(t)) = C(0)e^{-(d_4+r_2)t} &+ \frac{q r_1 A(0) e^{-(d_4+r_2)t}}{d_4+r_2-d_3-r_1}\left(e^{(d_4+r_2-d_3-r_1)t} - 1\right) \\ &+ \frac{q r_1 \sigma k_2}{(d_3+r_1)(d_4+r_2)}\left(1 - e^{-(d_4+r_2)t}\right) \\ &- \frac{q r_1 \sigma k_2 e^{-(d_4+r_2)t}}{(d_3+r_1)(d_4+r_2-d_3-r_1)}\left(e^{(d_4+r_2-d_3-r_1)t} - 1\right), \end{aligned} \tag{2.15}$$

while

$$\begin{aligned} E(I(t)) = I(0)e^{-(d_5+\delta)t} &+ \left[(1-\omega)(\mu + pk_1) + nk_2 + \frac{\sigma r_1 k_2 (d_4 + r_2 - q d_4)}{(d_3+r_1)(d_4+r_2)}\right] \\ &\times \left(\frac{1-e^{-(d_5+\delta)t}}{d_5+\delta}\right) + \frac{r_2 C(0) e^{-(d_5+\delta)t}}{d_5+\delta-d_4-r_2}\left(e^{(d_5+\delta-d_4-r_2)t} - 1\right) \\ &+ \frac{r_1 A(0)(d_4+r_2-d_3-r_1-q(d_4-d_3-r_1))e^{-(d_5+\delta)t}}{(d_4+r_2-d_3-r_1)(d_5+\delta-d_3-r_1)}\left(e^{(d_5+\delta-d_3-r_1)t} - 1\right) \\ &+ q r_1 r_2 e^{-(d_5+\delta)t}\left[\frac{\sigma k_2}{(d_4+r_2-d_3-r_1)(d_4+r_2)} - \frac{A(0)}{d_4+r_2-d_3-r_1}\right] \\ &\times \left(\frac{e^{(d_5+\delta-d_4-r_2)t} - 1}{d_5+\delta-d_4-r_2}\right) - \sigma k_2 r_1 e^{-(d_5+\delta)t}\left(\frac{e^{(d_5+\delta-d_3-r_1)t} - 1}{d_5+\delta-d_3-r_1}\right) \\ &\times \frac{d_4+r_2-d_3-r_1-q(d_4-d_3-r_1)}{(d_3+r_1)(d_4+r_2-d_3-r_1)}. \end{aligned} \tag{2.16}$$

3. SCENARIOS

In this section, we present different scenarios to understand the dynamical behavior of the proposed model.

1. When $d_3 + r_1 = 0$ and $k_2 > 0$, then the asymptotic behavior of $E(A(t))$, $E(C(t))$ and $E(I(t))$ is as

$$\lim_{t\to\infty} E(A(t)) = \infty, \lim_{t\to\infty} E(C(t)) = 0, \tag{3.1}$$
$$\lim_{t\to\infty} E(I(t)) = \frac{1}{d_5+\delta}((1-\omega)(\mu - pk_1) + nk_2),$$

and if $k_2 < 0$, then $E(A(t))$, $E(C(t))$ and $E(I(t))$ decrease and $E(A(t))$ vanishes at time $t = \frac{-A(0)}{\sigma k_2}$, vanishing time of $E(C(t))$ and $E(I(t))$ can be obtained by using expressions (2.11) and (2.13).

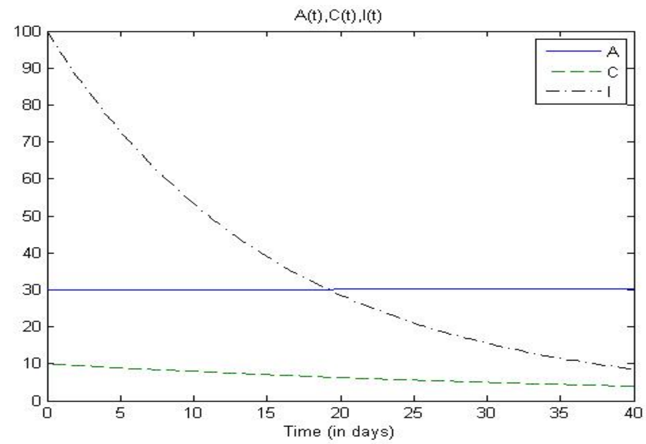

FIGURE 1. The plot shows the asymptotic behavior of A(t), C(t) and I(t) with parameters values $\beta = 0.48$, $m = 0.02$, $d_2 = 0.5$, $\gamma = 0.02$, $d_3 = 0.01$.

2. Similarly when $d_4 + r_2 = 0$ then

$$\lim_{t\to\infty} E(A(t)) = \frac{\sigma k_2}{d_3+r_1}, \quad \lim_{t\to\infty} E(C(t)) = \infty, \tag{3.2}$$
$$\lim_{t\to\infty} E(I(t)) = \frac{1}{d_5+\delta}((1-\omega)(\mu - pk_1) + nk_2) + \frac{(1-q)r_1\sigma k_2}{(d_3+r_1)^2}.$$

3. And when $d_5 + \delta = 0$ then

$$\lim_{t\to\infty} E(A(t)) = \frac{\sigma k_2}{d_3+r_1}, \lim_{t\to\infty} E(C(t)) = \frac{qr_1\sigma k_2}{(d_3+r_1)(d_4+r_2)}, \lim_{t\to\infty} E(I(t)) = \infty. \tag{3.3}$$

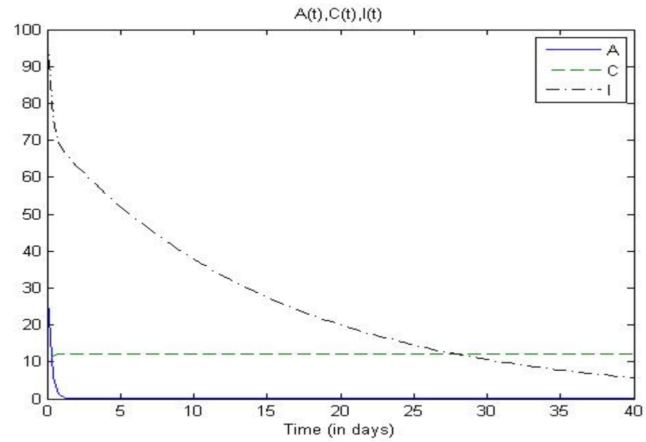

FIGURE 2. The plot shows the asymptotic behavior of A(t), C(t) and I(t) with parameters values $\beta = 0.48$, $m = 0.02$, $d_2 = 0.5$, $\gamma = 0.02$, $d_3 = 0.01$.

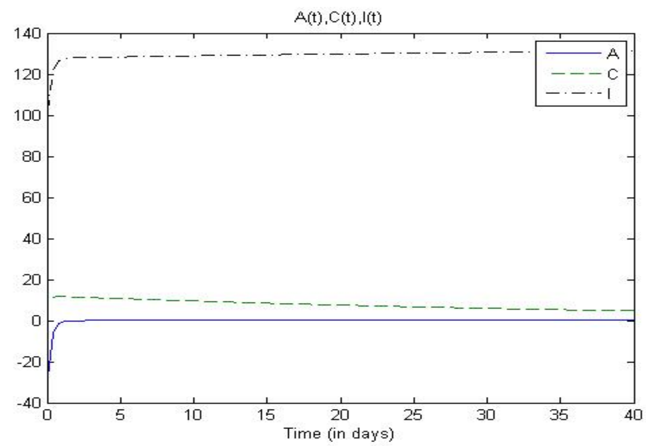

FIGURE 3. The plot represents the asymptotic behavior of A(t), C(t) and I(t) with parameters values $\beta = 0.48$, $m = 0.02$, $d_2 = 0.5$, $\gamma = 0.02$, $d_3 = 0.01$.

4. Moreover, for positive $d_4 + r_2$, $d_3 + r_1$, and $d_5 + \delta$, with arbitrary equality or inequality relation between them, we have the asymptotic stability as

(3.4)

$$\lim_{t\to\infty} E(A(t)) = \frac{\sigma k_2}{d_3 + r_1}, \lim_{t\to\infty} E(C(t)) = \frac{q r_1 \sigma k_2}{(d_4 + r_2)(d_3 + r_1)},$$
$$\lim_{t\to\infty} E(I(t)) = \frac{1}{d_5 + \delta}\left[(1-\omega)(\mu + pk_1) + nk_2 + \frac{q r_1 r_2 \sigma k_2}{(d_3 + r_1)(d_4 + r_2)} + \frac{(1-q) r_1 \sigma k_2}{d_3 + r_1}\right].$$

$d_1 = d_2 = d_3 = d_4 = d_5 = \lambda = \mu = \frac{1}{70}$, $\omega = .05$, $\sigma = 6$, $\delta = .05$, $r_1 = 4$, $r_2 = .009$, $\epsilon = .8$, $q = .07$, $p = .5$, $\beta = 5$, $\alpha = .5$, $S(0) = 40$, $L(0) = 20$, $A(0) = 30$, $C(0) = 10$, $I(0) = 100$, $k_1 = k_2 = .001$, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 = .01$.

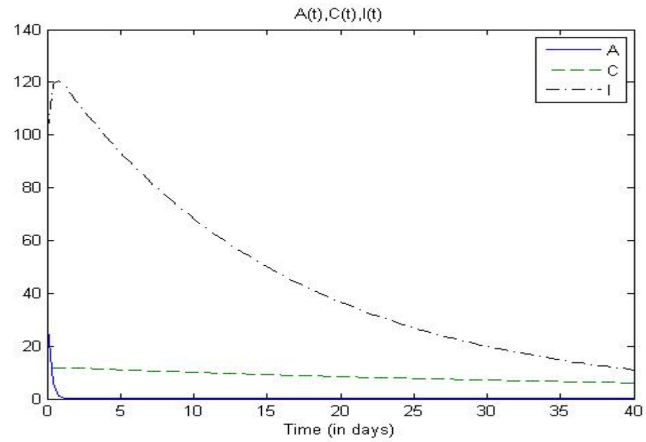

FIGURE 4. The graph shows the asymptotic behavior of A(t), C(t) and I(t) with parameters values $\beta = 0.48$, $m = 0.02$, $d_2 = 0.5$, $\gamma = 0.02$, $d_3 = 0.01$.

4. CONCLUSION

In this work, we introduced a hepatitis B virus stochastic transition model with vaccination and awareness campaign. First, we investigated the asymptotic behavior of vaccination and awareness campaign, through the unique positive solution of our stochastic model. Then we studied different scenarios in order to identify the best control. Graphical justification is also presented.

5. ACKNOWLEDGMENT

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REFERENCES

[1] J. Pang, J. A. Cui and X. Zhou, *Dynamical behavior of a Hepatitis B virus transmission model with vaccination*, *J. Theo. Bio.*, 265(4):572–578, 2002.

[2] T. Khan, G. Zaman and O. Algahtani, *Transmission dynamic and vaccination of Hepatitis B epidemic model, Wulfenia Journal*, 22:230–241, 2015.

[3] L. Zou, W. Zhang and S. Ruan, *Modeling the transmission dynamics and control of Hepatitis B virus in China*, *J. Theo. Bio.*, 262(2):330-338, 2010.

[4] S. Zhao, Z. Xu and Y. Lu, *A mathematical model of Hepatitis B virus transmission and its application for vaccination strategy in China*, *Int. J. Epi.*, 29(4):744–752, 2010.

[5] K. Wang, W. Wang and S. Song, *Dynamics of an HBV model with diffusion and delay*, *J. Theo. Bio.*, 253:36–44, 2008.

[6] R. M. Anderson and R. M. May, *Infectious Disease of Humans: Dynamics and Control*, Oxford University Press, Oxford, UK, 1991.

[7] R. Xu and Z. Ma, *An HBV model with diffusion and time delay, J. Theo. Bio.*, 257:499–509, 2009.

[8] F. Ahmad and G.R. Foster, *Global hepatitis, migration and its impact on Western healthcare*, *Gut* 59:1009–1011, 2010.

[9] G. F. Medley, N. A. Lindop, W. J. Edmunds and D. J. Nokes, *Hepatitis-B virus endemicity: heterogeneity, catastrophic dynamics and control*, *Natu. Medi.*, 7:619–624, 2001.

[10] S. Thornley, C. Bullen and M. Roberts, *Hepatitis B in a high prevalence New Zealand population: A mathematical model applied to infection control policy*, *J. Theo. Bio.*, 254:599–603, 2008.

[11] K. P. Maier, *Hepatitis-Hepatitisfolgen*, Georg Thieme Verlag, Stuttgart, Germany, 2000.

[12] C. W. Shepard, E. P. Simard, L. Finelli, A. E. Fiore and B. P. Bell, *Hepatitis B virus infection: epidemiology and vavvination*, *Epid. Rev.*, 28:112–125, 2006.

[13] R. Z. Hasminskii, *Stochastic Stability of Differential Equations*, Sijthoof and Noordhoof, Alphen aan Den Rijn, The Netherlands 1980.

[14] A. Lahrouz, L. Omari, D. Kiouach and A. Belmati, *Deterministic and stochastic stability of a mathematical model of smoking*, *Statistics and Probability Letters*, 81:1276–1284, 2011.