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DYNAMIC ANALYSIS OF A STOCHASTIC TRANSMISSION MODEL FOR ECHINOCOCCOSIS

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ABSTRACT. In this paper we study a stochastic transmission model for Echinococcosis. The model is proposed from the corresponding deterministic model presented in [18] by introducing random perturbations around the disease-free equilibrium and endemic equilibrium. By constructing a suitable Lyapunov function of quadratic form, we obtain the sufficient condition on the stochastic stability of the disease-free equilibrium and endemic equilibrium. Numerical simulations have been performed to verify/extend our analytical results.

Key Words: Echinococcosis, Stochastic epidemic model, Equilibrium, Stochastic stability, Lyapunov function

1. PRELIMINARIES

Echinococcosis (hydatidosis or hydatid disease), which is a group of infectious diseases caused by the larval stage of tapworms of the genus Echinococcosis, is a parasitic disease that affects both humans and other mammals, such as sheep, dogs, rodents, and horses [1]. Among the reported Echinococcosis, the two most clinically relevant species are E. granulosus, leading to cystic echinococcosis (CE) and E. multilocularis causing alveolar echinococcosis (AE). Humans are accidental hosts and, in most cases, do not contribute to continuance of the parasite life cycle, except under unique circumstances. It is estimated that there are more than three million people in the world who are infected with echinococcosis while 0.38 million cases exist in P.R. China [2–4]. Among the reported data, about 90% of all cases of echinococcosis in China are cystic echinococcosis caused by E. granulosus, whereas the reminder are from alveolar echinococcosis, which is caused by E. multilocularis and produces more severe pathological changes [5, 6]. Infection with E. granulosus results in the development of one or several unilocular hydatid cysts which develop mostly in the liver (70%) and the lungs (20%) [7]. CE also causes great loss of animal husbandry

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as sheep, goats and cattle. E. multilocularis metastasising lesion almost exclusively in the liver (98-100%).

Echinococcosis has a wide geographical distribution in P.R. China, mainly in western provinces and regions. According to the reporting system on diseases control and prevention established by China CDC, there are 27 provinces (autonomous regions, mulnicipalities) reported echinococcosis cases [9], and Xinjiang, Qinghai, Gansu, Ningxia, Tibet, Inner Monogolia, and Sichuan provinces are reported with a relatively high prevalence [8–12]. In addition, the rate of incidence of Echinococcosis has increased in the past decade. The operability of Echinococcosis exceeds 10/100000 in each year. High-risk group subject to Echinococcosis reaches up to 50 million, and the number of domestic animal amount being faced with infection of Echinococcosis is more than one hundred million, in which the amount of dogs is at least 5 million [13].

Mathematical modeling has become an important tool in analyzing the epidemiological characteristics of infectious disease and can provide useful control measures. Various models have been used to study different aspects of echinococcosis [14–18]. In [18], in order to explore effective control and prevention measures the authors proposed a deterministic model to study the transmission dynamics of echinococcosis in Xinjiang. The results showed that the dynamics of the model was completely determined by the basic reproduction number R_0. Du et al [19] introduced an echinococcosis transmission model with saturation incidence. In addition, they established a threshold type result, which states that when $R_0 < 1$, the disease will die out; the disease will persists, when $R_0 > 1$ and the recovery rate of dogs is very small.

In fact, the transmission process of epidemics are inevitably influenced by random variation in the noisy world. There is no exception for echinococcosis. In terms of the spread of echinococcosis, dogs are the definitive host, human, sheep and cattle are the intermediate host; the dogs engulf organs of sheep and cattle with hydatidosis, which contain huge larval of Echinococcosis granulosus; each of the individual protoscoleces may develop into an adult worm in about 7 or 8 weeks, which dischage eggs with the feces of dogs contaminating the soil, grassland, water and the living place of dogs; touching or taking articles, nutriment or water polluted by eggs will develop echinococcosis. As eggs can survive about 11 days in a dry environment, while in a suitable, moist and low temperature environment the infectivity may contain as long as a year. Therefore, the variation of season and temperature, which can be seen as the random fluctuation, have notably influence on the transmission of echinococcosis. Thus, in order to describe the transmission dynamics and trend more precisely, it is necessary to consider all kinds of uncertainty and stochastic factors. Random fluctuations in temperature will therefore be translated to fluctuations around disease-free equilibrium and endemic equilibrium [23].

However, few of the existing literatures formulate mathematical models to study the effect of unpredictable fluctuations in the environment on the hydatidosis. Thus, the purpose of the present paper is to extend the deterministic model of echinococcosis, introduced by Wang et al [18], allowing the random perturbation around equilibrium state.

In order to study stability properties of the considered model, we use well known method based on construction of appropriate Lyapunov functions. Though many authors have studied the stability of stochastic epidemic models by Lyapunov method, the Lyapunov of quadratic form has been rarely used. So, in this paper we want to construct a Lyapunov function of quadratic form and obtain some relative results.

The paper is organized in the following way. In Section 2, we briefly outline the deterministic model presented in [18] on the basis of which we construct the stochastic models. Some basic preliminaries are presented in Section 3. In Section 4, by constructing a suitable Lyapunov function of quadratic form, we investigate stochastic asymptotic of stability of the disease-free equilibrium and endemic equilibrium of the considered model. In Section 5, we present a numerical simulation result to show that the stochastic model of Echinococcosis disease transmission, with quantities which are reliable data, is compatible with the mathematical results obtained through the paper. We also give some discussion about the result.

2. THE MODEL

In this section, we briefly present the results by Wang et al [18]. They introduce a deterministic model of the hydatid disease and present the spread among human, definitive host (dogs and other canidae), intermediate host (sheep, goat, swine, etc) and eggs in the environment. Also, they found that the parameters of the human do not affect the dynamical behaviors of the echinococcosis. Hence in this paper we only consider definitive hosts, intermediate hosts and eggs. The dynamics of dogs, livestock and Echinococcosis eggs population is given by the following model

$$
\left\{
\begin{aligned}
\frac{\mathrm{d}S_D(t)}{\mathrm{d}t} &= A_1 - \beta_1 S_D(t) I_L(t) - d_1 S_D(t) + \sigma I_D(t),\\
\frac{\mathrm{d}I_D(t)}{\mathrm{d}t} &= \beta_1 S_D(t) I_L(t) - (d_1 + \sigma) I_D(t),\\
\frac{\mathrm{d}S_L(t)}{\mathrm{d}t} &= A_2 - \beta_2 S_L(t) x(t) - d_2 S_L(t),\\
\frac{\mathrm{d}I_L(t)}{\mathrm{d}t} &= \beta_2 S_L(t) x(t) - d_2 I_L(t),\\
\frac{\mathrm{d}x(t)}{\mathrm{d}t} &= a I_D(t) - d x(t).
\end{aligned}
\right.
\tag{2.1}
$$

with the initial condition $S_D(0) = S_{D0}$, $I_D(0) = I_{D0}$, $S_L(0) = S_{L0}$, $I_L(0) = I_{L0}$, $x(0) = x_0$. The total dogs population at t, given by $N_1(t)$, they partitioned into $S_D(t)$, $I_D(t)$, of individuals who are susceptible, and infectious, respectively. In the same way, they divide intermediate hosts population into two subclasses: the susceptible population $S_L(t)$, the infected population $I_L(t)$, and $N_2(t) = S_L(t) + I_L(t)$. All parameters of model (2.1) are assumed positive. For the dog population, A_1 describes the annual recruitment rate; d_1 is the natural death rate; σ denotes the recovery rate of transition from infected to noninfected dogs, including natural recovery rate and recovery due to anthelmintic treatment; $\beta_1 S_D I_L$ describe the transmission of Echinococcosis between susceptible dogs and infectious livestock after the ingestion of cyst-contain organs of infected livestock. For the livestock population, A_2 is the annual recruitment rate; d_2 is the death rate; $\beta_2 S_L x$ describes the transmission of Echinococcosis to livestock by the ingestion of Echinococcosis eggs in the environment. For Echinococcosis eggs, a denote the released rate from infected dogs; d is the mortality rate of eggs.

The total dog population size is $N_1(t)$ and can be determined as a solution of the differential equation $\dot{N}_1(t) = A_1 - d_1 N_1(t)$, which is obtained by adding the first two equations in model (2.1). Similarly, the total population size of livestock is $N_2(t)$ where $N_2(t)$ is a solution of the differential equation $\dot{N}_2(t) = A_2 - d_2 N_2(t)$ obtained by adding the third and the forth equation in model (2.1). Because $\lim_{t\to\infty} N_1(t) = \frac{A_1}{d_1}$, and $\lim_{t\to\infty} N_2(t) = \frac{A_2}{d_2}$, we can assume, without loss of generality, that the total dog population and the total livestock population are constants, that is $N_1(t) = \frac{A_1}{d_1}$ and $N_2(t) = \frac{A_2}{d_2}$, and therefore, model (2.1) becomes equivalent with the following model

$$
\begin{cases}
\dfrac{\mathrm{d}I_D(t)}{\mathrm{d}t} = -\beta_1 I_D(t) I_L(t) - (d_1+\sigma) I_D(t) + \dfrac{\beta_1 A_1}{d_1} I_L(t), \\
\dfrac{\mathrm{d}I_L(t)}{\mathrm{d}t} = -\beta_2 I_L(t) x(t) - d_2 I_L(t) + \dfrac{\beta_2 A_2}{d_2} x(t), \\
\dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = a I_D(t) - d x(t).
\end{cases} \tag{2.2}
$$

with initial values $I_D(0) = I_{D0}$, $I_L(0) = I_{L0}$ and $x(0) = x_0$. An important quantity of model (2.1) is the basic reproduction number

$$R_0 = \sqrt[3]{\frac{\beta_1 \beta_2 A_1 A_2 a}{(d_1+\sigma) d_1 d_2^2 d}}.$$

It is a measure of potential for disease spread in deterministic epidemic. Epidemiologically, R_0 is interpreted as the expected number of secondary infectious produced by index case in a completely susceptible host. It controls the number of equilibria of model (2.1). In [18] two equilibrium states of model (2.1) were obtained: the trivial stable state $E_0 = (\frac{A_1}{d_1}, 0, \frac{A_2}{d_2}, 0, 0)$ which represents the disease-free case, and the positive equilibrium state $E^* = (S_D^*, I_D^*, S_L^*, I_L^*, x^*)$, which represents the endemic case, provided that $R_0 > 1$. Therefore, model (2.2) owns a disease-free equilibrium $P_0 = (0, 0, 0)$, when $R_0 \le 1$, otherwise, it has a unique endemic stable state

$P^* = (I_D^*, I_L^*, x^*)$. Denote $\bar{P} = (\bar{I}_D, \bar{I}_L, \bar{x})$, with $\bar{P} = P_0$ if $R_0 \le 1$ and $\bar{P} = P^*$ if $R_0 > 1$.

As is well know, environmental fluctuations have a significant influence on all aspects of real life. So it is reasonable to investigate how these fluctuations affect the epidemic model presented in the previous section. Thus we introduce the stochastic perturbation terms into model (2.2).

It is highlight that most studies on biological and epidemiological models are devoted to stability analysis of their equilibrium states. For this purpose the standard approach is to assume that stochastic perturbations of the state variables around their steady state P^* and P_0 are Gaussian white noise type and that they are proportional to the distances of S_D, I_L, x from $\bar{S}_D$, $\bar{I}_L$, $\bar{x}$, respectively. This approach ensures that the equilibrium states P_0 or P^* of deterministic model (2.2) also be the equilibrium of stochastic model when $R_0 < 1$ or $R_0 > 1$, that is, $\bar{P}$ is the equilibrium of the stochastic model.

Hence, we obtain the stochastic Echinococcosis epidemic model of the form

$$
\begin{cases}
\dfrac{dI_D(t)}{dt} = \beta_1(\dfrac{A_1}{d_1} - I_D(t))I_L(t) - (d_1+\sigma)I_D(t) + \sigma_1(I_D(t) - \bar{I}_D)\dot{B}_1(t), \\
\dfrac{dI_L(t)}{dt} = \beta_2(\dfrac{A_2}{d_2} - I_L(t))x(t) - d_2 I_L(t) + \sigma_2(I_L(t) - \bar{I}_L)\dot{B}_2(t), \\
\dfrac{dx(t)}{dt} = aI_D(t) - dx(t) + \sigma_3(x(t) - \bar{x})\dot{B}_3(t).
\end{cases} \tag{2.3}
$$

with the initial conditions $I_D(0) = I_{D0}$, $I_L(0) = I_{L0}$ and $x(0) = x_0$, where $B(t) = (B_1(t), B_2(t), B_3(t))$ represents a three-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ with the filtration $\{\mathcal{F}_t\}_{t\ge 0}$, satisfying the usual conditions (it is right continuous and increasing, while $\{\mathcal{F}_0\}$ contains all P-null sets) and $\sigma_i > 0, i = 1, 2, 3$ denote the white noise intensity.

In order discuss the dynamic properties of model (2.3), we introduce new variables $x_1 = I_D - \bar{I}_D$, $x_2 = I_L - \bar{I}_L$ and $x_3 = x - \bar{x}$. Thus, we obtain the following model

$$
\begin{cases}
\dfrac{dx_1(t)}{dt} = -\beta_1 x_1(t)x_2(t) - (\beta_1 \bar{I}_L + d_1 + \sigma)x_1(t) + \beta_1 \bar{S}_D x_2(t) + \sigma_1 x_1 \dot{B}_1(t), \\
\dfrac{dx_2(t)}{dt} = -\beta_2 x_2(t)x_3(t) - (\beta_2 \bar{x} + d_2)x_2(t) + \beta_2 \bar{S}_L x_3(t) + \sigma_2 x_2(t)\dot{B}_2(t), \\
\dfrac{dx_3(t)}{dt} = ax_1(t) - dx_3(t) + \sigma_3 x_3(t)\dot{B}_3(t).
\end{cases} \tag{2.4}
$$

with the initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$ and $x_3(0) = x_{30}$. Obviously, the stability of equilibrium $\bar{P}$ of model (2.3) is equivalent to the stability of the trivial solution of model (2.4).

3. PRELIMINARIES

In order to discuss the stability of trivial solution of model (2.4), we give some results on the Lyapunov matrix equation (see [20], for instance) and Itô stochastic differential equations (see [21]).

Lemma 3.1. *Let A is a real constant matrix. If all characteristic roots of A have negative real parts, then for any negative definite matrix $C = C^T$, there exists a positive definite matrix $B = B^T$ such that $A^T B + BA = C$.*

Consider the stochastic differential equation

$$dx(t) = f(t, x(t))dt + g(t, x(t))dB(t), \tag{3.1}$$

with initial conditions $x(0) = x_0 \in \mathbb{R}^d$. We assume, with no emphasis on conditions, that there exists a unique global solution $x(t; 0, x_0)$ of Eq. (3.1), and $f(t, 0) = 0$ and $g(t, 0) = 0$ for $t \geq 0$. Thus, Eq. (3.1) has the trivial solution $x(t) \equiv 0$, corresponding to the initial condition $x_0 = 0$.

Definition 3.2. The trivial solution of Eq. (3.1) is said to be stochastically stable (or stable in probability) if for every $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, 0) > 0$ such that when $|x_0| < \delta$,

$$P\{|x(t; 0, x_0)| < r, t \geq 0\} \geq 1 - \varepsilon.$$

Definition 3.3. The trivial solution of Eq. (3.1) is said to be stochastically asymptotically stable if it is stochastically stable and, for every $\varepsilon \in (0, 1)$, there exists a $\delta = \delta(\varepsilon, r, 0) > 0$ such that when $|x_0| < \delta$,

$$P\{\lim_{t\to\infty} x(t; 0, x_0) = 0\} \geq 1 - \varepsilon.$$

Denote by $C([0, +\infty) \times \mathbb{R}^d; \mathbb{R}_+)$ the space of all non-negative functions $V(t, x) : [0, +\infty) \times \mathbb{R}^d \to \mathbb{R}_+$ such that they are continuous differential with respect to t and twice differential with respect to x, where $\mathbb{R}_+ = (0, +\infty)$. For any $V(t, x) \in C([0, +\infty) \times \mathbb{R}^d; \mathbb{R}_+)$, define the differential operator L associated with Eq. (3.1) by

$$LV(t, x) = V_t(t, x) + V_x(t, x)f(t, x) + \frac{1}{2} trace[g^T(t, x)V_{xx}g(t, x)].$$

The following lemma gives the conditions for stochastic asymptotic stability of trivial solution of Eq. (3.1) in terms of Lyapunov function.

Lemma 3.4. *Suppose that there exist non-negative function $V(t, x) \in C([0, \infty) \times \mathbb{R}^d; \mathbb{R}_+)$, continuous functions $a, b : \mathbb{R}_+ \to \mathbb{R}_+$, positive on $\mathbb{R}_+$ and a positive constant K such that $a(|x|) \leq V(t, x) \leq b(|x|)$ for all $|x| \leq K$ and $t \geq 0$.*

(a) If $LV \leq 0$ with $|x| < K$ and $t \geq 0$, then the trivial solution of Eq. (3.1) is stochastically stable.

(*b*) *If there exists a continuous function* $c : \mathbb{R}_+ \to \mathbb{R}_+$, *such that* $LV \leq -c(|x|)$ *with* $|x| \leq K$ *and* $t \geq 0$, *then the trivial solution of Eq.* (3.1) *is stochastically asymptotically stable.*

Since many problems concerning the stability of the equilibrium states of non-linear stochastic model can be reduced to those about stability of solutions of linear associated equation. We consider linear equation of Eq. (3.1)

$$dx(t) = Fx(t)dt + Gx(t)dB(t), \quad t \geq 0. \tag{3.2}$$

Lemma 3.5. *If linear system* (3.2) *is stochastically asymptotically stable, and the coefficients of equation* (3.1) *and* (3.2) *satisfy the inequality*

$$|f(t,x) - Fx| + |g(t,x) - Gx| < \delta|x|, \tag{3.3}$$

in a sufficiently small neighborhood of $x = 0$ *with a sufficiently small constant* δ, *then trivial solution* $x(t) = 0$ *of Eq.* (3.1) *is also stochastically asymptotically stable.*

4. STABILITY ANALYSIS FOR STOCHASTIC MODEL

In order to show the stochastic asymptotic stability of trivial solution of model (2.4), we consider the linearized system of model (2.4),

$$\begin{cases} \dfrac{d\tilde{x}_1(t)}{dt} = -(\beta_1\bar{I}_L + d_1 + \sigma)\tilde{x}_1(t) + \beta_1\bar{S}_D\tilde{x}_2(t) + \sigma_1\tilde{x}_1(t)\dot{B}_1(t), \\ \dfrac{d\tilde{x}_2(t)}{dt} = -(\beta_2\bar{x} + d_2)\tilde{x}_2(t) + \beta_2\bar{S}_L\tilde{x}_3(t) + \sigma_2\tilde{x}_2(t)\dot{B}_2(t), \\ \dfrac{d\tilde{x}_3(t)}{dt} = a\tilde{x}_1(t) - d\tilde{x}_3(t) + \sigma_3\tilde{x}_3(t)\dot{B}_3(t). \end{cases} \tag{4.1}$$

Firstly, take into account the deterministic part of system (4.1), and get the following lemma.

Let

$$A = \begin{pmatrix} -(\beta_1\bar{I}_L + d_1 + \sigma) & \beta_1\bar{S}_D & 0 \\ 0 & -(\beta_2\bar{x} + d_2) & -\beta_2\bar{S}_L \\ a & 0 & -d \end{pmatrix}.$$

Lemma 4.1. *Assume* $\bar{P} = P_0$ *if* $R_0 < 1$ *and* $\bar{P} = P^*$ *if* $R_0 > 1$. *Then all characteristic roots of* A *have negative real parts.*

Proof. By simple calculation, we get the corresponding characteristic equation of matrix A

$$\Phi(\lambda) = \lambda^3 + b_2\lambda + b_1\lambda + b_0,$$

where

$$b_0 = d(\beta_1 \bar{I}_L + d_1 + \sigma)(\beta_2 \bar{x} + d_2) - a\beta_1 \bar{S}_D \beta_2 \bar{S}_L,$$

$$b_1 = \beta_1 \bar{I}_L + d + d_1 + d_2 + \sigma,$$

$$b_2 = (\beta_1 \bar{I}_L + d_1 + \sigma)(\beta_2 + d_2) + d(\beta_1 \bar{I}_L + d_1 + d_2 + \sigma).$$

It is easy to see $b_1 > 0$, $b_2 > 0$. Noticing that

$$\begin{cases} A_1 = \beta_1 \bar{S}_D I_L + d_1 \bar{S}_D + \sigma \bar{I}_D, \\ A_2 = \beta_2 \bar{S}_L \bar{x} + d_2 \bar{I}_L + d_2 \bar{S}_L, \\ a\beta_1 \bar{S}_D \beta_2 \bar{S}_L = dd_2(d_1 + \sigma). \end{cases}$$

It follows that when $R_0 < 1$, we have $b_0 = dd_2(d_1 + \sigma)(1 - R_0^3) > 0$ and

$$\begin{aligned} b_1 b_2 - b_0 \; &= d^2(d_1 + d_2 + \sigma) + [(d_1 + \sigma)d_2 + d(d_1 + d_2 + \sigma)](d_1 + d_2 + \sigma) \\ &\quad + \frac{a\beta_1\beta_2 A_1 A_2}{d_1 d_2} > 0, \end{aligned}$$

and that when $R_0 > 1$, we have $b_0 = dd_2(d_1 + \sigma)(R_0^3 - 1) > 0$ and

$$\begin{aligned} b_1 b_2 - b_0 \; &= d^2(\beta_1 I_L^* + \beta_2 x^* + d_1 + d_2 + \sigma) + [(\beta_1 I_L^* + d_1 + \sigma)(\beta_2 x^* + d_2) \\ &\quad + d(\beta_1 I_L^* + \beta_2 x^* + d_1 + d_2 + \sigma)](\beta_1 I_L^* + \beta_2 x^* + d_1 + d_2 + \sigma) \\ &\quad + a\beta_1 S_D^* \beta_2 S_L^* > 0. \end{aligned}$$

Therefore, by Routh-Herwitz criteria, all roots of $\Phi(\lambda)$ have negative real parts. That is, all characteristic roots of A have negative real parts. $\square$

By Lemma 4.1 it is easy to obtain the following result, which is useful in constructing Lyapunov in the main result of this paper.

Lemma 4.2. *Assume $\bar{P} = P_0$ if $R_0 < 1$ and $\bar{P} = P^*$ if $R_0 > 1$. Taking matrix*

$$C = \begin{pmatrix} -c_1 & 0 & \\ 0 & -c_2 & 0 \\ 0 & 0 & -c_3 \end{pmatrix},$$

where $c_i > 0$, $i = 1, 2, 3$, which is a definite negative matrix. Then, there exists a positive definite symmetric quadratic $B = B^T = (b_{ij})_{3\times 3}$ and $b_{ij}, i, j = 1, 2, 3$ are constants to be determined later, such that $A^T B + BA = 2C$.

Proof. According to Lemma 3.1 and Lemma 4.1, there exists a positive definite matrix $B = B^T$, which is unique, such that $A^T B + BA = 2C$, which equals to the following linear equations

$$
\text{(4.2)}\qquad \begin{cases} -(\beta_1\bar{I}_L+d_1+\sigma)b_{11}+ab_{13}=-c_1, \\ \beta_1\bar{S}_D b_{11}-(\beta_1\bar{I}_L+\beta_2\bar{x}+d_1+d_2+\sigma)b_{12}+ab_{23}=0, \\ \beta_2\bar{S}_L b_{12}-(\beta_1\bar{I}_L+d+d_1+\sigma)b_{13}+ab_{33}=0, \\ \beta_1\bar{S}_D b_{12}-(\beta_2\bar{x}+d_2)b_{22}=-c_2, \\ \beta_1\bar{S}_D b_{13}+\beta_2\bar{S}_L b_{22}-(\beta_2\bar{x}+d+d_2)b_{23}=0, \\ \beta_2\bar{S}_L b_{23}-db_{33}=-c_3. \end{cases}
$$

Therefore, there exists a unique solution to system (4.2), which depend on the value of the elements of C. Using Gramer's principle, we obtain

$$
\begin{aligned}
b_{11}&=\frac{c_1}{a_1}+\frac{c_1a\beta_1\bar{S}_D\beta_2\bar{S}_L}{a_1^2a_2a_3}+\frac{Ga^2\beta_2\bar{S}_L}{a_1a_2a_3}n,\\
b_{12}&=\frac{c_1\beta_1\bar{S}_D}{a_1a_2}+(\frac{a}{a_2}+\frac{a^2\beta_1\bar{S}_D\beta_2\bar{S}_L}{a_1a_2^2a_3})n+\frac{ac_1(\beta_1\bar{S}_D)^2\beta_2\bar{S}_L}{a_1^2a_2^2a_3},\\
b_{13}&=\frac{c_1\beta_1\bar{S}_D\beta_2\bar{S}_L}{a_1a_2a_3}+\frac{a\beta_2\bar{S}_L}{a_2a_3}n,\\
b_{22}&=\frac{E}{a_4}+\frac{a\beta_1\bar{S}_D}{a_2}n,\quad b_{23}=n,\quad b_{33}=\frac{mc_3+G\beta_2\bar{S}_L}{mE},
\end{aligned}
$$

where

$$
\begin{aligned}
&a_1=\beta_1\bar{I}_L+d_1+\sigma,\quad a_2=\beta_1\bar{I}_L+\beta_2\bar{x}+d_1+d_2+\sigma,\\
&a_3=\beta_1\bar{I}_L+d+d_1+\sigma,\quad a_4=\beta_2\bar{x}+d_2,\ a_5=\beta_2\bar{x}+d+d_2,\\
&m=\frac{a_4a_5(\beta_1\bar{I}_L+\beta_2\bar{x}+d_2)+a_5\beta_2\bar{x}(d_1+\sigma)+d_2(d_1+\sigma)(\beta_2\bar{x}+d_2)}{a_2a_4},\\
&E=c_2+\frac{c_1(\beta_1\bar{S}_D)^2}{a_1a_2},\quad G=\frac{c_1(\beta_1\bar{S}_D)^2\beta_2\bar{S}_L}{a_1a_2a_3}+\frac{E\beta_2\bar{S}_L}{a_4},\\
&n=\frac{G}{m}+\frac{a\beta_1\bar{S}_D(mc_3+G\beta_2\bar{S}_L)}{m^2Ha_3},\quad H=d+\frac{a\beta_1\bar{S}_D\beta_2\bar{S}_L}{a_3m}.
\end{aligned}
$$

Therefore, we get the definitely positive matrix B. □

In the sequel, by a suitable Lyapunov function method, which was proposed by Kolmanovskii and Shaikhet, we obtain the conditions for stochastic asymptotic stability of trivial solution of linearized system (4.1) around the equilibrium.

Theorem 4.3. *Assume that there exist positive constants c_1, c_2 and c_3 such that*

$$
\text{(4.3)}\qquad 0\le\sigma_1^2\le\frac{2c_1}{b_{11}},\quad 0\le\sigma_2^2\le\frac{2c_2}{b_{22}},\quad 0\le\sigma_2^2\le\frac{2c_3}{b_{33}},
$$

where b_{11}, b_{22}, b_{33} are determined in Lemma 4.2. Then the trivial solution of model (2.4) is stochastically asymptotically stable.

Proof. Firstly, we prove that the trivial solution of system (4.1) is stochastically asymptotically stable.

Denote

$$D = \begin{pmatrix} \sigma_1\tilde{x}_1 & 0 & 0 \\ 0 & \sigma_2\tilde{x}_2 & 0 \\ 0 & 0 & \sigma_3\tilde{x}_3 \end{pmatrix}.$$

Regarding the complexity of system (4.1) we cannot apply any standard Lyapunov function. Consider Lyapunov function defined in Lemma 4.2, $V(\mathbf{x}) = \mathbf{x}^T B\mathbf{x}$. If we apply the generating operator L on $V(\mathbf{x})$, it is not difficult to show that

$$\begin{aligned} LV &= \mathbf{x}^T(A^T B + BA)\mathbf{x} + \frac{1}{2}trace(D(2B)D^T) \\ &= \dot{V} + (b_{11}\sigma_1^2\tilde{x}_1^2 + b_{22}\sigma_2^2\tilde{x}_2^2 + b_{33}\sigma_3^2\tilde{x}_3^2) \\ &= 2x^T Cx + (b_{11}\sigma_1^2\tilde{x}_1^2 + b_{22}\sigma_2^2\tilde{x}_2^2 + b_{33}\sigma_3^2\tilde{x}_3^2) \\ &= -(2c_1 - b_{11}\sigma_1^2)\tilde{x}_1^2 - (2c_2 - b_{22}\sigma_2^2)\tilde{x}_2^2 - (2c_3 - b_{33}\sigma_3^2)\tilde{x}_3^2. \end{aligned}$$

which is negative, in regard of (4.3). Hence, by virtue of Lemma 3.5, it follows that the trivial solution of system (4.1) is stochastically asymptotically stable.

On the other hand, by Lemma 3.4 and Lemma 3.5, in order to prove the theorem it suffices to verify condition (3.3). The left-side of (3.3) becomes

$$\begin{aligned} &\sqrt{(-\beta_1 x_1 x_2)^2 + (-\beta_2 x_2 x_3)^2} \\ &\leq \sqrt{\beta_1^2\varepsilon^2 x^2 + \beta_2^2\varepsilon^2 x_2^2} \leq M\varepsilon\sqrt{x_1^2 + x_2^2} \leq M\varepsilon|\mathbf{x}|. \end{aligned}$$

providing that $\mathbf{x} = (x_1, x_2, x_3)$ belongs to the small neighborhood $|\mathbf{x}| < \varepsilon$ and $M = \sqrt{\max\{\beta_1^2, \beta_2^2\}}$. Since condition (3.3) holds, the proof is completed. □

Directly from Theorem 4.3 we further have the following corollaries.

Corollary 4.4. *When $R_0 < 1$, disease-free equilibrium P_0 of model* (2.3) *is stochastically asymptotically stable.*

Corollary 4.5. *When $R_0 > 1$, endemic equilibrium P^* of system* (2.3) *is stochastically asymptotically stable.*

5. NUMERICAL SIMULATIONS

In this section we show that the simulation of the stochastic model of a Echinococcosis disease transmission is compatible with the mathematical results obtained in Section 4. In order to verify the stability results for model (2.3), we use the Euler-Maruyama approximate method (see [22]) to simulate the solution of the considered equations.

In view of [18], we fix $A_1 = 2 \times 10^4\ yr^{-1}$, $d_1 = 0.08\ yr^{-1}$, $\beta_1 = 5.8 \times 10^{-8}\ yr^{-1}$, $\sigma = 2\ yr^{-1}$, $A_2 = 1.05 \times 10^8\ yr^{-1}$, $d_2 = 0.33\ yr^{-1}$, $\beta_2 = 7.4 \times 10^{-8}\ yr^{-1}$, $a = 9.7\ yr^{-1}$

and $d = 10.42\ yr^{-1}$. For such a choice of the model parameters, we obtain the reproduction number $R_0 = 0.7736 < 1$, which ensure $\bar{P} = P_0$, *i.e.* the perturbation is around the disease-free equilibrium of system (2.3). We set $c_1 = 10000$, $c_2 = \frac{1}{5}$ and $c_3 = 100$, then we can obtain that $0 \leq \sigma_1^2 \leq 0.4304$, $0 \leq \sigma_2^2 \leq 0.1168$ and $0 \leq \sigma_3^2 \leq 0.1762$ from condition (4.3). We choose $\sigma_1^2 = 0.23$, $\sigma_2^2 = 0.11$ and $\sigma_3^2 = 0.16$, then by Corollary 4.4 disease-free equilibrium E_0 is stochastically asymptotically stable, which can be seen in Figure 1, where the initial conditions are given by $I_D(0) = 8 \times 10^5$, $I_L(0) = 5.7 \times 10^7$ and $x(0) = 1.44 \times 10^7$. From Figure 1 (a) we can see under environmental noises echinococcosis in dogs will die out about eighty years later, however, in deterministic model echinococcosis in dogs dies out about twenty years later. It followed that environmental noises prolong existing time of echinococcosis in dogs. We can get similar results from (b) and (c).

On the other hand, from [18] we can also get that if we change A_1 to $2 \times 10^5 yr^{-1}$, then $R_0 = 1.667 > 1$ becomes to which is greater than unit one, which means $\bar{P} = P^*$, *i.e.* the perturbation is around the endemic equilibrium of model (2.3). For this we choose $c_1 = 500$, $c_2 = 1$ and $c_3 = 50$, and we can obtain $0 \leq \sigma_1^2 \leq 2.6567$, $0 \leq \sigma_2^2 \leq 0.7994$ and $0 \leq \sigma_3^2 \leq 0.8054$ from condition (4.3). We take $\sigma_1^2 = 1.1$, $\sigma_2^2 = 0.8$ and $\sigma_3^2 = 0.45$, then from Corollary 4.5 endemic equilibrium P^* of system (2.3) is stochastically asymptotically stable, which can be seen in Figure 2. Figure 2 showed that when $R_0 > 1$ deterministic model and stochastic counterpart has very little difference in dogs and livestock, but environmental noises can delay echinococcosis eggs attain endemic equilibrium about twenty years.

Furthermore, we present some examples in which Theorem 4.3 remain true when condition (4.3) is not satisfied. More precisely, if we use quantities of the model parameters from [18] once more, for $A_1 = 2 \times 10^4\ yr^{-1}$, $d_1 = 0.08\ yr^{-1}$,$\beta_1 = 5.8 \times 10^{-8}\ yr^{-1}$, $\sigma = 2\ yr^{-1}$, $A_2 = 1.05 \times 10^8\ yr^{-1}$, $d_2 = 0.33\ yr^{-1}$, $\beta_2 = 7.4 \times 10^{-8}\ yr^{-1}$, $a = 9.7\ yr^{-1}$ and $d = 10.42\ yr^{-1}$, then $R_0 = 0.7736 < 1$. If we choose that intensities of noise are $\sigma_1^2 = 0.6$, $\sigma_2^2 = 0.3$ and $\sigma_3^2 = 0.4$, then condition (4.3) is not satisfied, but from Figure 3 we see that disease-free equilibrium P_0 of model (2.3) may be stochastically asymptotically stable. In addition, from [18], if we change A_1 to $2 \times 10^5 yr^{-1}$, then $R_0 = 1.667 > 1$. In this case, we choose $\sigma_1^2 = 2.8$, $\sigma_2^2 = 1.0$ and $\sigma_3^2 = 0.81$, then endemic equilibrium P^* of system (2.3) is stochastically asymptotically stable without condition (4.3) as well from Figure 4. Figure 3 (a) showed that echinococcosis in dogs go to extinction about ten years later and in corresponding deterministic model the disease dies out about twenty years later, which means the white noise make echinococcosis in dogs die out much faster. However, from Figure 3 (b) and (c) we can get that the environmental noises prolong the existing time of echinococcosis in livestock and eggs. From Figure 4, it is observed that when $R_0 > 1$ deterministic model and stochastic counterpart has very little difference in dogs and livestock but environmental noises can delay echinococcosis eggs attain endemic equilibrium.

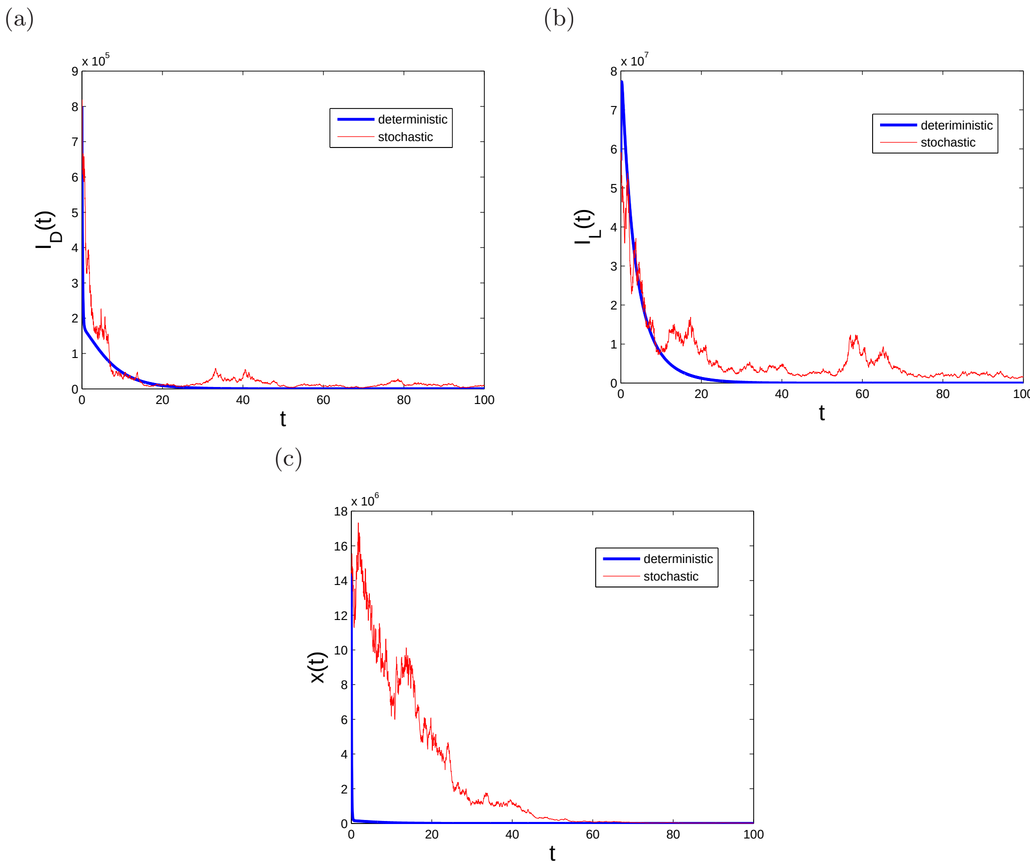

Figure 1. Deterministic and stochastic trajectories of model (2.3) with $R_0 < 1$.

6. DISCUSSION

In this paper, in order to explore the effect on the transmission dynamics and trend of echinococcosis of white noise, we extended the epidemic model of echinococcosis presented in [18] by introducing fluctuation around disease-free equilibrium and endemic equilibrium in them. We propose to examine how environmental fluctuations affect the stability of system (2.2). By constructing a Lyapunov function of quadratic form, which has never been used in the analysis of stochastic differential equations, we obtaind sufficient conditions for the stochastic asymptotic stability of the disease-free equilibrium as well as the endemic equilibrium.

Combining analytical results we have give some numerical simulations. From Figure 1 it is observed that environmental noises prolong existing time of echinococcosis. When $R_0 < 1$ and condition (4.3) is not satisfied from Figure 3 we can get that the white noise make echinococcosis in dogs die out much faster but prolong the existing time of echinococcosis in livestock and eggs. Figure 2 and Figure 4 showed that

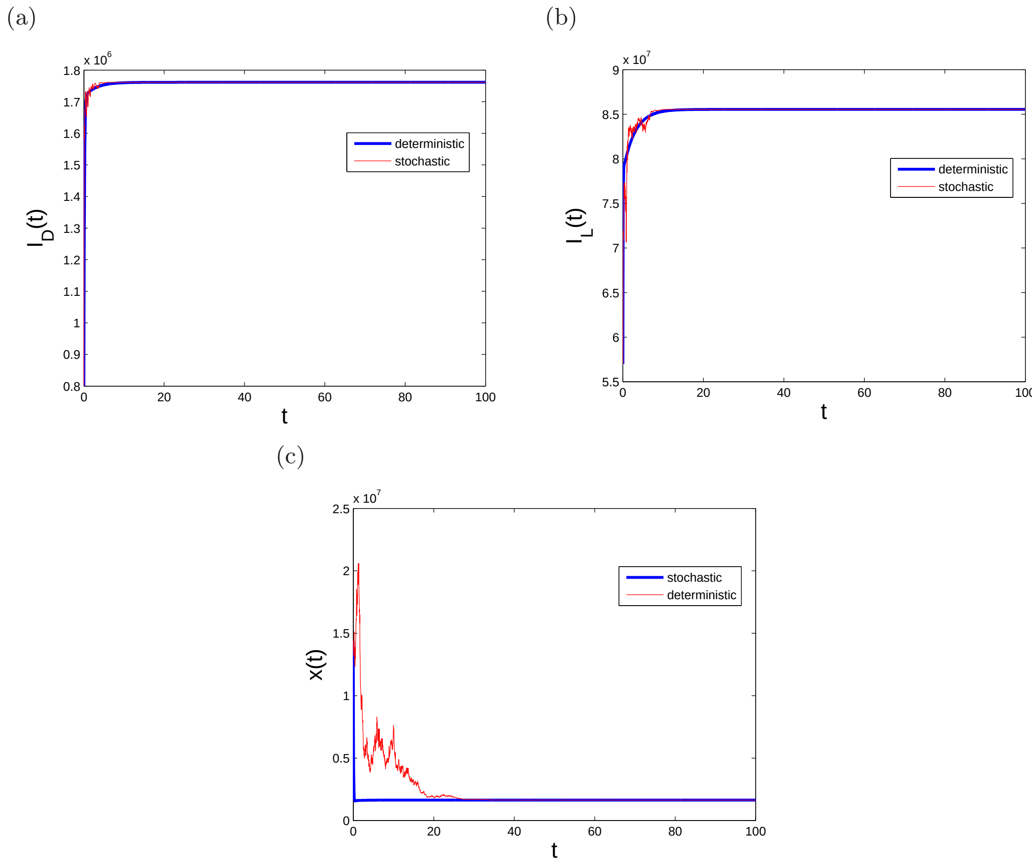

Figure 2. Deterministic and stochastic trajectories of model (2.3) with $R_0 > 1$.

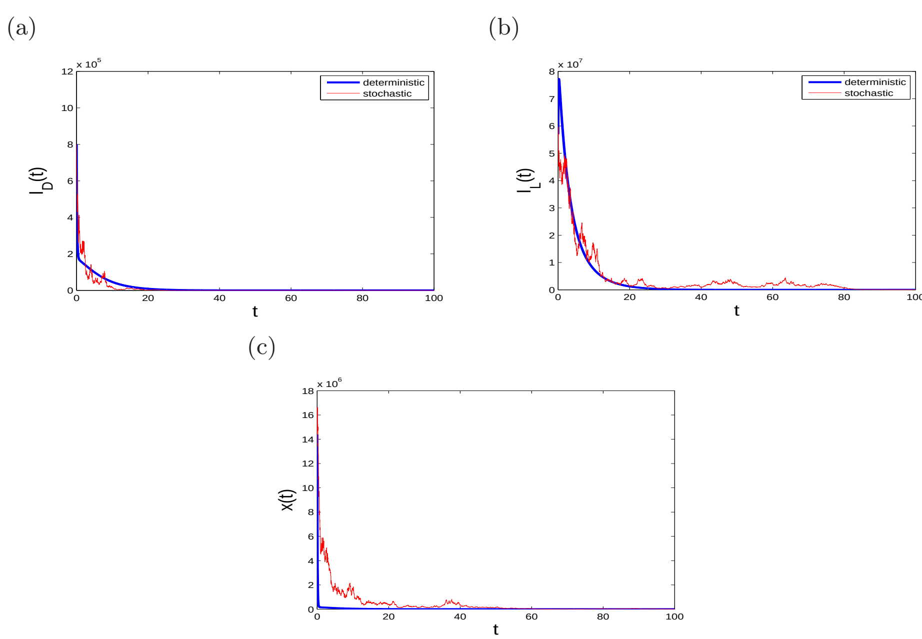

Figure 3. Deterministic and stochastic trajectories of model (2.3) with $R_0 < 1$ and without condition (4.3).

when $R_0 > 1$ deterministic model and stochastic counterpart has very little difference in dogs and livestock but environmental noises can delay echinococcosis eggs attain endemic equilibrium. Furthermore, according to Figure 3 and Figure 4 we can see that Theorem 4.3 give sufficient but not necessary condition for the trivial solution of considered system to be stochasticaaly asymptotically stable, in other words, the stability conditions for the disease-free equilibrium and endemic equilibrium are weaker than condition (4.3). Therefore, in our further work, we wish to give a sufficient and necessary condition for stochastic asymptotic stability of stochastic echinococcosis epidemic model considered in this paper.

(a) (b)

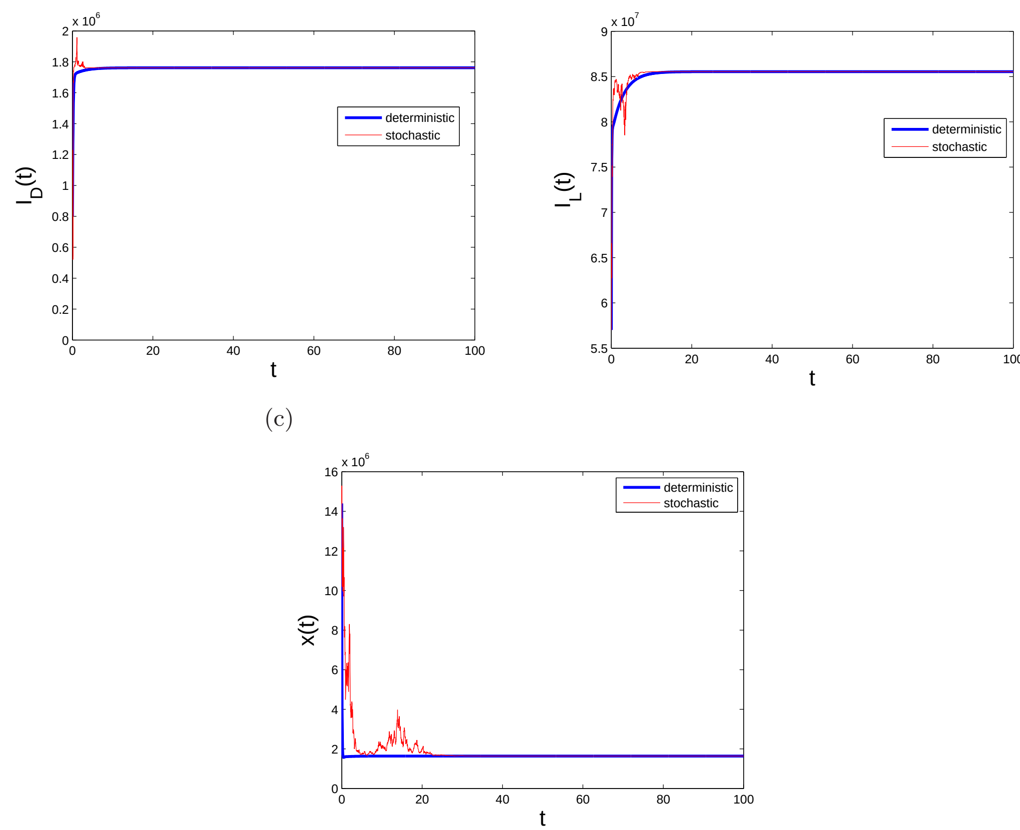

Figure 4. Deterministic and stochastic trajectories of model (2.3) with $R_0 > 1$ and without condition (4.3).

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