Neural, Parallel, and Scientific Computations 24 (2016) 305-315

AN EFFICIENT PDE-BASED NONLINEAR ANISOTROPIC DIFFUSION MODEL FOR IMAGE DENOISING

SANTOSH KUMAR, MOHD. SARFARAZ, AND M. K. AHMAD

Department of Mathematics, Aligarh Muslim University, Aligarh, India

ABSTRACT. In this paper, we propose a new nonlinear anisotropic diffusion model for image denoising. The main idea is to apply a priori smoothness on the solution image. We present proof of the viscosity solution for our model. The results of our model using explicit numerical schemes are compared with other known image denoising models.

Keywords: Image denoising, nonlinear diffusion

AMS subject classification: 65M06, 76R50, 68U10

1. Introduction

The nonlinear diffusion techniques and PDE-based variational models are very popular in image denoising and restoration. The nonlinear diffusion method for image denoising and edge detection was first introduced by Perona and Malik [13]. This method is based on a diffusion process governed by a partial differential equation (PDE), where diffusion amount depends on the gradient of images.

Mathematically, $u_0 : \Omega \to \mathbb{R}$ represents a noisy version of a true image, and it is obtained by the following imaging process

$$u_0(x) = u(x) + n(x), \tag{1.1}$$

where $u(x)$ denotes the desired clean image, $u_0(x)$ denotes the pixel values of a noisy image for $x \in \Omega$, $\Omega \subset \mathbb{R}^2$ is a bounded domain, usually a rectangle and $n(x)$ is additive white noise assumed to be close to Gaussian. The values $n(i, j)$ of n at the pixels (i, j) are independent random variables, each with a Gaussian distribution of zero mean and variance σ^2.

In our tests, we will use the peak signal to noise ratio (PSNR) as a criteria for the quality of restoration:

$$\mathrm{PSNR} = 10\log_{10}\left(\frac{R^2}{\frac{1}{mn}\sum_{i,j}^{n}(u(i,j) - u_{new}(i,j))^2}\right), \tag{1.2}$$

where $\{u(i, j) - u_{new}(i, j)\}$ is the difference of the pixel values between the restored and original images.

Received February 24, 2016

The choice of the diffusivity c is very important in controlling the smoothing and even enhancement of edges. The Charbonnier diffusivity $c(s) = \frac{1}{\sqrt{1+(|s|^2/K^2)}}$, that is related to the convex regularizer $\psi(s^2) = \sqrt{K^4 + K^2 s^2} - K^2$, see references [8, 22], is used in our experiments.

In this paper, we propose a new nonlinear anisotropic diffusion model which incorporates adaptive information computed from the image at scale t. Following [1, 4], well-posedness of the proposed scheme is proved using the theory of viscosity solutions. We present proof of the viscosity solution of our model. We have tested our algorithm on various types of images. To quantify the results, the experimental values in terms of PSNR are given in Tables 1–3.

2. Total variation based denoising algorithms

In general, variational deblurring and denoising of an image can be achieved by minimizing the energy functional presented in [23],

$$E(u) = \int_\Omega \psi(|\nabla u|^2)\, dx + \frac{\lambda}{2} \int_\Omega (u - u_0)^2\, dx. \tag{2.1}$$

The Euler-Lagrange equation associated with (2.1) with homogeneous Neumann boundary conditions is given by

$$\begin{aligned} &0 = -\operatorname{div}(\psi'(|\nabla u|^2)\nabla u) + \lambda\, (u - u_0), \quad x \in \Omega, \\ &\frac{\partial u}{\partial \vec{n}} = 0, \quad x \in \partial\Omega, \end{aligned} \tag{2.2}$$

where $\partial\Omega$ is the boundary of Ω and $\vec{n}$ is the outward normal to $\partial\Omega$.

The resulting gradient descent equation is

$$u_t = \operatorname{div}(c(|\nabla u|)\nabla u) - \lambda\, (u - u_0), \tag{2.3}$$

with $u(x, 0)$ given as initial data (the original noisy image $u_0(x)$ used as initial guess), homogeneous Neumann boundary conditions, i.e., $\frac{\partial u}{\partial \vec{n}} = 0$ on the boundary of the domain. It is also known as diffusion-reaction equation where the diffusion term with diffusivity $c(s) = \psi'(s^2)$ is related to the regulariser in the energy functional.

Applying a priori smoothness on the solution image, our nonlinear anisotropic diffusion model becomes,

$$u_t = \operatorname{div}(c(|\nabla G_\sigma * u|)\nabla G_\sigma * u) - \lambda\, (G_\sigma * u - u_0). \tag{2.4}$$

Witkin [24] noticed that the convolution of the signal with Gaussians at each scale was equivalent to solving the heat equation with the signal as initial datum. The term $(G_\sigma * \nabla u)(x, t) = (\nabla G_\sigma * u)(x, t)$, which appears inside the divergence term of (2.4), is simply the gradient of the solution at time σ of the heat equation with $u(x, 0)$ as initial datum.

In order to preserve the notion of scale in the gradient estimate, it is convenient that this kernel G_σ depends on a scale parameter [12]. In fact, the function G_σ can be considered as "low-pass filter" or any smoothing kernel, i.e., a denoising technique is used before solving the nonlinear diffusion problem [1, 5].

We use the following class of functions for the diffusion equation, given in [3, 20],

$$c(x, |\nabla u|) = \alpha(x) c_g(|\nabla u|). \tag{2.5}$$

Here α is the adaptive parameter estimated at each pixel $x \in \Omega$. The function c_g depends on the gradient image $|\nabla u|$ and can be chosen similar to $c(s)$. If we choose $\alpha(x) = 1$, $c_g = c(s)$ and $G_\sigma * u$ as u then the model (2.4) can be written as:

$$\frac{\partial u}{\partial t} = \operatorname{div}(c(x, |\nabla u|)\nabla u) - \lambda(u - u_0). \tag{2.6}$$

3. **Theoretical considerations**

In this section, motivated by Alvarez et al. [1] and Prasath et al. [14], we want to present the viscosity solution for model (2.6).

$$\begin{aligned} &\frac{\partial u}{\partial t} = \operatorname{div}(c(x, |\nabla u|)\nabla u) - \lambda(u - u_0), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}_+, \\ &u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n. \end{aligned} \tag{3.1}$$

Let us first introduce two auxiliary functions depending on x and p from $\mathbb{R}^n$, a symmetric matrix-valued one a and a vector one χ. We denote

$$a_{ij}(x, p) = c(x, |p|)\delta_{ij} + c_y(x, |p|)\frac{p_i p_j}{|p|}, \tag{3.2}$$

$$\chi_i(x, p) = \frac{\partial c(x, |p|)}{\partial x_i}. \tag{3.3}$$

Here δ_{ij} is Kronecker's delta, and c_y is the partial derivative of $c(x, y)$ with respect to the second variable.

Motivated by Alvarez et al. [2], we consider the case of spatially periodic boundary conditions. We will assume that there is an orthogonal basis b_i in $\mathbb{R}^n$ so that

$$u(., x + b_i) = u(., x), \quad x \in \mathbb{R}^n,\ i = 1, 2, \ldots, n. \tag{3.4}$$

Let u_0 is Lipschitz and satisfies (3.4). Of course, c (and thus a and χ) should also satisfy the same spatial periodicity restriction (with respect to x but not to y or p). Functions a and χ are continuous, bounded, periodic and continuously differentiable in x and their x-derivatives are uniformly (w.r.t. p) bounded,

$$a_{ij}(x, p)\xi_i\xi_j \geq C\left[\operatorname{mod}\left(\frac{\partial a(x, p)}{\partial x_k}\right)\right]_{ij}\xi_i\xi_j, \quad k = 1, \ldots, n,\ \xi, x, p \in \mathbb{R}^n. \tag{3.5}$$

Here $\lambda \geq 0$ and below C stands for a generic positive constant, which can take different values in different lines.

We first recall the definition of viscosity sub-/supersolution of (3.1), if for any $\phi \in C^2([0,T]\times\mathbb{R}^n)$ and any point $(x_0,t_0)\in(0,T]\times\mathbb{R}^n$, at which $u-\phi$ attains local maximum/minimum [10].

$$\frac{\partial\phi(x_0,t_0)}{\partial t} - \mathrm{div}(c(x_0,|\nabla\phi(x_0,t_0)|)\nabla\phi(x_0,t_0)) + \lambda(u(x_0,t_0)-u_0(x_0)) \le 0/\ge 0. \tag{3.6}$$

A viscosity solution is a function which is both a subsolution and a supersolution.

Lemma 3.1. *Let A and B be quadratic matrices of order n. Assume that B is symmetric and there is a constant $M \ge 0$ such that*

$$MA_{ij}\xi_i\xi_j \ge \mathrm{mod}(B)_{ij}\xi_i\xi_j, \quad \forall\ \xi\in\mathbb{R}^n. \tag{3.7}$$

Then for any matrix U (of the same order but not necessarily symmetric) one has

$$Tr^2(BU^\top) \le M\|B\|Tr(UAU^\top), \tag{3.8}$$

where $\|.\|$ denotes the operator norm of a matrix and $\mathrm{mod}(B)$ *be the matrix whose entries are the absolute values of the entries of B.*

Proof. Formula (3.7) and (3.8) are invariant with respect to orthogonal changes of bases. Thus, without loss of generality, we may assume that B is already diagonalized by an orthogonal transform. Then

$$\begin{aligned}Tr^2(BU^\top) &= (B_{ii}U_{ii})^2 \le \|B\||B_{ii}U_{ii}^2\\ &= \|B\|(\mathrm{mod}(B)_{ii}U_{ii}^2 \le \|B\|(\mathrm{mod}(B)_{ii}U_{ki}U_{kj}\\ &= \|B\|(\mathrm{mod}(B)_{ij}U_{ki}U_{kj} \le M\|B\|A_{ij}U_{ki}U_{kj} = M\|B\|Tr(UAU^\top).\end{aligned}$$

□

Theorem 3.2. *The problem* (3.1) *has a unique viscosity solution u in $C([0,T]\times\mathbb{R}^n)\cap L^\infty(0,T,W^{1,\infty}(\mathbb{R}^n))$ for any $T\in[0,\infty)$, provided that u_0 is Lipschitz continuous on $\mathbb{R}^n$, and if $v\in C(\mathbb{R}^n\times[0,T))$ is a viscosity solution of* (3.1) *with u_0 replaced by a Lipschitz continuous function v_0, then for all $T\in[0,\infty)$, there exists a constant $C>0$, depending only on u_0, v_0 and T, such that*

$$\sup_{0\le t\le T}\|u(x,t)-v(x,t)\|_{L^\infty(\mathbb{R}^n)} \le C\|u_0-v_0\|_{L^\infty(\mathbb{R}^n)}. \tag{3.9}$$

Moreover, $\inf_{\mathbb{R}^n} u_0 \le u(x,t) \le \sup_{\mathbb{R}^n} u_0$.

Proof. If u is a viscosity solution of equation (3.1) on $\mathbb{R}^n\times\mathbb{R}_+$, then

$$\inf_{\mathbb{R}^n} u_0 \le u(x,t) \le \sup_{\mathbb{R}^n}\ u_0, \quad \text{on } \mathbb{R}^n\times\mathbb{R}_+. \tag{3.10}$$

Let $\phi(x,t)=\delta t$, then, at the point (x_0,t_0), $t_0>0$, of the global maximum of $u(x,t)-\delta t$, (3.6) gives $\delta+\lambda(u(t_0,x_0)-u_0(x_0))\le 0$, when $u(x_0,t_0)<u_0(x_0)$, so we get a contradiction since $u(x_0,t_0)-\delta t_0 \ge u_0(x_0)$ due to the fact that (x_0,t_0) is the

global maximum point of $u(x,t)-\delta t$; thus the function $u(x,t)-\delta t$ attains its global maximum at $t=0$, and it remains to let $\delta \to 0^+$, we get (3.10).

Now, we establish a formal a priori estimate for $\sup_{\mathbb{R}^n} |\nabla u|$. Observe that (3.1) is equivalent to

$$\frac{\partial u}{\partial t} = [a_{ij}(x,\nabla u)u_{x_i x_j} + \chi_i(x,\nabla u)u_{x_i}] - \lambda(u-u_0). \tag{3.11}$$

Differentiating (3.11) with respect to each $x_k,\ k=1,\dots,n$, multiplying by $2u_{x_k}$ and taking a summation w.r.t. k, we get

$$\begin{aligned}\gamma(|\nabla u|^2) := {}& \frac{\partial |\nabla u|^2}{\partial t} - a_{ij}(x,\nabla u)\frac{\partial^2}{\partial x_i \partial x_j}|\nabla u|^2 - \frac{\partial a_{ij}(x,\nabla u)}{\partial p_l}u_{x_i x_j}\frac{\partial}{\partial x_l}|\nabla u|^2 \\ & - \chi_i(x,\nabla u)\frac{\partial}{\partial_i}|\nabla u|^2 - \frac{\partial \chi_i(x,\nabla u)}{\partial p_l}u_{x_i}\frac{\partial}{\partial x_l}|\nabla u|^2 + 2\lambda(u_{x_k} - (u_0)_{x_k})u_{x_k} \\ = {}& -2a_{ij}(x,\nabla u)u_{x_k x_i}u_{x_k x_j} + 2\frac{\partial a_{ij}(x,\nabla u)}{\partial x_k}u_{x_i x_j}u_{x_k} \\ & + 2\frac{\partial \chi_{ij}(x,\nabla u)}{\partial x_k}u_{x_i}u_{x_k}.\end{aligned} \tag{3.12}$$

The Lemma 3.1 gives opportunity to discharge the undesired influence of the second term in the right-hand side of (3.12). For the second term, due to the Lemma 3.1 and Cauchy's inequality, we have

$$\begin{aligned}\left|2\frac{\partial a_{ij}(x,\nabla u)}{\partial x_k}u_{x_i x_j}u_{x_k}\right| &\le C|u_{x_k}|\sqrt{a_{ij}(x,\nabla u)u_{x_k x_i}u_{x_k x_j}} \\ &\le a_{ij}(x,\nabla)u_{x_k x_i}u_{x_k x_j} + C|\nabla u|^2.\end{aligned} \tag{3.13}$$

The sum of the absolute values of the subsequent terms of the right-hand side of (3.12) does not exceed $C(1+|\nabla u|^2)$. Thus,

$$\gamma(|\nabla u|^2) \le C(1+|\nabla u|^2), \tag{3.14}$$

so

$$\gamma(e^{-Ct}(1+|\nabla u|^2)) \le 0. \tag{3.15}$$

From the weak maximum principle for the weakly parabolic operator γ one easily concludes that

$$|\nabla u|^2 \le C. \tag{3.16}$$

Using (3.10) and (3.16), by means of the approach from [2] we can get the uniform Hölder estimate

$$|u(x,t) - y(x,s)|^2 \le C|t-s|. \tag{3.17}$$

From (3.10), (3.16) and (3.17), the solutions of these problems are uniformly bounded and equicontinuous on $\mathbb{R}^n \times [0,T]$. Then we can select a uniformly converging sequence of approximate solutions, and pass to the limit in the viscosity sense using the general consistency/stability results from [9]. The uniqueness of solutions follows from the stability estimate (3.9). This bound may be shown by revisiting the proof of a similar bound in [17]. We only point out that the matrix τ [17] is replaced by

$$\tau = \begin{pmatrix} D_1 & \sqrt{D_1}\sqrt{D_2} \\ \sqrt{D_1}\sqrt{D_2} & D_2 \end{pmatrix}, \tag{3.18}$$

where

$$D_1 = a\left(x_0, \frac{|x_0 - y_0|^2(x_0 - y_0)}{\delta}\right), \quad D_2 = a\left(y_0, \frac{|x_0 - y_0|^2(x_0 - y_0)}{\delta}\right),$$

and the notation within is taken from [17]. note that the $2n\times 2n$ matrix τ is symmetric and positive-semidefinite. □

4. The Discrete Scheme

We still write $G_\sigma * u$ as u. Let u_{ij}^n be the approximation to the value $u(x_i, y_j, t_n)$, where

$$x_i = i\Delta x, \quad y_j = j\Delta x, \quad i, j = 1, 2, \ldots, N,$$
$$N\Delta x = 1, \quad t_n = n\Delta t, \quad n \geq 1,$$

where $\Delta x, \Delta y$ and Δt are the spatial step sizes and the time step size respectively.

The explicit partial derivatives of models (2.3) and (2.4) can be expressed as:

$$\begin{aligned} u_{ij}^t = &\frac{1}{2\Delta x}((c_{i+1,j}^n + c_{i,j}^n)(u_{i+1,j}^n - u_{i,j}^n) - (c_{i,j}^n + c_{i-1,j}^n)(u_{i,j}^n - u_{i-1,j}^n)) \\ &+ \frac{1}{2\Delta x}\left((c_{i,j+1}^n + c_{i,j}^n)(u_{i,j+1}^n - u_{i,j}^n) - (c_{i,j}^n + c_{i,j-1}^n)(u_{i,j}^n - u_{i,j-1}^n)\right) - \lambda(u_{ij}^n - u_{ij}^0), \end{aligned}$$

where the diffusivity $c(|\nabla u|)$ is discretised by,

$$c_{ij}^n = \psi'\left(\left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{\Delta x}\right)^2 + \left(\frac{u_{i,j+1}^n - u_{i,j-1}^n}{\Delta x}\right)^2\right),$$

with homogeneous Neumann boundary conditions.

The explicit method is stable and convergent for $\frac{\Delta t}{\Delta x^2} \leq 0.5$, see [11].

5. Numerical implementation

We have used two gray scale images as shown in Figure 1. The pixel values of all images lie in interval [0, 255]. The Gaussian white noise is added by the normal imnoise function imnoise (I,'Gaussian', M, σ^2), i.e., the mean M and variance σ^2 in Matlab. We first scale the intensities of the images into the range between zero and one before we begin our experiments. We have taken $\Delta t/\Delta x^2 = 0.4$, Charbonnier diffusivity $K = 5$ and Lagrange multiplier $= 0.85$ as in [6] and [7] in our all experiments.

(A) (B)

FIGURE 1. Original Test Images used for different experiments (A) Lena: 256×256, (B) Boat: 256×256.

Table 1. Results obtained by using models (2.3) and (2.4) applied to the images in Figures 2(A) and 3(A) with Gaussian white noise ($\sigma^2 = 0.002$).

Images	PSNR Noisy Image	Images	PSNR (Model-2.3)	Images	PSNR (Model-2.4)
Fig. 2(A)	27.02	Fig. 2(D)	30.33	Fig. 2(G)	30.48
Fig. 3(A)	27.05	Fig. 3(D)	29.75	Fig. 3(G)	30.01
-	-	No. of iterations	400	No. of iterations	200

Table 2. Results obtained by using models (2.3) and (2.4) applied to the images in Figures 2(B) and 3(B) with Gaussian white noise ($\sigma^2 =0.004$).

Images	PSNR Noisy Image	Images	PSNR (Model-2.3)	Images	PSNR (Model-2.4)
Fig. 2(B)	24.08	Fig. 2(E)	27.14	Fig. 2(H)	28.78
Fig. 3(B)	24.09	Fig. 3(E)	26.80	Fig. 3(H)	28.45
-	-	No. of iterations	400	No. of iterations	200

Table 3. Results obtained by using models (2.3) and (2.4) applied to the images in Figures 2(C) and 3(C) with Gaussian white noise ($\sigma^2= 0.006$).

Images	PSNR Noisy Image	Images	PSNR (Model-2.3)	Images	PSNR (Model-2.4)
Fig. 2(C)	22.36	Fig. 2(F)	25.17	Fig. 2(I)	27.46
Fig. 3(C)	22.33	Fig. 3(F)	24.96	Fig. 3(I)	27.28
-	-	No. of iterations	400	No. of iterations	200

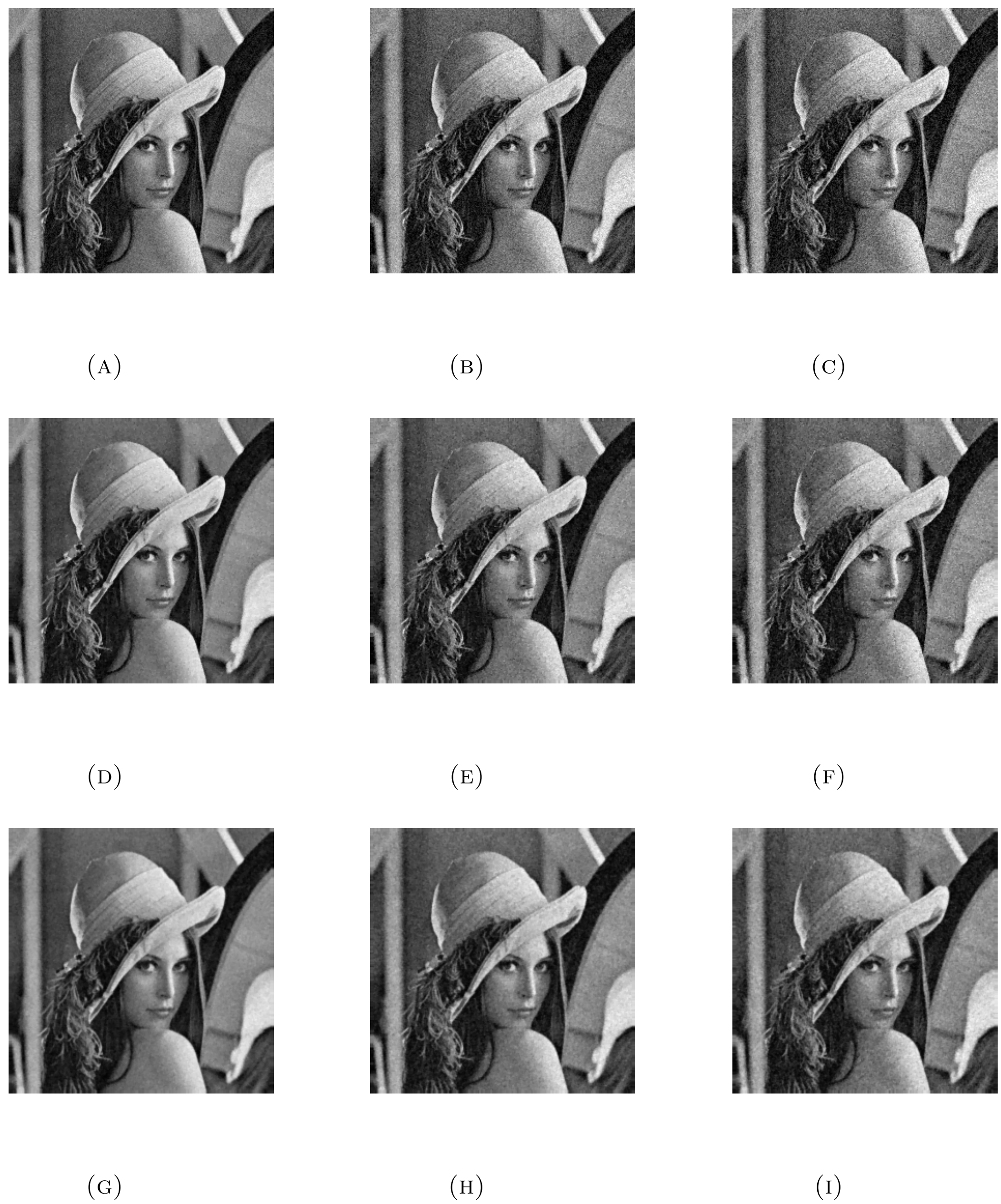

Figure 2. (top row) Noisy Lena images with different levels of Gaussian noise (A)-(C), $\sigma^2 = 0.002,\ 0.004,\ 0.006$, respectively; (second row) (D)-(F) corresponding denoised images by model (2.3); (third row) (G)-(I) by model (2.4).

6. Concluding Remarks

We have presented a second order partial differential equation based new nonlinear diffusion model for image denoising. The main idea is to apply a priori smoothness on the solution image. The forward-backward difference schemes are used to discretize models (2.3) and (2.4). The model (2.4) gives larger PSNR values than that of model

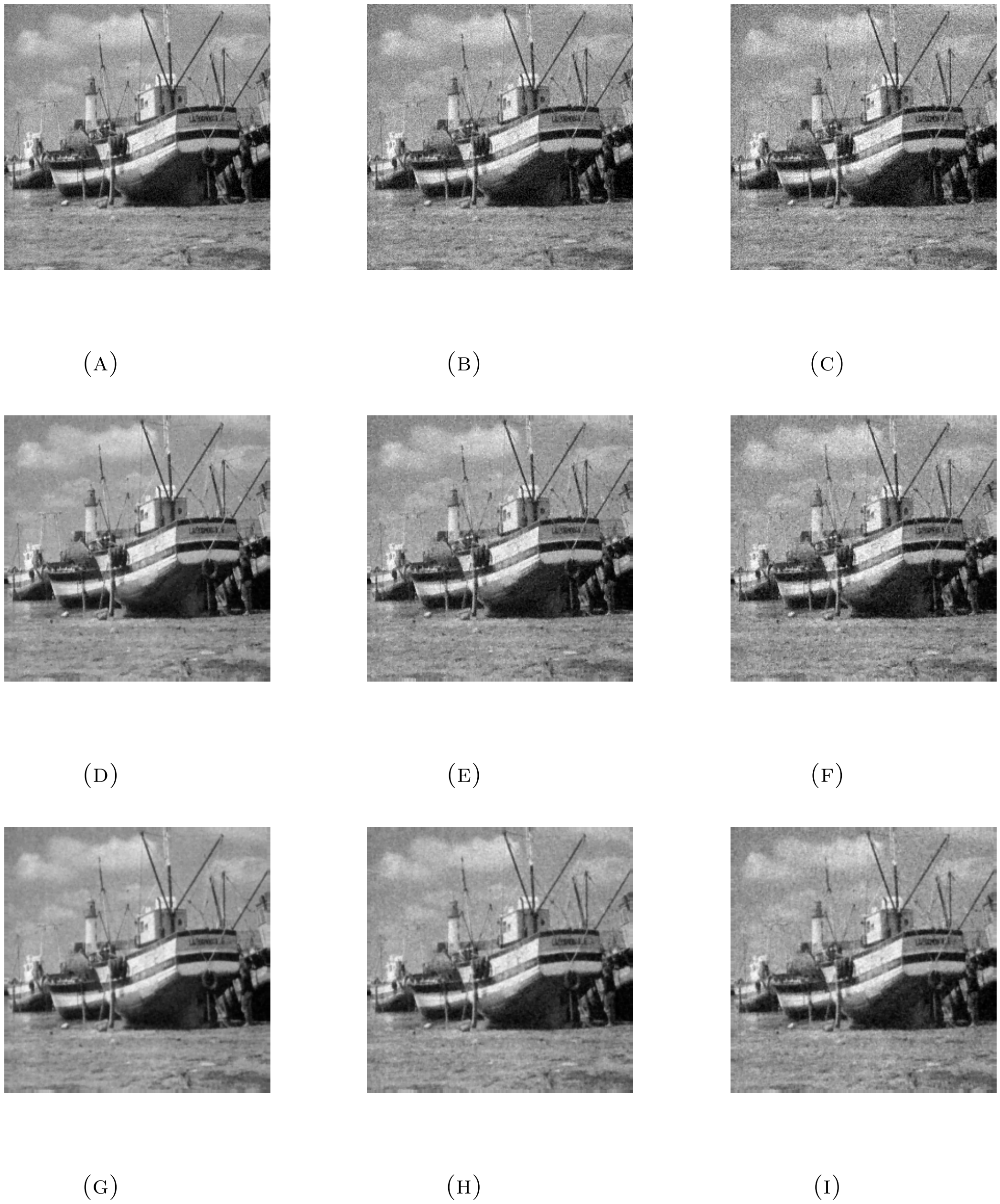

FIGURE 3. (top row) Noisy Boat images with different levels of Gaussian noise (A)-(C), $\sigma^2 = 0.002,\ 0.004,\ 0.006$, respectively; (second row) (D)-(F) corresponding denoised images by model (2.3); (third row) (G)-(I) by model (2.4).

(2.3) even at relatively small iteration numbers. Thus we can say the model (2.4) gives better denoised images than that of model (2.3).

REFERENCES

[1] L. Alvarez, P. L. Lions and J. M. Morel: Image selective smoothing and edge detection by nonlinear diffusion II*, *SIAM J. Numer. Anal.*, 29(3): 845–866, 1992.

[2] L. Alvarez and J. Esclarín: Image quantization using reaction-diffusion equations, *SIAM Journal on Applied Mathematics*, 57(1): 153–175, 1997.

[3] T. Barbu, V. Barbu, V. Biga, and D. Coca: A PDE variational approach to image denoising and restoration, *Nonlinear Analysis: Real World Applications*, 10(3): 1351–1361, 2009.

[4] C. A. Z. Barcelos, M. Boaventura and E. C. Jr. Silva: A well-balanced flow equation for noise removal and edge detection, *IEEE Transactions on Image Processing*, 12(7): 751–763, 2003.

[5] F. Catte, P. L. Lions, J. M. Morel and T. Coll: Image selective smoothing and edge detection by nonliear diffusion*, *SIAM J. Numer. Anal.*, 29(1): 182–193, 1992.

[6] T. F. Chan, G. H. Golub and P. Mulet: A nonlinear primal-dual method for total variation-based image restoration, *SIAM J. Sci. Comput.*, 20(6): 1964–1977, 1999.

[7] Q. Chang and I-Liang Chern: Acceleration methods for total variation-based image denoising, *SIAM J. Sci. Comput.*, 25(3): 982–994, 2003.

[8] P. Charbonnier, L. Blanc-Feraud, G. Aubert and M. Barlaud: Two deterministic half-quadratic regularization algorithms for computed imaging, in *Proceedings of IEEE International Conference on Image Processing*, Vol. 2, pp. 168–172, *IEEE Computer Society Press*, Austin, Tex, USA, November 1994.

[9] M. G. Crandall, H. Ishii and P. L. Lions: User's guide to viscosity solution of second order partial differential equations, *Bull. Amer. math. Soc.*, 27(1): 1–67, 1992.

[10] L. C. Evans and J. Spruck:, Motion of level sets by mean curvature I, *J. Differ. Geom.*, 33: 635–681, 1991.

[11] L. Lapidus and G. F. Pinder: Numerical solution of partial differential equations in science and engineering, *SIAM Rev.*, 25(4): 581–582, 1983.

[12] A. Marquina: Inverse scale space methods for blind deconvolution, *UCLA CAM Report*, 06–36, 2006.

[13] P. Perona and J. Malik: Scale space and edge detection using anisotropic diffusion, *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(7): 629–639, 1990.

[14] V. B. S. Prasath and D. Vorotnikov: Weighted and well-balanced anisotropic diffusion scheme for image denoising and restoration, *Nonlinear Analysis: Real World Applications*, 17: 33–46, 2014.

[15] J. G. Rosen: The gradient projection method for nonlinear programming, Part II, nonlinear constraints, *J. Soc. Indust. Appl. Math.*, 9(4): 514–532, 1961.

[16] L. Rudin, S. Osher and E. Fatemi: Nonlinear total variation based noise removal algorithm, *Phys. D.*, 60: 259–268, 1992.

[17] Y. Shi and Q. Chang: New time dependent model for image restoration, *Applied Mathematics and Computation*, 179(1): 121–134, 2006.

[18] G. Strang: Accurate Partial Difference Methods II, *Non Linear Problems. Numer. Math.*, 6(1): 37–46, 1964.

[19] G. Strang: On the construction and comparison of difference schemes, *SIAM J. numer. Anal.*, 5(3): 506–517, 1968.

[20] D. Strong: Adaptive total variation minimizing image restoration, Ph.D. Thesis, *UCLA Mathematics Department*, USA, August, 1997.

[21] J. Weickert: *Anisotropic Diffusion in Image Processing*, *Teubner Stuttgart*, 1998.

[22] J. Weickert: A Review of nonlinear diffusion filtering. *In Scale Space, LNCS*, Springer Berlin, 1252: 1–28, 1997.

[23] M. Welk, D. Theis, T. Brox and J. Weickert: PDE-based deconvolution with fordward-backward diffusivities and diffusion tensors, *In Scale Space, LNCS*, Springer Berlin, 3459: 585–597, 2005.

[24] A. P. Witkin: Scale-space filtering, *Proc. IJCAI Karlsruhe*, 1019–1021, 1983.