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MODELLING PEDESTRIANS' IMPACT ON THE PERFORMANCE OF A ROUNDABOUT

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ABSTRACT. In this paper we investigate the relationship between short time stay of pedestrian on the crosswalk situated at entrance and exiting roads of the roundabout. In addition, we study its implication on the performance of the roundabout in regulating traffic flow problems. To this aim, the roundabout is modeled as network of roads with 2×1 and 1×2-type junctions and external incoming and outgoing roads. The evolution of traffic flow on the road network at the roundabout is modeled by Lighthill-Whitham-Richards model with extended flux to capture the presence of pedestrians on crosswalk. We introduce three cost functionals that measure the total mass of vehicle , average velocity, and total flux respectively on the network. Then we analyze the performance of the roundabout with and without pedestrian through numerical simulation using Godunov scheme.

Keywords: Traffic flow, roundabout performance, scalar conservation laws, pedestrians.

AMS (MOS) Subject Classification. 35L65, 33F05, 90B20.

1. Introduction

A wide variety of mathematical models describing traffic flow and separately pedestrian motion has been intensively investigated in the recent years using microscopic and macroscopic approach. The macroscopic fluid dynamic type description of traffic flow model was introduced in the 1950s by Lighthill and Whitham [21] and independently by Richards [23], on an infinite single road using a non-linear scalar hyperbolic conservation laws. This model is commonly referred to as LWR model. Further improvements are achieved in [1, 2, 15] and successfully extended to networks in recent years, see for example [8, 10, 12, 13, 17] and references therein. These models have also been utilized for the optimization of vehicular traffic flow on road networks through various approaches, see for example [6, 7, 16, 22] and references therein.

The first macroscopic pedestrian flow model was due to the seminal work of Hughes [24] where the preferred direction of motion is given by an eikonal equation. Furthermore, in [9] pedestrian traffic modeling using scalar conservation laws based on

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the solution of the eikonal equation has been investigated. More recently, the coupling of traffic flow networks with pedestrian motion on the street has been achieved in [3, 4].

Modern roundabouts are now considered as an alternative traffic control device that can improve safety and operational efficiency at intersections compared to other conventional intersection controls [11].

In the present work we are interested to investigate the relation between short stay of pedestrians on the crosswalk situated on entrance and exiting arms of the roundabout and its implication on the performance of the roundabout in regulating traffic flow problems. Compared to [6, 22], the model presented in this paper use extended flux due to pedestrians motion on the crosswalk and focus on investigating the performance of the roundabout rather than optimizing the traffic flow.

Roundabouts can be seen as particular road networks and can be modeled as alternatively periodic sequences of 2×1 and 1×2-type junctions. In this paper, we consider a roundabout joining m-incoming and m-outgoing roads, $m \in \mathbb{N}$, $m \geq 2$, with pedestrian crosswalk situated both on the entrance and exit roads. The evolution of the traffic flow on the whole network of the roundabout is described by nonlinear scalar hyperbolic partial differential equations.

We introduce three cost functionals that measure the total mass of vehicle, average velocity, and total flux on the network to analyze the performance of the roundabout with and without pedestrian through numerical simulation.

The paper is structured as follows. In Section 2 we describe mathematical model for the road networks with roundabout, governing equations and the solution of Riemann problem at junctions. Section 3 presents detailed numerical simulations and comparisons. Finally we conclude in Section 4.

2. The Model

In this work we consider a roundabout joining m-incoming and m-outgoing roads, $m \in \mathbb{N}$, $m \geq 2$, with pedestrian crosswalk situated both on the entrance and exit arms as illustrated in Figure 1. A roundabout can be seen as a directed graph in which roads are represented by arcs and junctions by vertexes. The arcs are modeled by intervals $I_i = [a_i, b_i] \subset \mathbb{R}$, $a_i < b_i$, $i = 1, 2, \ldots, 2m$. In the case of incoming and outgoing roads either b_i or a_i can be extended to $+\infty$ or $-\infty$. Each junction J_i is described by its incoming and outgoing roads, where we denote the set of all incoming roads of junction J_i by $\mathrm{Inc(J_i)}$ and the set of all outgoing roads of junction J_i by $\mathrm{Out(J_i)}$ for $i = 1, 2, \ldots, 2m$. In this setting, the roundabout illustrated in Figure 1 can be decomposed into alternatively periodic sequences of 2×1 and 1×2-type junctions (compare Figure 2).

To recover the behavior of the roundabout we introduce Coclite, Garavello and Piccoli (CGP) condition at junctions.

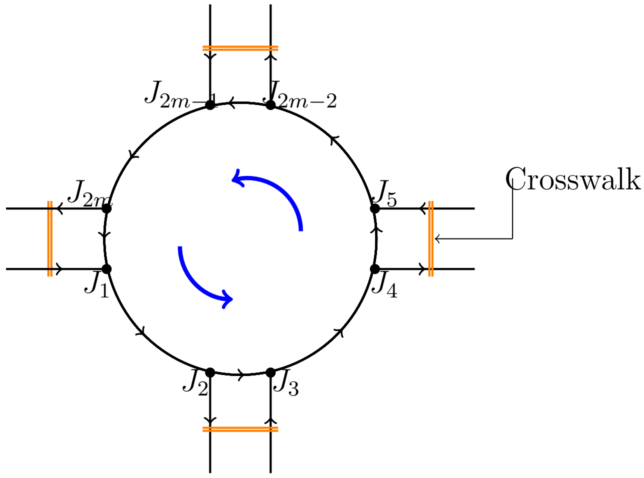


FIGURE 1. Sketch of the roundabout considered in the article.

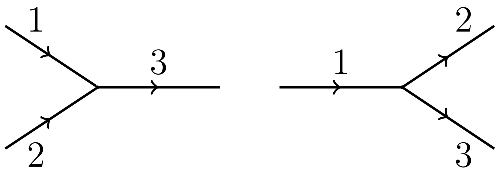


FIGURE 2. A junction with two incoming and one outgoing road (left), and a junction with one incoming and two outgoing roads (right).

For the definition of road networks and detail description we refer to [12, 18].

Definition 2.1. A road network is a couple $(\mathcal{I}, \mathcal{J})$, where $\mathcal{I} = \{I_i = [a_i, b_i] \subseteq \mathbb{R}, i = 1, 2, \ldots, 2m\}$ represents a finite set of edges (roads), and $\mathcal{J}$ is the collection of vertexes. Each vertex J is the union of two nonempty subsets **Inc(J)** and **Out(J)** of $\{1, 2, \ldots, 2m\}$ representing respectively, the incoming and the outgoing roads.

In the context of the present work, each junction can be identified as either 2×1 or 1×2 type. The evolution of the traffic flow on each arc of the roundabout is given by the scalar hyperbolic conservation law

$$\partial_t \rho_i + \partial_x f(\rho_i) = 0, \quad (t, x) \in \mathbb{R}^+ \times I_i, \quad i = 1, 2 \ldots, 2m, \tag{2.1}$$

where $\rho_i = \rho_i(t, x) \in [0, \rho_{\max}]$ is the mean traffic density and $\rho_{\max}$ the maximal density on the road. The flux function $f_i : [0, \rho_{\max}] \to \mathbb{R}^+$ is given by the following flux-density relation

$$f_i(\rho_i) = \rho_i v(\rho_i)$$

where $v : [0, \rho_{max}] \to \mathbb{R}^+$ is a smooth decreasing Lipschtiz continuous function denoting the mean traffic speed. As given in [18], the basic assumption we will make for the evolution of traffic on the roundabout is that the velocity function is dependent only on the density ρ of the cars. The mean traffic speed v attains its maximal value

v_{max} for small density and when ρ increases to some maximum capacity ρ_{max}, the mean traffic speed vanishes. For future use we normalize the vehicle density $\rho(t,x)$ so that $0 \leq \rho \leq \rho_{max} = 1$. Furthermore, as presented in [12], we make the following assumption on the flux function

(A_1) f_i is smooth strictly concave C^2 function.
(A_2) $f_i(0) = f_i(1) = 0$.

Assumption (A_1) and (A_2) guarantees the existence and uniqueness of critical density $\rho_c \in (0,1)$ such that $f_i'(\rho_c) = 0$. A typical example of flux function satisfying assumption (A_1) and (A_2) is

$$f(\rho) = \rho v_{max}(1-\rho) \tag{2.2}$$

which is commonly referred as fundamental diagram in the transportation literature (see Figure 3). For the theory of scalar hyperbolic conservation laws we refer to [5].

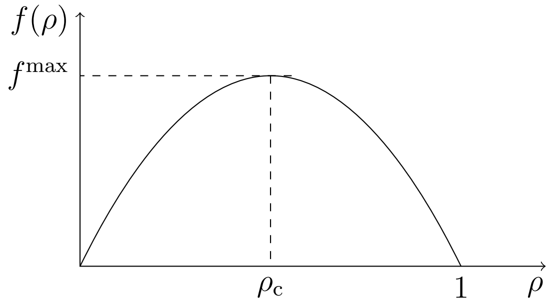


FIGURE 3. Flux function considered.

More recently, the coupling of traffic flow networks with pedestrian motion on the street has been achieved in [3, 4] using scalar hyperbolic conservation laws in combination with the solution of the eikonal equation. In the present work we are interested to investigate the relation between short stay of pedestrian motion on the crosswalk on entrance and exiting arms of the roundabout and its implication on the performance of the roundabout in regulating traffic flow problems. To achieve this goal, we assume that the crosswalk is a two-dimensional space orthogonal to the driving direction and extend the traffic flux function at the exit and entrance by

$$\tilde{f}(\rho_i) = \rho_i v(\rho_i) g(p), \quad i = 1, 2, \ldots, m \tag{2.3}$$

where the function g is defined by

$$g(p,t) = \begin{cases} 0 & \text{if p} = 1 \text{ }, t \in [t_1, t_2] \\ 1 & \text{if p} = 0 \end{cases} \tag{2.4}$$

and $p \in \{0,1\}$. The state $p = 1$ corresponds to the situation when pedestrians are on the crosswalk and $p = 0$ corresponds to no pedestrians are on the crosswalk. The time t_1 and t_2 are randomly chosen. The duration $t = t_2 - t_1$ is the time taken by pedestrians to cross the road. The situation $g(p, t) = 0$ indicates the occupancy of road by pedestrians and the interruption of traffic flow. The state $g(p, t) = 1$ corresponds to the absence of pedestrians on the crosswalk at the exit and entrance of the roundabout and the traffic flow behaves normally as given in the LWR model. We solve the Cauchy problem

$$\begin{cases} \partial_t \rho_i + \partial_x \tilde{f}(\rho_i) = 0, & (t, x) \in \mathbb{R}^+ \times I_i, \\ \rho_i(0, x) = \rho_{i,0}(x) & x \in I_i, \quad 1, 2, \ldots, 2m. \end{cases} \tag{2.5}$$

on each I_i where $\rho_{i,0}(x)$ is the initial density on the road of the roundabout. We consider entropic solutions on each of the single roads of the roundabout. A weak solution of the network and Riemann problem has been given in [12, 18] and references therein as a solution in the sense of distributions with test functions, which are smooth at the junctions. We use entropic solutions on each of the single roads of the roundabout in the sense of the following definition.

Definition 2.2. Consider a roundabout as in Figure 1. Let A be a distribution matrix at a fixed junction of the roundabout as given in [12]. A collection of function

$$\rho = (\rho_i)_{i=1,\ldots,2m} \in \prod_{i=1}^{2m} \mathcal{C}^0 \left(\mathbb{R}^+; \mathbf{L^1} \cap \mathrm{BV}(I_i)\right)$$

is an admissible solution to (2.5) if

1. ρ_i satisfies the Kružhkov entropy condition [20] on $(\mathbb{R}^+ \times I_i)$, that is, for every $k \in \mathbb{R}$ and for all $\varphi \in \mathcal{C}_c^1(\mathbb{R} \times I_i)$, $t > 0$,

$$\begin{aligned} &\textstyle\int_{\mathbb{R}^+} \int_{I_i} (|\rho_i - k| \partial_t \varphi + \mathrm{sgn}\,(\rho_i - k)(\tilde{f}(\rho_i) - \tilde{f}(k)) \partial_x \varphi) dx dt \\ &\quad + \textstyle\int_{I_i} |\rho_{i,0} - k| \varphi(0, x) dx \geq 0; \quad i = 1, 2, \ldots, 2m. \end{aligned} \tag{2.6}$$

2. $\tilde{f}(\rho_i(t, b_i^-)) = \sum\limits_{j=1}^{n} \beta_{j,i} \tilde{f}(\rho_i(t, a_i^+))$, at each junction of the roundabout;
3. $\tilde{f}(\rho_i(t, b_i^-))$ must be maximized subject to (1) and (2).

Here b_i^- denotes left side of b_i in the interval whereas a_i^+ indicates the right side of a_i. Condition (1) is concerned with conservation of cars whereas condition (2)and (3) correspond to the preferences of drivers and the maximization procedure. In the definition, $n = 1$ or 2 depending on the junction type of the roundabout.

For future use we define the function τ as follows, for further properties see [12].

Definition 2.3. Let $\tau : [0, 1] \to [0, 1]$ be the map such that

- $\tilde{f}(\tau(\rho)) = \tilde{f}(\rho)$ for every $\rho \in [0, 1]$;
- $\tau(\rho) \neq \rho$ for every $\rho \in [0, 1] \setminus \{\rho_c\}$.

Definition 2.4. A Riemann problem at a junction J is a Cauchy problem for constant initial data on each road i.

Definition 2.5. A Riemann solver for junction J is a map $RS : [0,1]^2 \times [0,1] \to [0,1]^2 \times [0,1]$ that associates the Riemann data $\rho_0 = (\rho_{1,0}, \rho_{2,0}, \rho_{3,0})$ at J to a vector $\hat{\rho} = (\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)$ such that the solution on the incoming road $I_i, i = 1, 2$ is given by the wave $(\rho_i, \hat{\rho}_i)$ and on the outgoing road $I_j, j = 3$, the solution is given by the wave $(\hat{\rho}_j, \rho_j)$. Consistency condition: $RS(RS(\rho_0)) = RS(\rho_0)$.

For a road $i \in Inc(J)$, the solution $\rho_i(t,x)$ over its spacial domain $x < b_i$ is given by the solution to the following Riemann problem

$$\begin{cases} \partial_t \rho_i + \partial_x \tilde{f}(\rho_i) = 0, & (t,x) \in \mathbb{R}^+ \times I_i, \\ \rho_i(0,x) = \begin{cases} \rho_{i,0} & \text{if } x < b_i, \\ \hat{\rho}_i & \text{if } x \geq b_i. \end{cases} \end{cases}$$

The Riemann problem for outgoing road is defined similarly except for $\rho_i(0, x > b_i) = \rho_{i,0}$ and $\rho_i(0, x \leq b_i) = \hat{\rho}_i$. If $\rho = (\rho_1, \ldots, \rho_{n+m})$, $\rho_i \in [0, +\infty] \times I_i$ is a weak solution as in [12, 18] at the junction such that each $x \mapsto \rho_i(t,x)$ has a bounded variation, then precisely the conservation of cars through the junction J

$$\sum_{i=1}^{n} \tilde{f}(\rho_i(t, b_i)) = \sum_{j=n+1}^{n+m} \tilde{f}(\rho_i(t, a_j)) \tag{2.7}$$

holds where for a junction type 1×2, $n = 1, m = 2$ and for 2×1, $n = 2, m = 1$ in the context of the roundabout.

Definition 2.5 does not guarantee the uniqueness of solution at each junction. Hence, to ensure uniqueness of the solution we shall introduce a parameter $q \in]0,1[$ in the system. That is, assume that q is a priority parameter that defines the amount of flux that enters the outgoing main lane from each incoming road. In particular, when the priority parameter is applied, $q\tilde{f}(\rho(t,b^-))$ represents the flux allowed from the incoming mainline into the outgoing main lane, and $(1-q)\tilde{f}(\rho(t,b^-))$ represents the flux from the incoming secondary road of the roundabout.

For a junction with 1×2 type, we consider traffic split ratio $\alpha \in (0,1)$ describing the distribution of traffic among outgoing roads depending on the preference of drivers at each junction J. We denote the Riemann initial data by $\rho_{i,0} = \rho_{i,0}(b_i)$ for incoming arcs and $\rho_{i,0} = \rho_{i,0}(a_i)$ for outgoing arcs for a single junction. Assuming a unique solution for the problem at the junction, we denote the solution at the junction, i.e., at $x = b_i$ for incoming and at $x = a_i$ for outgoing roads, by

$$(\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3).$$

Given the constant initial values $\rho_{i,0}$, we need to determine a unique solution $\hat{\rho}_i$ satisfying the coupling condition in the context of Coclite, Garavello and Piccoli (CGP) approach at junction. The possible values of $\hat{\rho}_i$ are necessarily as follows.
On the incoming road

$$\hat{\rho}_i \in \begin{cases} \{\rho_{i,0}\} \cup (\tau(\rho_{i,0}), 1] & \text{if } 0 \leq \rho_{i,0} < \rho_c \\ [\rho_c, 1], & \text{if } \rho_c \leq \rho_{i,0} \leq 1 \end{cases} \tag{2.8}$$

and on the outgoing road

$$\hat{\rho}_i \in \begin{cases} [0, \rho_c] & \text{if } 0 \leq \rho_{i,0} \leq \rho_c \\ \{\rho_{i,0}\} \cup [0, \tau(\rho_{i,0})) & \text{if } \rho_c \leq \rho_{i,0} \leq 1. \end{cases} \tag{2.9}$$

The values of i are easily fixed depending on the junction type in the context of the present work.

Remark 2.6. On the incoming road:

- if $\rho_{i,0} < \rho_c < \hat{\rho}_i < 1$, $\tilde{f}(\rho_{i,0}) > \tilde{f}(\hat{\rho}_i)$, and $\rho_c < \rho_{i,0} < 1$, the solution of the Riemann problem consists of a shock wave with a negative speed. Moreover,
- if $\rho_{i,0} < \rho_c < \hat{\rho}_i < 1$ and $\tilde{f}(\rho_{i,0}) = \tilde{f}(\hat{\rho}_i)$, the solution consists of contact wave.

On the outgoing road:

- if $\rho_{i,0} < \rho_c$ the solution of the Riemann problem consists of a shock wave with a positive speed.
- if $\hat{\rho}_i < \rho_c < \rho_{i,0} < 1$, the solution of the Riemann problem consists of a shock wave with positive speed and contact wave when $\tilde{f}(\rho_{i,0}) = \tilde{f}(\hat{\rho}_i)$.

For the following discussion we refer to [14, 16] for detail.

A. **Coupling conditions for junction type** 2×1:

We consider a junction with two incoming arcs and one outgoing arc. The initial densities on each roads i are given by $\rho_{i,0}$ with $i = 1, 2, 3$. The corresponding fluxes are denoted by $\gamma_{i,0} = \tilde{f}(\rho_{i,0})$. Denote the maximum of the flux by $\tilde{f}(\rho_c)$. We denote the sets of valid resulting fluxes γ_i by Ω_i. For the incoming roads $i = 1, 2$ this is

$$\begin{aligned} \rho_{i,0} \leq \rho_\mathrm{c} &\Rightarrow \Omega_i = [0, \gamma_{i,0}], \\ \rho_{i,0} \geq \rho_\mathrm{c} &\Rightarrow \Omega_i = [0, f(\rho_\mathrm{c})]. \end{aligned} \tag{2.10}$$

For the outgoing road $i = 3$,

$$\begin{aligned} \rho_{i,0} \leq \rho_\mathrm{c} &\Rightarrow \Omega_i = [0, \tilde{f}(\rho_\mathrm{c})], \\ \rho_{i,0} \geq \rho_\mathrm{c} &\Rightarrow \Omega_i = [0, \gamma_{i,0}]. \end{aligned} \tag{2.11}$$

Moreover, we can define c_i such that

$$\Omega_i = [0, c_i].$$

The fluxes at the junction are found in the following way, distinguishing two cases:

(1) $c_1 + c_2 \leq c_3$: In this case, we have to look for γ_1, γ_2 such that

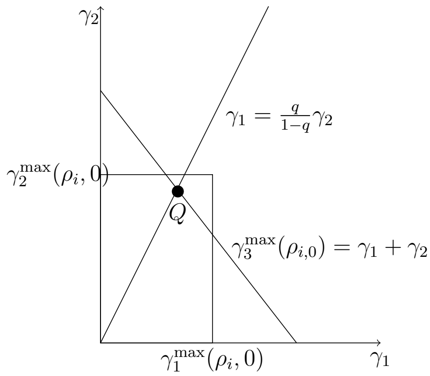
$$\max \gamma_1 + \gamma_2 \quad \text{w.r.t.}$$
$$0 \leq \gamma_1 \leq c_1, \quad 0 \leq \gamma_2 \leq c_2, \quad \gamma_1 + \gamma_2 \leq c_3.$$

The unique solution is found to be $\gamma_1 = c_1, \gamma_2 = c_2, \gamma_3 = c_1 + c_2$.

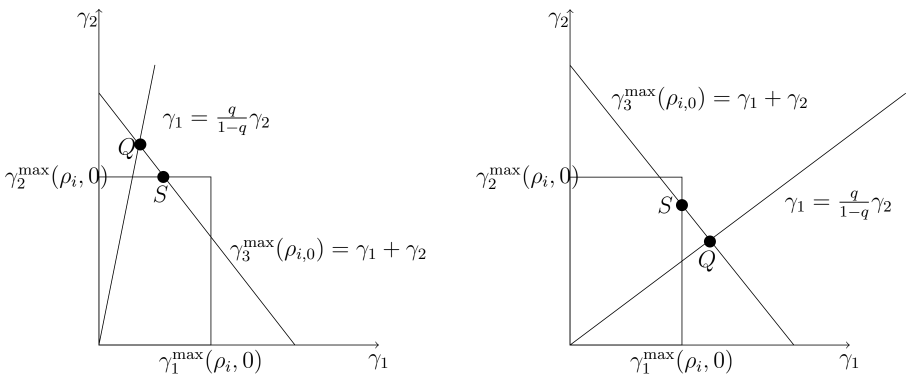
(2) $c_1 + c_2 \geq c_3$: In this case, we have to look for γ_1, γ_2 such that

$$\max \gamma_1 + \gamma_2 \quad \text{w.r.t.}$$
$$\gamma_1 = \tfrac{q}{1-q}\gamma_2$$
$$0 \leq \gamma_1 \leq c_1, \quad 0 \leq \gamma_2 \leq c_2, \quad \gamma_1 + \gamma_2 = c_3.$$

where $q \in (0, 1)$ is the priority parameter introduced at the merging junction as given in [12]. Figure 4 illustrates feasible set for the solution of the Riemann solver. For detail theory regarding Riemann solver at junction we refer the reader to Section(5.2.2) of [12]. The purpose of the priority parameters is to regulate the condition that neither impose insufficient flows nor send excess vehicles than the carrying capacity of the main link of the roundabout.



(A) Intersection inside Ω_i



(B) Intersection outside Ω_i

FIGURE 4. Solutions of the Riemann Solver at the junction.

Since $c_1 + c_2 > c_3$, $\hat{\gamma}_3 = \min(c_1 + c_2, c_3) = c_3$. For $c_2 > (1-q)c_3$ and $c_1 > qc_3$, we set $Q_1^i = 1 - \frac{c_2}{c_3}$ and $Q_2^i = \frac{c_1}{c_3}$ such that $Q_2^i - Q_1^i = \frac{c_1}{c_3} - (1 - \frac{c_2}{c_3}) > 0$. Under these conditions, the unique solution at junction is found to be

- $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) = (c_3 - c_2, c_2, c_3) \quad$ if $q \in (0, Q_1^i)$;
- $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) = (qc_3, (1-q)c_3, c_3) \quad$ if $q \in [Q_1^i, Q_2^i]$;
- $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3) = (c_1, c_3 - c_1, c_3) \quad$ if $q \in (Q_2^i, 1)$.

B. **Coupling conditions for junction type** 1×2:

We consider a junction with one incoming and two outgoing arcs. We use the same notation as before; i.e., we define $\gamma_{i,0}$ and the sets Ω_i depending on whether incoming or outgoing roads are considered. Using traffic distribution rates $\alpha_{2,1}, \alpha_{3,1} \in (0,1)$ with $\alpha_{2,1} + \alpha_{3,1} = 1$, then the CGP-conditions are

(1) $\gamma_1 \in \Omega_1, \alpha_{j,1}\gamma_1 \in \Omega_j$ for j = 2,3;
(2) Maximize γ_1 w.r.t. (1);
(3) $\gamma_j = \alpha_{j,1}\gamma_1, j = 2, 3$.

Using $\Omega_i = [0, c_i]$, $i = 1, 2, 3$, we obtain

$$\gamma_1 = \min\{c_1, \frac{c_2}{\alpha_{2,1}}, \frac{c_3}{\alpha_{3,1}}\}.$$

Remark 2.7. Condition (B) is exactly what is known as the FIFO (first in, first out) rule of a dispersing junction in the traffic literature, see for example [14].

2.1. **Analytical Study.** In this subsection we give analysis for traffic evolution on the roundabout network with and without the presence of pedestrians on the cross-walk. The analysis is only limited to a roundabout having four incoming and four outgoing roads for later numerical study and simplification purpose.

2.1.1. *In the Absence of Pedestrian.* Let $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4)$ be the traffic flux on the incoming roads towards the roundabout and $f(\rho_c)$ be the maximal traffic flux on the road network. Assume that $\alpha f(\rho_c)$ of the traffic is flowing out of the main roads of the roundabout through their corresponding exiting arms while the remaining $(1-\alpha)f(\rho_c)$ proceed to flow on the main road of the roundabout towards the next junction. Suppose that $q = (q_1, q_2, q_3, q_4)$ is the applied priority parameters vector at the respective merging junctions of the roundabout where $\alpha = (\alpha_{ji})$ is the splitting rate. In the case of supply limited situation,

$$\tilde{f}_i + (1-\alpha)f(\rho_c) > f(\rho_c), \quad i = 1, 2, 3, 4. \tag{2.12}$$

On the other hand from presentation under condition (A) we know that $(1-\alpha)f(\rho_c) < c_1$ and $\tilde{f}_i \leq c_2$. Then we have the following conditions

(a) $q_i \in (0, Q_1^i)$

(b) $q_i \in [Q_1^i, Q_2^i]$
(c) $q_i \in (Q_2^i, 1)$, $i = 1, 2 \ldots, 4$.

If $q_i \notin [Q_1^i, Q_2^i]$, $i = 1, 2 \ldots, 4$, then either $q_i \in (0, Q_1^i)$ or $q_i \in (Q_2^i, 1)$. If $q_i \in (0, Q_1^i)$, all the traffic on the incoming external roads of the roundabout enter the junction while excess vehicles are waiting on the main link of the roundabout. In such a situation, backward propagating shock waves are produced on the main link of the roundabout while no wave occur on the incoming secondary road at each junction. On the contrary, if $q_i \in (Q_2^i, 1)$ all the traffic on the main link of the roundabout enter their corresponding junctions while queues are formed on the external incoming road of the roundabout. In both situation, the priority rule is violated due to the limited traffic demand entering the junction from either the main road of the roundabout or from the incoming external road of the roundabout. However, if $q_i \in [Q_1^i, Q_2^i]$ the priority rule is satisfied well due to sufficient demand from both main and the incoming secondary roads of the roundabout at each junction. In this case, backward propagating shock will be formed on the network and congestion get raised. Furthermore, some of the junctions could be congested while the others stay demand limited. This fact is due to the volume of inflow traffic on the incoming edges.

2.1.2. *Pedestrian Involvement.* This situation includes all the cases given under Subsection 2.1.1 in addition to the presence of pedestrian on the crosswalk. We assume that the crosswalk is situated both on the incoming and outgoing roads of the roundabout without any traffic light.

Consider vehicular traffic and pedestrian flow during peak hours. Assume for short period of time the crosswalk on the exit arm of the roundabout is occupied by pedestrians. Consequently, the original traffic flux is altered and behaves as given by equation 2.3. For the random duration $t_1 \leq t \leq t_2$ the flux is equal to zero on the crosswalk since the road is occupied by pedestrians. The interruption in flux function on the outgoing road results in a backward propagating shock wave. Thus, depending on the amount of traffic volume and duration of the pedestrians staying on the crosswalk the operational performance of the roundabout could be reduced.

On the contrary, when the pedestrians occupy the crosswalk on the entrance arm of the roundabout in the random time interval $t_1 \leq t \leq t_2$, queue would be formed on the incoming edge behind the crosswalk while the roundabout operate with less traffic compared to its carrying capacity. Further, when the probability of the pedestrians to be on the crosswalk both at entrance and exit road of the roundabout equal to 1, some of the circulatory road of the roundabout becomes congested due to backward propagating shock wave being demand limited on the entrance side. Also, as a result of priority parameter, drivers wait at the give-way lines for appropriate acceptable gaps between vehicles already circulating on the roundabout. This could also contribute in producing a backward propagating shock wave. However, under low traffic flow any vehicle can proceed through the roundabout without delay.

From these scenarios, one can infer that traffic congestion plays a fundamental role in the formation of delay because vehicles spend longer periods of time near the roundabout while queuing, decelerating or accelerating due to the presence of pedestrian on the crosswalk.

3. **Numerical Approximation**

In this section we consider the traffic regulation problem for a road network given as in Figure 1. We analyze the impact of pedestrians' motion on the traffic evolution on the networks of a roundabout. In particular, we want to compare the performance of a roundabout with and without the involvement of pedestrians during peak hours.

3.1. **Network topology.** The roundabout will be modeled by

- 8 roads from the circle: $\mathcal{I}_5, \mathcal{I}_6, \mathcal{I}_7, \mathcal{I}_8, \mathcal{I}_9, \mathcal{I}_{10}, \mathcal{I}_{11}, \mathcal{I}_{12}$ coupled with CGP condition;
- 8 roads connecting the roundabout with the rest of the network: 4 incoming roads and 4 outgoing ones.

3.2. **Numerical scheme.** From the topology, it can be noted that all the junctions in the roundabout can be represented as alternatively periodic sequences of 2×1 and 1×2-type for which it might be necessary to define respectively a right-of-way parameter q and distribution rate α. The first step is then to discretize the junction model. We define a numerical grid in $(0, T) \times \mathbb{R}$ using the following notation.

- Δx is the fixed space grid size;
- Δt is the time step given by the CFL condition;
- $(t^n, x_j) = (n\Delta t, j\Delta x)$ for $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ are the grid points.

Each road is divided in N + 1 cells numbered from 0 to N. The first and last cell of an edge are always a junction and we assume that these cells are ghost cells.

3.3. **Godunov Scheme.** The Godunov scheme as introduced in [19] is based on exact solutions to Riemann problems. The main idea of this method is to approximate the initial datum by a piecewise constant function, then the corresponding Riemann problems are solved exactly and a global solution is simply obtained by piecing them together. Finally, one takes the mean on the cell and proceeds by induction. Under the CFL condition

$$\Delta t \max_{j\in\mathbb{Z}} \left| \lambda^n_{j+\frac{1}{2}} \right| \leq \Delta x, \tag{3.1}$$

the waves generated by different Riemann problems do not interact. In the above inequality, $\lambda^n_{j+\frac{1}{2}}$ is the wave speed of the Riemann problem solution at the interface $x_{j+\frac{1}{2}}$ at time t^n. Under the condition (3.1) the scheme can be written as

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}(F^G(\rho_i^n, \rho_{i+1}^n) - F^G(\rho_{i-1}^n, \rho_i^n)), i = 2, 3, \ldots, \mathrm{N} - 1 \; \forall n \tag{3.2}$$

where the numerical flux F^G takes the following expression:

$$F^G(u, v) = \begin{cases} \min(f(u), f(v)) & \text{if } u \le v, \\ \max(f(u), f(v)) & \text{if } v < u \ < \rho_c \vee \rho_c < v < u, \\ f(\rho_c) & \text{if } v < \rho < u. \end{cases} \tag{3.3}$$

for concave flux f. We introduce the following cost functionals that indicate the total mass of vehicle on the road networks of the roundabout, average velocity, and flux respectively to analyze the performance of the roundabout with and without pedestrian motion on the crosswalk.

$$\begin{aligned} J_1(t) &= \sum_{i=1}^{m} \int_{I_i} \rho(t, x)\ dx, \\ J_2(t) &= \sum_{i=1}^{m} \int_{I_i} v(\rho_i(t, x))\ dx, \\ J_3(t) &= \sum_{i=1}^{m} \int_{I_i} \tilde{f}(\rho_i(t, x))\ dx \end{aligned} \tag{3.4}$$

For a fixed time horizon $[0, T]$ our aim is to compare $\int_0^T J_1(t)dt$,$\int_0^T J_2(t)dt$, and $\int_0^T J_3(t)dt$ for an appropriate fixed distribution matrix and priority parameters.

3.3.1. *Comparison of roundabout with and without pedestrian motion.* We consider approximation obtained by Godunov scheme with space step size $\Delta x = 0.333$ and the time step determined by the CFL condition. The traffic and pedestrian flow on the road network is simulated in a time interval $[0, T_{max}]$, where $T_{\max} = 10$. For the initial condition on the roads of the network, we assume that at initial time $t = 0$ all the roads are empty and influx at boundary of incoming edges is equal to 0.2. In order to show the different state of traffic evolution on the network, we assume

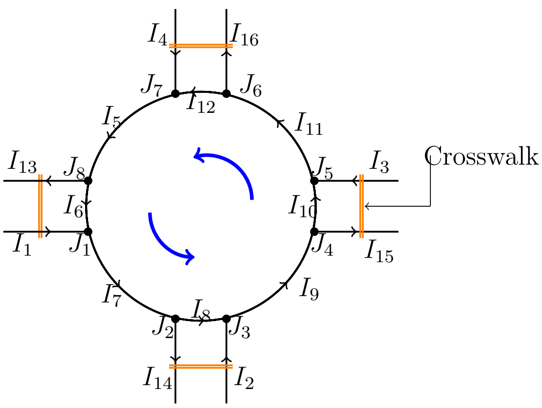
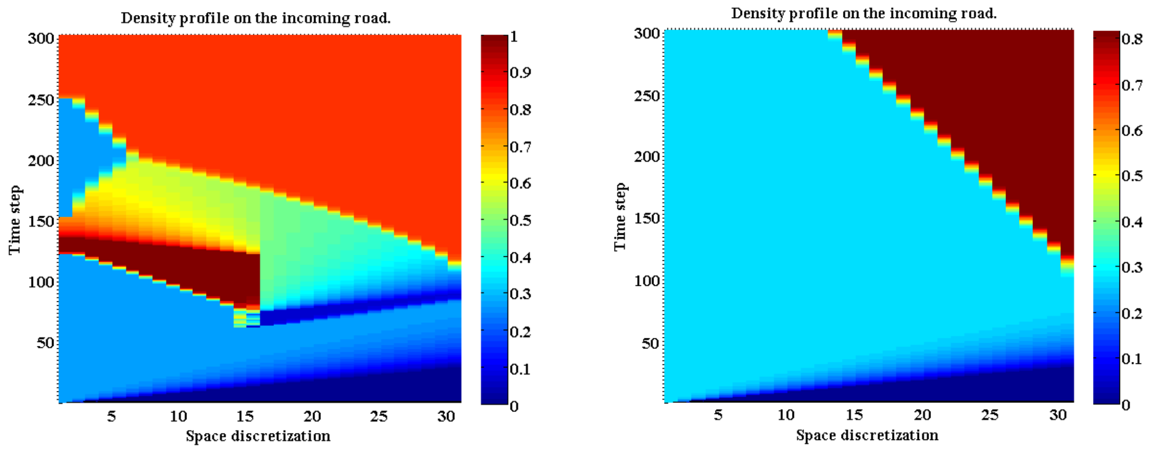


FIGURE 5. For comparison.

that the crosswalk is marked orthogonally at the midpoint of incoming and outgoing external roads. There is no crosswalk on the main roads forming the roundabout under consideration. Further, we assume that the crosswalk on the incoming and outgoing roads are occupied by pedestrians for short period of time step $t = t_2 - t_1$, see Figure 5. We now compare the results of the traffic flow with and without the pedestrians. The corresponding pictures on all the external incoming and outgoing roads, roads forming the roundabout look the same and therefore we just compare one from each of them. For the first few time steps $t < 62$, the evolution of traffic



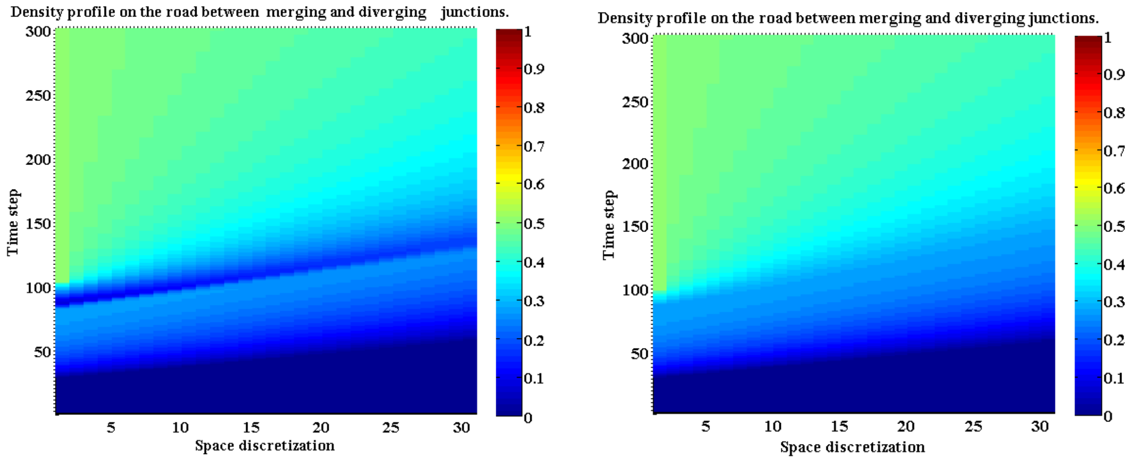
(A) Roundabout with pedestrians on the crosswalk.

(B) Roundabout without pedestrians on the crosswalk.

FIGURE 6. Traffic evolution on the incoming roads of the roundabout.

on the incoming roads behave similar in both cases due to the absence of pedestrians on the crosswalk. As soon as the pedestrians interrupt flow on the incoming roads at cell position $x = 15$ at time step $t = 62$, the situation is immediately changed as illustrated in Figure 6. Different colors in the figure correspond to different states of traffic evolution over simulation period. The blue color corresponds to demand limited case whereas the red color corresponds to congested state. The shock occurred due to pedestrians motion on the crosswalk propagating back on the incoming road. The part of the road between crosswalk and roundabout stay demand limited until the pedestrians cleared on the road at time step $t = 77$. Then rarefaction wave fill this portion of the road. Due to priority at merging junctions of the roundabout, new shock wave is produced on the incoming road. This shock wave moves back on the incoming road as depicted in Figure 6a.

Traffic congestion can occur at merging junctions in the case of roundabout without pedestrian involvement, see Figure 6b. Similar to the other case shock wave propagating back on the incoming roads. Comparing these two states of the roundabout one can easily observe the difference in the magnitude of traffic jam. Shock formed due to priority at merging junction do not reach the influx boundary in the absence of pedestrian involvement. In Figure 7a, the blue color at about time step $t = 100$

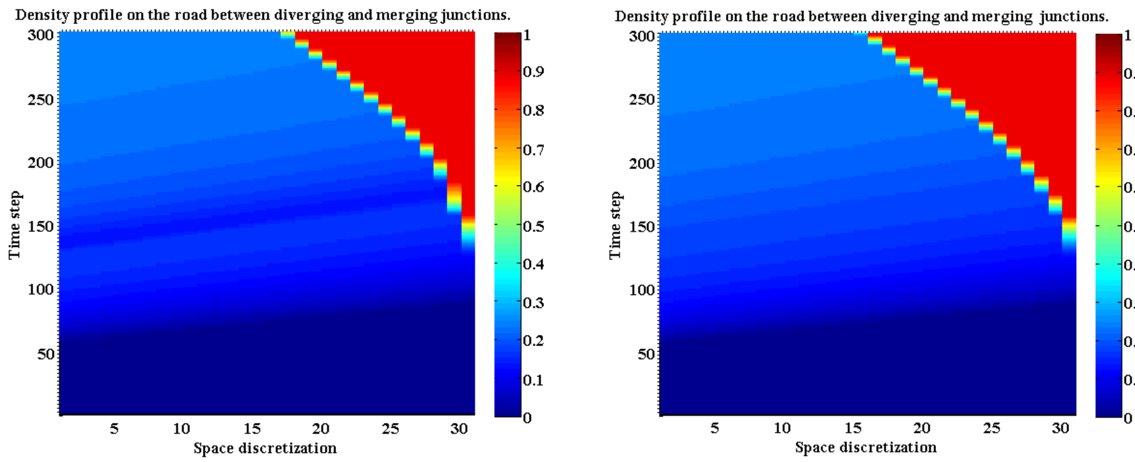


(A) Roundabout with pedestrians on the crosswalk.

(B) Roundabout without pedestrians on the crosswalk.

FIGURE 7. Traffic evolution on the main road between merging and diverging junctions of the roundabout.

reveals that the interrupted flow reach the main road between merging and diverging junctions of the roundabout. Rarefaction waves on the main road between merging and diverging junctions of the roundabout increases the density to its critical density. The evolution of traffic on this portion of the roundabout remains smooth.

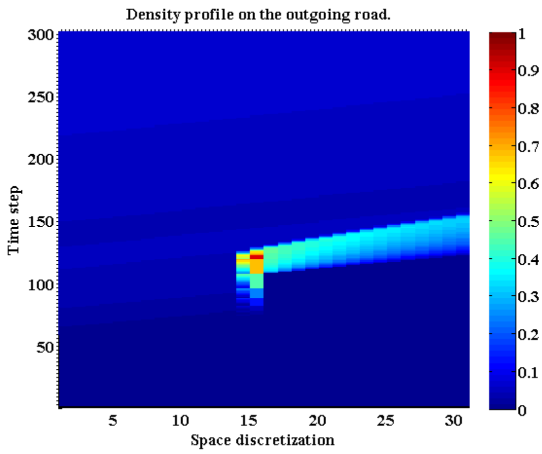


(A) Roundabout with pedestrians on the crosswalk.

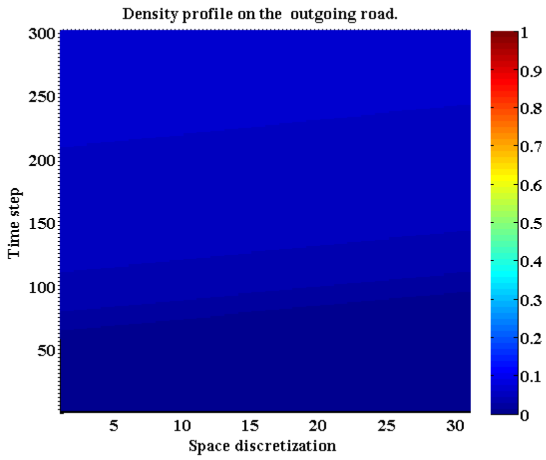
(B) Roundabout without pedestrians on the crosswalk.

FIGURE 8. Traffic evolution on the main road between diverging and merging junctions of the roundabout.

Traffic congestion appearing on the main road forming the roundabout at merging junctions. The shocks are moving back as it can be seen from Figure 8. Further, the impact induced by pedestrians is not clearly reflected in this portion of the roundabout. This could be due to αf of the traffic exiting the roundabout through outgoing roads and the influence of priority parameter value at merging junctions.



(A) Roundabout with pedestrians on the crosswalk.



(B) Roundabout without pedestrians on the crosswalk.

FIGURE 9. Traffic evolution on the outgoing roads of the roundabout.

Traffic jam arises on the outgoing edges when pedestrians are moving on the crosswalk (compare Figure 9). The influence due to this jam on the traffic circulating on the inner road of the roundabout is insignificant because of the short stay of pedestrians on the crosswalk. The outgoing secondary roads of the roundabout without pedestrians interference remain demand limited.

3.3.2. *Comparison.* In this subsection we compute changes in total density, average velocity and total flux of the cost functional introduced in equation (3.4); that is, in the case of traffic evolution on the roundabout without pedestrian and with pedestrian involvement. More precisely, we consider fixed distribution rate in both cases and different simulations cases which vary according to the values of the priority parameter $q \in \{0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8\}$. Then we compute separately the values of the cost functional and take their respective differences for comparison.

From Table 1 one can easily infer that due to presence of pedestrian on the crosswalk there are more vehicles waiting on the network. In the case of roundabout without pedestrian involvement the role played by priority parameters are insignificant in altering total density. This is due to the fact that congestion which propagate backwards on the incoming roads do not reach the other end over the given time interval. In the contrary, in the case of roundabout with pedestrian involvement the priority parameter plays remarkable role in reflecting changes in the cost functionals.

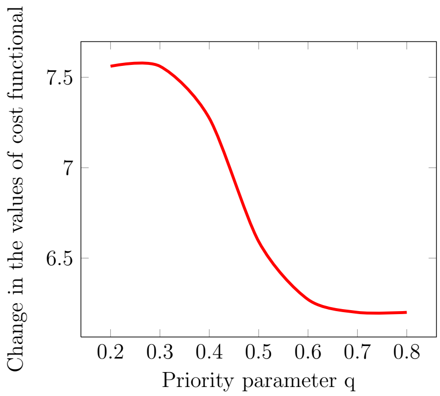
(A) In the absence of pedestrians

q	$\sum_{t=0}^{T} J_1$	$\sum_{t=0}^{T} J_2$	$\sum_{t=0}^{T} J_3$
0.2	36.1013	119.0809	19.1204
0.3	36.1013	119.0809	19.5410
0.4	36.1013	119.0809	19.6240
0.5	36.1013	119.0809	19.4765
0.6	36.1013	119.0809	19.2031
0.7	36.0878	119.0944	18.988
0.8	36.0877	119.0945	18.9765

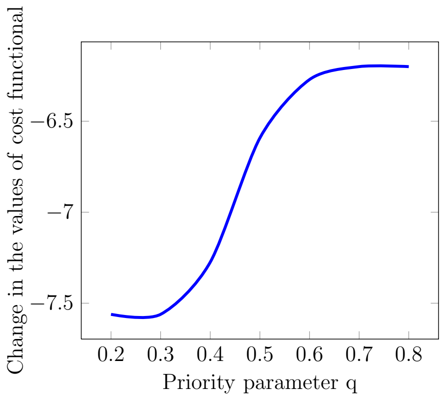
(B) In the presence of pedestrians

q	$\sum_{t=0}^{T} J_{1P}$	$\sum_{t=0}^{T} J_{2P}$	$\sum_{t=0}^{T} J_{3P}$
0.2	43.6625	111.5197	19.0936
0.3	43.6625	111.5197	19.1127
0.4	43.3753	111.8065	18.9086
0.5	42.6935	112.4887	18.6102
0.6	42.3726	112.8096	18.3428
0.7	42.2879	112.8943	18.1639
0.8	42.2879	112.8943	18.1574

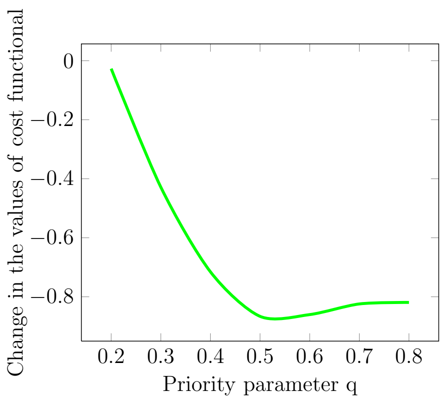
TABLE 1. Values of cost functional over time horizon. J_1, J_2 and J_3 respectively denotes cost functionals that measure total density, average velocity and total flux on the network.



(A) Change in total density



(B) Change in average velocity



(C) Change in total Flux

FIGURE 10. Difference in the values of cost functional measuring total density of vehicles, average velocity and total flux on the network.

To minimize traffic congestion, we give more priority for cars circulating on main road of the roundabout in both cases. Consequently, the change in the total density of vehicles initially constant and then it starts decreasing, see Figure 10a. Furthermore, for $q \geq 0.7$ the change in the total mass of vehicles on the network becomes constant. Similarly, the change in the average velocity of vehicles is initially constant and then it starts increasing as depicted in Figure 10b. Figure 10c describes the change in the total flux due to priority parameters on the whole network.

Comparing these tables we can deduce that the interruption by pedestrian decreases the average velocity of vehicles on the network. Similarly, it reduces the traffic flux on the whole network. Thus, the simulation result indicates that the presence of pedestrian on the crosswalk influence the performance of the roundabout in controlling traffic flow problem.

4. **Conclusions**

In this article we studied the performance of the roundabout in regulating traffic flow problems in the presence and absence of pedestrians on the crosswalk located at entrances and existing roads. The evolution of traffic flow on the whole road network of the roundabout is described by nonlinear scalar hyperbolic partial differential equations. After we descritized the equations via the Godunov scheme we computed the values of cost functionals which measure the total mass of vehicles on the road networks of the roundabout, average velocity and total flux for both cases. Then we compared the values of cost functionals. The simulation result indicated that the presence of pedestrian on the crosswalk reduce the performance of roundabout in controlling traffic flow problem. Optimization and validation of the model with real data will be considered in the future work.

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