# DYNAMICS OF A SYMBIOTIC MODEL WITH HERD BEHAVIOUR AND STRONG ALLEE EFFECT

PROSENJIT SEN, ALAKES MAITI, AND G. P. SAMANTA

Department of Mathematics, Indian Institute of Engineering Science and
Technology, Shibpur, Howrah - 711 103, India, E-mail: prosenjitsen83@gmail.com
Department of Mathematics, Vidyasagar Evening College, Kolkata – 700006,
India, E-mail: alakesh_maity@hotmail.com
Department of Mathematics, Indian Institute of Engineering Science and
Technology, Shibpur, Howrah - 711 103, India, E-mail: g_p_samanta@yahoo.co.uk,
gpsamanta@math.iiests.ac.in

**ABSTRACT.** In this paper we will consider the Allee effect on a symbiotic population system with herd behaviour. Criteria for stability of equilibrium points are derived. The effect of discrete time-delay on the model is investigated. Computer simulation of various solutions is presented to illustrate our mathematical findings. The biological implications of our analytical and numerical findings are discussed critically.

**Key Words** Symbiotic model, Herd, Allee effect, Stability, Time-delay.

## 1. INTRODUCTION

*Symbiotic relationship* is a most widespread phenomenon that is very common in nature. In this relationship, the organisms live together and influence their mutual livelihoods. Symbiotic relationship may be *obligate* or *facultative*. In obligate symbiosis, two organisms cannot survive without each other. In facultative symbiosis, the species live together by choice.

So far as the growth of a single-species population is concerned, it has long been recognised that the famous *logistic growth function* is a logical choice. The function is introduced in 1838 by the Belgian mathematician Pierre Francois Verhulst [42] and later it is rediscovered in 1920 by American biologists Reymon Pearl and Lowell Reed [36]. If $x(t)$ denotes the population density at time $t$, then the logistic growth equation is given by

$$(1.1) \qquad \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right),$$

where $r$ is the intrinsic per capita growth rate and $k$ is the carrying capacity of the environment. The logic behind this is very simple. As the resources (e.g., space,

food, essential nutrients) are limited, every population grows into a saturated phase from which it cannot grow further; the ecological habitat of the population can carry just so much of it and no more. This suggests that the per capita growth rate is a decreasing function of the size of the population, and reaches zero as the population achieved a size $k$ (in the saturated phase). Further, any population reaching a size that is above this value will experience a negative growth rate. The term $-rx^2/k$ may also be regarded as the loss due to symbiotic population. Although logistic growth function became extremely popular, but, in real life situations, researchers found many evidences where the populations show a reverse trend in low population density [13, 18, 21, 22, 34, 37]. This phenomenon of positive density dependence of population growth at low densities is known as the *Allee effect* [40, 22].

The phenomenon of *Allee effect* is named after the US Behavioral scientist Warder Clyde Allee (although Allee never used the term 'Allee effect'). Allee described this concept in three of his papers [4, 5, 6]. Actually, the term 'Allee effect' was introduced by Odum [35]. Since the late eighties of the 20th century, the concept gained importance but there were necessity of clear-cut definitions and clarification of concepts. The necessity was fulfilled when three reviews by Stephens et al. [40], Courchamp et al. [21], Stephens and Sutherland [39]. There are many reasons for Allee effect, such as difficulty in mate finding, reduced antipredator vigilance, problem of environmental conditioning, reduced defense against predators, and many others (for thorough reviews, see references [13, 22]).

The Allee effect can be divided into two main types, depending on how strong the per capita growth rate is depleted at low population densities. These two types are called the strong Allee effect [41, 44, 45] or critical depensation [16, 17, 29], and the weak Allee effect [40, 43] or noncritical depensation [16, 17, 29]. Usually, the Allee effect is modelled by a growth equation of the form

$$(1.2) \qquad \frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)(x - m),$$

where $x(t)$ denotes the population density at time $t$, $r$ is the intrinsic per capita growth rate, and $k$ is the carrying capacity of the environment. Here $0 < m \ll k$. When $m > 0$ and the population size is below the threshold level $m$, then the population growth rate decreases [8, 19, 23, 27], and the population goes to extinction. In this case, the equation describes the *strong Allee effect* [41, 44, 45]. On the contrary, the description of *weak Allee effect* is also available (see references [24, 44]). In this paper, we are concerned with strong Allee effect. The above growth is often said to have a *multiplicative Allee effect.* There is another mathematical form of the growth function featuring the *additive Allee effect.* In this paper, we are not interested in additive Allee effect (interested readers might see the works of Aguirre et al. [1, 2]).

A comparison of the logistic growth function of (1.1) and the function representing Allee effect in equation (1.2) can be found in [31].

A *herd* or *pack* is a social grouping of different animals of the same species. When a species shows herd behaviour, the individuals of the species show a collective social behavior, and each individual chooses a behaviour that corresponds to that of the majority of other members (for example, all moving in the same direction at a given time). There are several reasons for herd behaviour (see [10, 11, 12, 32, 31] and references therein).

Now, when a population lives forming groups, then all members of a group do not interact at a time. There are many reasons for this herd behaviour, such as searching for food resources, defending the predators, etc. Usually the species, which exhibits this social behaviour, allows the weakest individuals to occupy the interior of the herd, leaving the healthier and stronger animals around it. As a consequence, it is necessary to search for suitable function to describe this social behaviour. Only a few works have so far tried to enlighten this area. These works demonstrated an ingenious idea that suitable powers of the state variables can account for the social behaviour of the populations. For example, to explore the consequence of forming spatial group of fixed shape by predators, Cosner et al. [20] introduced the idea that the square root of the predator variable is to be used in the function describing the encounter rate in two-dimensional systems. Unfortunately, such an idea has not been used by the researchers for about a decade. The work of Chattopadhyay et al. [15] may be regarded as a strong recognition of this concept. Then came the most innovative work of Ajraldi et al. [3], which gave such modelling a new dimension. Their idea is very interesting. If $x$ is the density of a population that gathers in herds and the herd occupies an area $A$, then the number of individuals staying at outermost positions in the herd is proportional to the length of the perimeter of the patch where the herd is located. Clearly, its length is proportional to $\sqrt{A}$. Since $x$ is distributed over a two-dimensional domain, $\sqrt{x}$ would therefore count the individuals at the edge of the patch.

The Lotka-Volterra type mutualism model, where each of the species follows logistic growth dynamics in the absence of the other, is taken as the basis of mathematical study of symbiotic relationship. The details of the formulation and analysis of this model can be found in the classical book of Brauer and Castillo-Chavez [14]. Recently, Ajraldi et al. [3] have derived and analyzed a symbiotic model, where one of the species shows a herd behaviour. In this paper, we have considered a symbiotic model where one species shows herd behaviour, whereas the other species is subject to strong Allee effect.

In recent times, it is well understood that many of the processes, both natural and manmade, in biology, medicine, et cetera, involve time-delays or time-lags. Time-delays occur so often, in almost every situation, that to ignore them is to ignore reality. Kuang [28] mentioned that animals must take time to digest their food before further activities and responses take place and hence any model of species dynamics without delays is an approximation at best. Now it is beyond doubt that in an improved analysis, the effect of time-delay due to the time required in going from egg stage to the adult stage, gestation period, et cetera, has to be taken into account. Detailed arguments on importance and usefulness of time-delays in realistic models may be found in the classical books of Gopalsamy [25], Kuang [28] and MacDonald [30].

Hutchinson [26] pointed out that the logistic equation would be inappropriate for the description of population growth in the case where there is a time-delay in some of the processes involved. The delayed logistic equation or Hutchinson's equation is one of the first examples of a delay differential equation that has been thoroughly examined (see for example [7, 25, 28, 30], and the references within).

In this paper, we have considered a two-species symbiotic population model. The first species displays a herd behaviour. On the other hand, the second species is vulnerable to Allee effect. The paper is structured as follows. In section 2, the basic deterministic model (without time-delay) is introduced. In section 3, the behaviours including positivity, stability of equilibria are discussed. The effect of time-delay in the logistic growth of the first species is analyzed in section 4. In section 5, the analytical findings of sections 3 and 4 are verified through computer simulation. Section 6 contains the general discussions of the paper and biological implications of our mathematical findings.

## 2. **THE BASIC MATHEMATICAL MODEL**

The model we analyze in this paper is composed of two species in symbiotic relationship. The population densities of the species in time $t$ are denoted by $x$ and $y$. Before we introduce the model and dip into the depth of the rigorous analysis, we would like to present a brief sketch of the construction of the model which may indicate the biological relevance of it.

1. We assume that the first species (with density $x$) follows a logistic growth in the absence of the other species.

2. The second species is subject to a strong Allee effect.

3. There is a symbiotic relationship (or mutualism) between the two species.

4. The first species exhibits a herd behaviour.

These considerations motivate us to introduce the following prey-predator system under the framework of the following set of nonlinear ordinary differential equations:

(2.1)
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) + a\sqrt{x}y, \quad x(0) > 0,$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{c}\right)(y - m) + b\sqrt{x}y, \quad y(0) > 0.$$

Here $r$ is the intrinsic growth rate of the first species, $k$ is its carrying capacity. The parameter $s$ is the intrinsic growth rate of the second species, $c$ is its carrying capacity, $m$ indicates the Allee threshold for the first species (since the Allee effect is strong, we have $0 < m \ll c$). The term $b\sqrt{x}y$ in the first equation can be thought of as the contribution made by the second species to the growth rate of the first species. Similarly, $b\sqrt{x}y$ is the contribution in the growth rate of the second species caused by individuals of the first species.

## 3. DYNAMICS OF THE MODEL

In this section, we discuss the dynamical behaviours of the system (2.1). We first notice the positivity of solutions.

### 3.1. Positivity of solutions.

**Theorem 3.1.** *All solutions of the system* (2.1) *that start in* $\mathbb{R}^2_+$ *remain positive forever.*

The proof is simple and therefore it is omitted.

### 3.2. Equilibria.
In this section, we find the equilibrium points of the system (2.1) and study their stability. The eqilibrium points and the conditions of their existence are given in the following lemma.

**Lemma 3.2.** *The trivial equilibrium* $E_0(0,0)$ *of the system* (2.1) *always exists. There are three boundary equilibrium points* $E_1(k,0)$, $E_2(0,c)$ *and* $E_3(0,m)$, *each of which also exists unconditionally. The first component* $x^*$ *of the interior or coexistence equilibrium* $E^*(x^*, y^*)$ *exists if and only if the equation*

(3.1)
$$\frac{sr^2x}{a^2c}\left(\frac{x}{k} - 1\right)^2 - \frac{rs\sqrt{x}}{a}\left(1 + \frac{m}{c}\right)\left(\frac{x}{k} - 1\right) - b\sqrt{x} + sm = 0$$

*has a positive root. When this condition is satisfied, then* $x^*$ *is a positive root of* (3.1). *If further,* $x^* > k$, *then* $y^*$ *exists and is given by*

$$y^* = \frac{r\sqrt{x^*}}{a}\left(\frac{x^*}{k} - 1\right).$$

3.3. **Behaviour near the boundary equilibria** $E_0, E_2, E_3$. System (2.1) cannot be linearized at $E_0 = (0,0)$, $E_2 = (0,c)$ and $E_3 = (0,m)$, so local stability of $E_0$, $E_2$ and $E_3$ cannot be studied in the usual way. However, we provide some logical arguments to understand the behaviours of the system (2.1) near the above equilibrium points.

If the first populations disappears, the system reduces to one equation. In this circumstance, the surviving population $y$ follows the equation $\frac{dy}{dt} = sy\left(1 - \frac{y}{c}\right)(y-m)$ having three equilibrium points $0, m, c$. It is well known that the equilibrium points $0$ and $c$ are asymptotically stable, and $m$ is unstable. Therefore, if $y(0) < m$, then $y$ will approach $0$; otherwise it will grow toward its own carrying capacity $c$. Thus $E_2(0,c)$ is locally asymptotically stable, and $E_3(0,m)$ is unstable.

To study the behaviour near $E_0(0,0)$, it is reasonable to assume that $x \ll \min\{1,k\}$ and $y \ll \min 1, m$. Then $x < \sqrt{x}$ and $y < \sqrt{y}$, and $x, y$ are so small that, following the approach of Melchionda et al. [33], we have

$$\frac{dx}{dt} \approx r\sqrt{x} > 0, \quad \frac{dy}{dt} \approx s\sqrt{y} > 0.$$

This indicates that the origin is unstable.

3.4. **Stability of $E_1$ and $E^*$.** The Jacobian matrix of system (2.1) at an equilibrium point $(x,y)$ (when $x \neq 0$) is

$$J(x,y) = \begin{bmatrix} r - \frac{2r}{k}x + \frac{ay}{2\sqrt{x}} & a\sqrt{x} \\ \frac{by}{2\sqrt{x}} & (y-m)(s - \frac{2sy}{c}) + sy(1 - \frac{y}{c}) + b\sqrt{x} \end{bmatrix}.$$

**Theorem 3.3.** *The equilibrium point $E_1(k,0)$ of system is stable if and only if $b\sqrt{k} - ms < 0$.*

*Proof.* The Jacobian matrix at $E_1 = (k,0)$ is

$$J(E_1) = \begin{bmatrix} -r & a\sqrt{k} \\ 0 & b\sqrt{k} - ms \end{bmatrix}.$$

Clearly, its eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = b\sqrt{k} - ms$. Since $\lambda_1 < 0$, $E_1$ is stable if and only if $\lambda_2 < 0 \Leftrightarrow b\sqrt{k} - ms < 0$. Hence the theorem follows. $\square$

The Jacobian matrix at $E^*(x^*, y^*)$ is

$$J(E^*) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

where

$$a_{11} = r - \frac{2rx^*}{k} + \frac{ay^*}{2\sqrt{x^*}}$$

$$a_{12} = a\sqrt{x^*}$$

$$a_{21} = \frac{by^*}{2\sqrt{x^*}}$$

$$a_{22} = (y^* - m)\left(s - \frac{2sy^*}{c}\right) + sy^*\left(1 - \frac{y^*}{c}\right) + b\sqrt{x^*}.$$

The characteristic equation of $J(E^*)$ is

$$\lambda^2 + P\lambda + Q = 0,$$

where $P = -\text{tr } J(E^*) = -(a_{11} + a_{22})$ and $Q = \det J(E^*) = a_{11}a_{22} - a_{12}a_{21}$. Then we have the following theorem guaranteeing the stability of $E^*$.

**Theorem 3.4.** *If the equilibrium point $E^*(x^*, y^*)$ exists with $P > 0$ and $Q > 0$, then $E^*$ is locally asymptotically stable.*

*Proof.* It is easy to notice that, if the conditions of the theorem are satisfied, then all the eigenvalues of $J(E^*)$ have negative real parts. Hence the theorem follows. $\square$

## 4. EFFECT OF DISCRETE TIME-DELAY

It is mentioned that time-delay is a vital factor in biological systems. In particular, delay in logistic growth function is extremely important [7, 30, 25, 26, 28, 38]. As a starting point of this section, we consider the following generalization of the model (2.1) involving discrete delay:

(4.1)
$$\frac{dx}{dt} = rx\left[1 - \frac{x(t - \tau)}{k}\right] + a\sqrt{x}y,$$
$$\frac{dy}{dt} = sy\left(1 - \frac{y}{c}\right)(y - m) + b\sqrt{x}y.$$

Here it is assumed that the net per capita rate of change might depend on the state of the system $\tau$ time units in the past [7].

The system (4.1) has the same equilibria as in the previous case. The main purpose of this section is to study the stability behaviour of $E^*(x^*, y^*)$ in presence of delay $(\tau \neq 0)$.

We linearize the system (4.1) by using the following transformations:

$$x = x^* + x_1, \quad y = y^* + y_1.$$

Then the linear system is given by

(4.2)
$$\frac{du}{dt} = Au(t) + Bu(t - \tau)$$

where $u(t) = [x_1 \ y_1]^T$, $A = (a_{ij})_{2\times 2}$, $B = (b_{ij})_{2\times 2}$, and $a_{11} = 0$, $a_{12} = 0$, $a_{21} = \frac{by^*}{(2\sqrt{x^*})}$, $a_{22} = (y^* - m)(s - \frac{2sy^*}{c}) + sy^*(1 - \frac{y^*}{c}) + b\sqrt{x^*}$, $b_{11} = -\frac{rx^*}{k}$, $b_{12} = 0$, $b_{21} = 0$, $b_{22} = 0$.

We look for solution of the model (4.2) of the form $u(t) = \rho e^{\lambda t}$, $0 \neq \rho \in \mathbb{R}^2$. This leads to the following characteristic equation:

$$(4.3) \qquad \lambda^2 + a_1\lambda + (a_2\lambda + a_3)e^{-\lambda\tau} = 0,$$

where $a_1 = -a_{22}$, $a_2 = -b_{11}$, $a_3 = b_{11}a_{22}$.

It is well known that the signs of the real parts of the solutions of (4.3) characterize the stability behaviour of $E^*$. Therefore, substituting $\lambda = \xi + i\eta$ in (4.3), we obtain real and imaginary parts, respectively, as

$$(4.4) \qquad \xi^2 - \eta^2 + a_1\xi + [\{a_2\xi + a_3\}\cos\eta\tau + a_2\eta\sin\eta\tau]\, e^{-\xi\tau} = 0,$$

and

$$(4.5) \qquad 2\xi\eta + a_1\eta + [a_2\eta\cos\eta\tau - \{a_2\xi + a_3\}\sin\eta\tau]\, e^{-\xi\tau} = 0.$$

A necessary condition for a stability change of $E^*$ is that the characteristic equation (4.3) has purely imaginary solutions. Hence to obtain the stability criterion, we set $\xi = 0$ in (4.4) and (4.5) to obtain

$$(4.6) \qquad \eta^2 = a_3\cos\eta\tau + a_2\eta\sin\eta\tau,$$

and

$$(4.7) \qquad a_1\eta = a_3\sin\eta\tau - a_2\cos\eta\tau.$$

Eliminating $\tau$ by squaring and adding (4.6) and (4.7), we get the equation for determining $\eta$ as

$$(4.8) \qquad \eta^4 + d_1\eta^2 + d_2 = 0,$$

where $d_1 = a_1^2 - a_2^2$, $d_2 = -a_3^2$.

Substituting $\eta^2 = \sigma$ in (4.8), we get a quadratic equation given by

$$(4.9) \qquad \sigma^2 + d_1\sigma + d_2 = 0.$$

The quadratic (4.9) always has one and only one positive real root. Let $\sigma_0$ be the unique positive root of (4.9). So there exist $\sigma_0 = \eta_0^2$ that satisfy equation (4.8). In the following theorem, we have given a criterion for switching the stability behaviour of $E^*$.

**Theorem 4.1.** *Let $E^*(x^*, y^*)$ exists with $P > 0$ and $Q > 0$. Then $E^*$ is locally asymptotically stable for $0 \leq \tau < \tau^*$, and unstable for $\tau > \tau^*$ and the system exhibits a hopf bifurcation near $E^*$ for $\tau = \tau^*$, provided*

$$f(\eta_0) = \frac{1}{\eta_0}(a_1 a_3 \eta_0 \sin\eta_0\tau - a_1 a_2 \eta_0^2 \cos\eta_0\tau + 2a_2\eta_0^3 \sin\eta_0\tau + 2a_3\eta_0^2 \cos\eta_0\tau - a_2^2\eta_0^2) > 0,$$

*where*

$$\tau^* = \frac{1}{\eta_0}\arcsin\left[\frac{\eta_0^3 a_2 + a_1 a_3 \eta_0}{a_3^2 + a_2^2\eta_0^2}\right].$$

*Proof.* First we note that $\eta_0$ is a solution of (4.8). Solving (4.6) for $\cos \tau \eta_0$ and substituting in (4.7), we find that for $\tau = \tau^*$, the characteristic equation (4.3) have purely imaginary roots, $\pm i \eta_0$. Again it may be noted that if $\pm \eta_0$ is a solution of (4.6) and (4.7), then $\eta_0^2$ is a solution of (4.8). The theorem will be proved if we can show that $\left[ \frac{d\xi}{d\tau} \right]_{\tau=\tau^*} > 0$. To show this, we differentiate (4.6) and (4.7) with respect to $\tau$ and then set $\xi = 0$ to obtain

$$(4.10) \qquad\qquad C(\eta)\frac{d\xi}{d\tau} + D(\eta)\frac{d\eta}{d\tau} = G(\eta)$$

and

$$(4.11) \qquad\qquad -D(\eta)\frac{d\xi}{d\tau} + C(\eta)\frac{d\eta}{d\tau} = H(\eta),$$

where

$$
\begin{aligned}
C(\eta) &= a_1 + a_2 \cos \eta\tau - a_3\tau \cos \eta\tau - a_2\eta\tau \sin \eta\tau \\
D(\eta) &= -2\eta - a_3\tau \sin \eta\tau + a_2 \sin \eta\tau + a_2\eta\tau \cos \eta\tau \\
G(\eta) &= a_3\eta \sin \eta\tau - a_2\eta^2 \cos \eta\tau \\
H(\eta) &= a_2\eta^2 \sin \eta\tau + a_3\eta \cos \eta\tau.
\end{aligned}
$$

Solving (4.10) and (4.11) with $\tau = \tau^*$ and $\eta = \eta_0$, we get

$$\left[ \frac{d\xi}{d\tau} \right]_{\tau=\tau^*} = \frac{\eta_0^2 f(\eta_0)}{C^2(\eta_0) + D^2(\eta_0)},$$

which is positive under the condition of the theorem ($\because f(\eta_0) > 0$). Hence the theorem is established. $\qquad\square$

## 5. NUMERICAL SIMULATION

Analytical studies can never be complete without numerical verification of the results. In this section we present computer simulation of different solutions of the system (2.1) using MATLAB.

First we choose the parameters of the system as $a = 2.5$, $b = 0.4$, $c = 4.2$, $r = 1.2$, $s = 2.5$, $k = 3.5$, $m = 4$ and $x(0) = 3.5$, $y(0) = 2.5$. Then condition $(-b\sqrt{k} + ms = 5.94 > 0)$ of Theorem 3.3 is satisfied, and as such $E_1(k,0)$ is locally asymptotically stable. The behaviour of $x$ and $y$ with time is depicted in Figure 1. Clearly $x$ approaches $k = 3.5$ and $y$ approaches 0 in finite time.
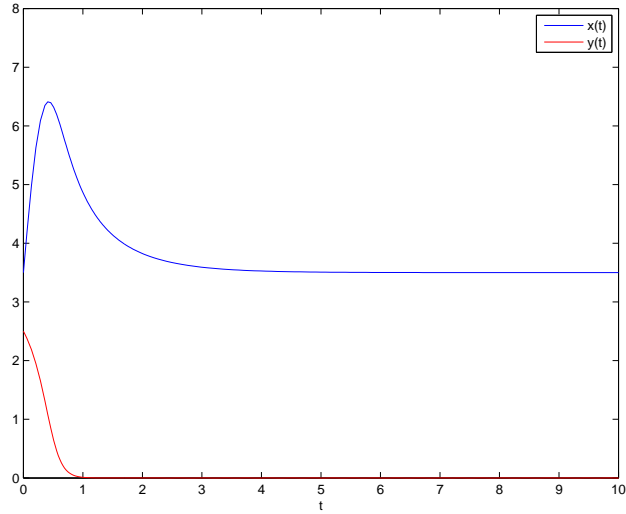
**Figure 1.** *Behaviour of x and y with t for the system* (2.1) *when $a = 2.5$, $b = 0.4$, $c = 4.2$, $r = 1.2$, $s = 2.5$, $k = 3.5$, $m = 4$ and $x(0) = 3.5$, $y(0) = 2.5$.*

Now we choose the parameters of the system as $a = 0.5$, $b = 0.5$, $c = 0.8$, $r = 0.03$, $s = 0.65$, $k = 0.5$, $m = 0.3$. Then condition of Theorem 3.4 is satisfied, and as such $E^*(6.4790, 1.8263)$ is locally asymptotically stable. The corresponding phase portrait with different initial choices is shown in Figure 2. The behaviour of $x$ and $y$ with time is shown in Figure 3 when $x(0) = 1$ and $y(0) = 1$.



**Figure 2.** *Phase portrait of the system* (2.1) *for different values of $x(0)$ and $y(0)$ when $a = 0.5$, $b = 0.5$, $c = 0.8$, $r = 0.03$, $s = 0.65$, $k = 0.5$, $m = 0.3$.*
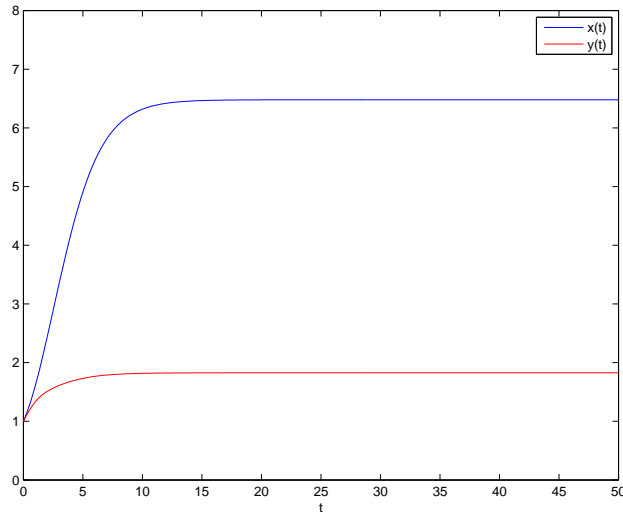
**Figure 3.** *Behaviour of x and y with t for the system* (2.1) *when the parameter values are same as in Figure 2 and* $x(0) = 1$, $y(0) = 1$.

For the above choices of parameters as in Figure 2, using Theorem 4.1, we see that $\tau^* = 5.12$. Therefore, by Theorem 4.1, $E^*(6.4790, 1.8263)$ is locally asymptotically stable for $0 \leq \tau < \tau^*$, and Hopf bifurcation occurs at $\tau = \tau^*$. We verify that, for $\tau = 5.1 < \tau^*$, $E^*$ is locally asymptotically stable. The corresponding phase portrait for different initial choices is shown in Figure 4. The stable behaviour of $x$ and $y$ with time is presented in Figure 5, when $x(0) = 0.2$, $y(0) = 0.3$.
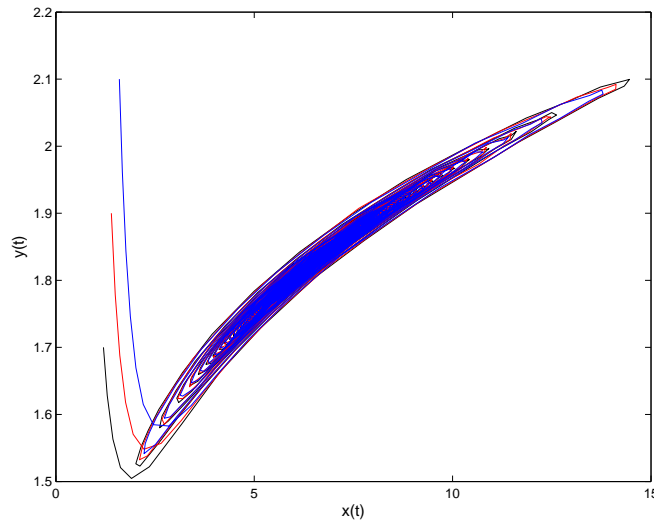


**Figure 4.** *Phase portrait of the system* (4.1) *for different values of* $x(0)$ *and* $y(0)$ *when* $a = 0.5$, $b = 0.5$, $c = 0.8$, $r = 0.03$, $s = 0.65$, $k = 0.5$, $m = 0.3$ *and* $\tau = 5.1$.
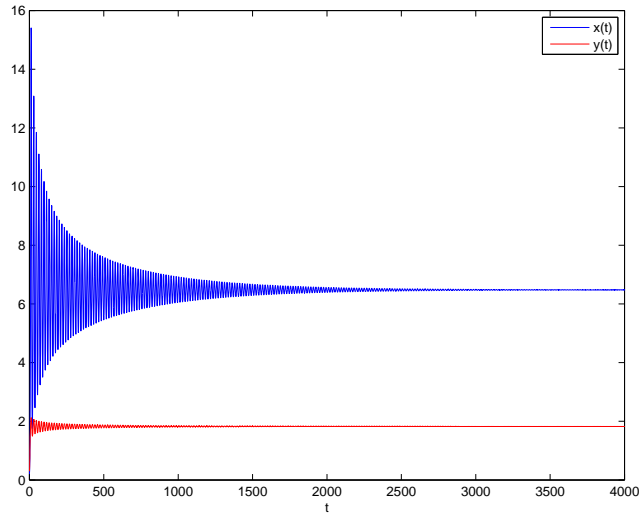
**Figure 5.** *Stable behaviour of x and y with t for the system* (4.1) *when the parameter values are same as in Figure 4 and $x(0) = 0.2$, $y(0) = 0.3$.*

On the other hand, if we take $\tau = 5.3 > \tau^*$ keeping other parameters fixed, then $E^*$ becomes unstable. The corresponding phase portrait for different initial choices is shown in Figure 6, which shows that there is a limit cycle that grows out of $E^*$. The oscillations of $x$ and $y$ with time is presented in Figure 7, when $x(0) = 0.2$, $y(0) = 0.3$.
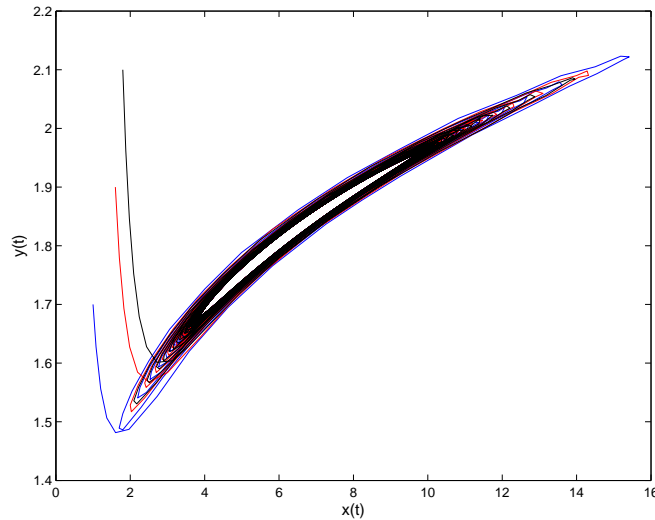


**Figure 6.** *Phase portrait of the system* (4.1) *for different values of $x(0)$ and $y(0)$ when the parameter values are same as in Figure 4 except $\tau = 5.3 > \tau^*$.*
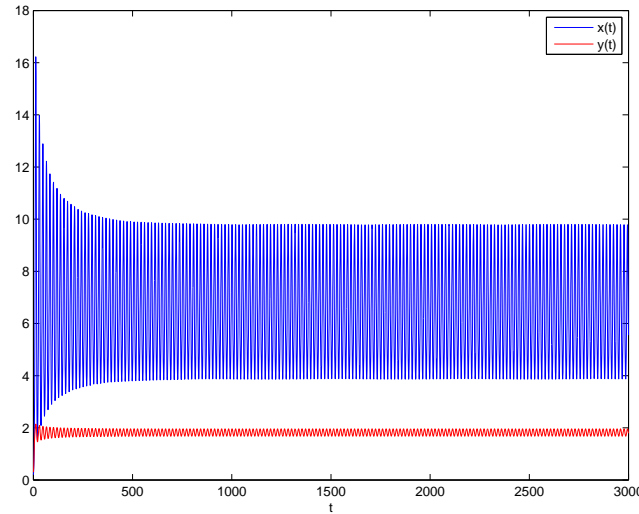
**Figure 7.** *Oscillations of $x$ and $y$ with $t$ for the system* (4.1) *when the parameter values are same as in Figure 6 and $x(0) = 0.2$, $y(0) = 0.3$.*

## 6. Concluding remarks

Many populations in nature live in herd for numerous reasons. When two species live in close physical association, they might have a symbiotic relationship (usually for the advantage of both). Although symbiosis is usually thought of as a mutually beneficial relationship, but there is difference of opinion. It is not our intention to enter into that debate, rather we take it in usual sense. Now, a population might be susceptible to Allee effect due to many reasons. There should be no denying that, studies of interplay between these phenomena would be very exciting and challenging. In this paper, we have constructed a two-species symbiotic model (2.1), where one species shows a herd behaviour and the other is subject to strong Allee effect.

It is observed (in Theorem 3.1) that the solutions of the system (2.1) remains non-negative forever. Analysis of stability of the equilibrium points is presented. The effect of discrete time-delay in the growth term of the first species is studied. It is shown that the delay of certain dimension might keep the system in a stable state, whereas the delay has the capability to induce instability into the system through Hopf bifurcation. In other words, delay has a regulatory impact on the underlying system. Our results are illustrated through computer simulation using MATLAB.

Nowadays, preservation of ecological balance in nature is an issue which the entire globe is concerned with. Stability analysis of the interior equilibrium might provide the conditions for ecological balance in nature. From our analysis, it is evident that if condition of Theorem 3.4 is satisfied, and $\tau$ lies below $\tau^*$, then ecological balance can be preserved.

# REFERENCES

[1] P. Aguirre, E. González-Olivares and E. Sáez, Two limit cycles in a Leslie-Gower predator-prey model with additive Allee effect, *Nonlinear Anal. Real World Applications*, 10:1401–1416, 2009.

[2] P. Aguirre, E. González-Olivares and E. Sáez, Three limit cycles in a Leslie-Gower predator-prey model with additive Allee effect, *SIAM J. Appl. Math.*, 69:1244–1262, 2009.

[3] V. Ajraldi, M. Pittavino and E. Venturino, Modeling herd behavior in population systems, *Nonlinear Anal. Real World Applications*, 12:2319–2338, 2011.

[4] W. Allee, Animal aggregations, *Quart. Rev. Biol.*, 2:367–398, 1927.

[5] W. Allee, Studies in animal aggregations: some physiological effects of aggregation on the brittle starfish, Ophioderma brevispina, *J. Exp. Zool.*, 48:475–495, 1927.

[6] W. Allee, Studies in animal aggregations: mass protection against colloidal silver among gold-fishes, *J. Exp. Zool.*, 61:185–207, 1932.

[7] J. Arino, L. Wang, G. S. K. Wolkowicz, An alternative formulation for a delayed logistic equation, *J. Theor. Biol.* 241:109–119, 2006.

[8] A. D. Bazykin, F. S. Berezovskaya, A. S. Isaev and R. G. Khlebopros, Dynamics of forest insect density: bifurcation approach, *J. Theor. Biol.*, 186:267–278, 1997.

[9] M. Begon, J. L. Harper and C. R. Townsend, *Ecology: Individuals, populations and communities*, Blackwell Science, 1996.

[10] S. P. Bera, A. Maiti and G. P. Samanta, Modelling herd behavior of prey: analysis of a prey-predator model, *World Journal of Modelling and Simulation*, 11:3–14, 2015.

[11] S. P. Bera, A. Maiti and G. P. Samanta, Stochastic analysis of a prey-predator model with herd behaviour of prey, *Nonlinear Analysis: Modelling and Control*, 21(23):345–361, 2016.

[12] S. P. Bera, A. Maiti and G. P. Samanta, Dynamics of a food chain model with herd behaviour of the prey, Model. Earth Syst. Environ.,9 pages, 2016, DOI 10.1007/s40808-016-0189-4.

[13] L. Berec, E. Angulo and F. Courchamp, Multiple Allee effects and population management,*Trends Ecol. Evol.*, 22:185–191, 2006.

[14] F. Brauer, and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, New York, 2012.

[15] J. Chattopadhyay, S. Chatterjee and E. Venturino, Patchy agglomeration as a transition from monospecies to recurrent plankton blooms, *J. Theor. Biol.*, 253:289–295, 2008.

[16] C. W. Clark, *Mathematical Bioeconomics: The Optimal Mmanagement of Renewable Resources*, Wiley, New York, 1990.

[17] C. W. Clark, *The Worldwide Crisis in Fisheries: Economic Models and Human Behavior*, Cambridge University Press, Cambridge, 2007.

[18] T. H. Clutton-Brock, D. Gaynor, G. M. McIlrath, A. D. C. Maccoll, R. Kansky, P. Chadwick, M. Manser, J. D. Skinner and P. N. M. Brotherton, Predation, group size and mortality in a cooperative mongoose, Suricata suricatta, *J. Anim. Ecol.*, 68:672–683, 1999.

[19] E. D. Conway and J. A. Smoller, Global analysis of a system of predator-prey equations, *SIAM J. Appl. Math.*, 46: 630–642, 1986.

[20] C. Cosner, D. L. DeAngelis, J. S. Ault and D. E. Olson, Effects of spatial grouping on the functional response of predators, *Theor. Pop. Biol.*, 56:65–75, 1999.

[21] F. Courchamp, T. Clutton-Brock and B. Grenfell, Inverse dependence and the Allee effect, *Trends Ecol. Evol.*, 14:405–410, 1999.

[22] F. Courchamp, L. Berec and J. Gascoigne, *Allee Effects in Ecology and Conservation*, Oxford University Press, Oxford, 2008.

[23] J. D. Flores, J. Mena-Lorca, B. González-Yañez, and E. González-Olivares, Consequences of depensation in a Smith's bioeconomic model for open-access fishery, In R. Mondaini and R. Dilao (Eds.), *Proceedings of international symposium on mathematical and computational biology*, E-papers Servios Editoriais Ltda, 219–232, 2007.

[24] E. González-Olivares and A. Rojas-Palma, Multiple Limit Cycles in a Gause type predator-prey model with Holling type III functional response and Allee effect on prey, *Bull. Math. Biol.*, 73: 1378–1397, 2011.

[25] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer Academic, Dordrecht, 1992.

[26] G. E. Hutchinson, Circular causal systems in ecology, *Ann. NY Acad. Sci.*, 50:221–246, 1948.

[27] M. Kot, *Elements of Mathematical Biology*, Cambridge University Press, Cambridge, 2001.

[28] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993.

[29] M. Liermann and R. Hilborn, Depensation: evidence, models and implications, *Fish Fish.*, 2:33–58, 2001.

[30] N. Macdonald, *Biological Delay Systems: Linear Stability Theory*, Cambridge University Press, Cambridge, 1989.

[31] A. Maiti, P. Sen, D. Manna and G. P. Samanta, A Predator-prey system with herd behaviour and strong Allee effect, *Nonlinear Dynamics and Systems Theory*, 16:86–101, 2016.

[32] A. Maiti, P. Sen and G. P. Samanta, Deterministic and stochastic analysis of a prey-predator model with herd behaviour in both, *Systems Science and Control Engineering: An Open Access Journal, Taylor and Francis, U.K.*, 4(1):259–269, 2016.

[33] D. Melchionda, E. Pastacaldi, C. Perri and E. Venturino, Interacting population models with pack behavior, *arXiv:1403.4419v1 [math.DS]*, 18 Mar 2014.

[34] M. S. Mooring, T. A. Fitzpatrick, T. T. Nishihira and D. D. Reisig, Vigilance, predation risk, and the Allee effect in desert bighorn sheep, *J. Wildlife Management*, 68: 519–532, 2004.

[35] E. Odum, *Fundamentals of Ecology*, Saunders, Philadelphia, 1953.

[36] R. Pearl and L. Reed, On the rate of growth of the population of the United States, *Proc. Nat. Acad. Sci.*, 6:275–288, 1920.

[37] D. J. Rinella, M. S. Wipfli, C. A. Stricker, R. A. Heintz and M. J. Rinella, Pacific salmon (Oncorhynchus sp.) runs and consumer fitness: growth and energy storage in stream-dwelling salmonids increase with salmon spawner density, *Canad. J. Fish. Aquat. Sci*, 69:73–84, 2012.

[38] S. Ruan, Delay differential equations in single species dynamics, Arino et al. (eds.), *Delay Differential Equations and Applications*, Springer, Berlin,477–517, 2006.

[39] P. A. Stephens and W. J. Sutherland, Consequences of the Allee effect for behaviour, ecology and conservation, *Trends Ecol. Evol.*, 14:401–405, 1999.

[40] P. A. Stephens, W. J. Sutherland and R. P. Freckleton, What is the Allee effect? *Oikos*, 87:185–190, 1999.

[41] G. A. K. van Voorn, L. Hemerik, M. P. Boer and B. W. Kooi, Heteroclinic orbits indicate over exploitation in predator-prey systems with a strong Allee effect,*Math. Biosci.*, 209: 451–469, 2007.

[42] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, *Correspondance mathématique et physique*, 10:113–121, 1838.

[43] G. Wang, X. G Liang and F. Z Wang, The competitive dynamics of populations subject to an Allee effect, *Ecol. Model.*, 124:183–192, 1999.

[44] M. H. Wang and M. Kot, Speeds of invasion in a model with strong or weak Allee effects, *Math. Biosci.*, 171:83–97, 2001.

[45] J. Wang, J. Shi and J. Wei, Predator-prey system with strong Allee effect in prey, *J. Math. Biol.*, 62:291–331, 2011.