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A DELTA-SHAPED BASIS METHOD FOR ILL-POSED NONHOMOGENEOUS ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, a Delta-shaped basis method is coupled with the method of fundamental solutions and Tikhonov regularization for solving ill-posed nonhomogeneous elliptic boundary value problems. Delta-shaped basis functions are used to approximate the source function since they can effectively handle scattered data and give rapidly convergent approximation. This approach also results in an easy derivation of a particular solution for a general type elliptic operator. The associated homogeneous problem is solved by the method of fundamental solutions with Tikhonov regularization. The approach is mesh free and is effective for domains of irregular shapes. Numerical results show that this method is accurate and stable against perturbed data.

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1. INTRODUCTION

Inverse problems are concerned with determining causes for observed or desired effects. Problems of this type appear in many application fields in science and engineering. For example, data can be missing on parts of the boundary due to the issues of accessibility and cost of measurement [7]. In this paper, we study the inverse nonhomogeneous problem of an elliptic operator whose boundary data are given only on the part of the accessible boundary,

$$Lu = h(x,y) \text{ in } \Omega, \tag{1.1}$$

$$u = f(x,y) \text{ on } \Gamma, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = g(x,y) \text{ on } \Gamma, \tag{1.3}$$

where L is an elliptic differential operator, Ω is a bounded and simply connected domain in R^2, Γ is a part of the boundary $\partial\Omega$, $h(x,y)$ is the source function, $f(x,y)$ and $g(x,y)$ are respectively the Dirichlet and Neumann data specified on Γ, and n is the unit outward normal with respect to $\partial\Gamma$.

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Inverse problems are usually ill-posed. That is, they most often do not fulfill Hadamard's postulates of well-posedness: they might not have a solution in the strict sense, solutions might not be unique and/or might not depend continuously on the data. It has only been since the mid-1960s that inverse problems have been identified as a proper subfield of mathematics [8]. Problem (1.1)–(1.3) is the Cauchy problem for an elliptic equation which arises in many real applications [1, 2, 11]. It is ill-posed and its solution does not depend continuously on the data. Solution methods for ill-posed PDE problems have been actively investigated by researchers in the past couple of decades. Reinhardt et. al. [30] solved the Cauchy problem for Laplace's equation using standard five-point difference approximation. Hao and Lesnic [15] solved the problem using the conjugate gradient method. Hon and Wei [16, 38] transformed the Cauchy problem of Laplace equation into a classical moment problem to achieve the numerical approximation of the solution. Hon and Wei later solved the inverse heat conduction problems using the method of fundamental solutions (MFS) and Tikhonov regularization [17, 18]. Applications of the MFS with regularization in solving inverse problems of elliptic operators can be found in [26, 27, 39, 40]. The radial basis meshless collocation methods [7, 23, 24] were also used for solving inverse boundary value problems.

Following the framework of boundary methods, we first find a particular solution u_p that satisfies the equation (1.1), i.e., $Lu_p = h$. Once u_p is known, it gives rise to the following homogeneous equation subject to a new boundary condition,

$$Lu_h = 0 \text{ in } \Omega, \tag{1.4}$$

$$u_h = f(x,y) - u_p(x,y) \text{ on } \Gamma, \tag{1.5}$$

$$\frac{\partial u_h}{\partial n} = g(x,y) - \frac{\partial u_p(x,y)}{\partial n} \text{ on } \Gamma. \tag{1.6}$$

The solution of the original problem (1.1)–(1.3) is then obtained as $u = u_p + u_h$.

The dual reciprocity method (the DRM) [28, 29] has been a popular method to overcome the difficulties of evaluating u_p. In the framework of the DRM, the source function can be approximated by a variety of bases [4, 12, 32]. It is most commonly approximated by the radial basis functions (the RBFs) [4, 6, 25]. Despite the important interpolating properties of the RBFs, one of the drawbacks of their use is that it is difficult to obtain rapidly convergent interpolants. With the RBFs, the closed form approximate particular solution can be difficult to derive for a general differential operator L. A good candidate for overcoming these difficulties is the Delta-shaped basis [31, 33, 35] which is effective for scattered data and hence for problems on irregular-shaped domains. The characteristics of the Delta-shaped basis allows for not only an accurate approximation of the source function, but also an easy derivation of a closed form approximate particular solution.

Once a particular solution is found, the associated homogeneous problem is then solved by the method of fundamental solutions. The MFS was first proposed by Kupradze and Aleksidze [22]. As an efficient boundary method it has been applied to solve many well-posed and ill-posed partial differential equations. There are two well-known review papers [10, 20] of the MFS for well-posed and ill-posed problems.

In real applications, the data are usually obtained through measurement, which means that the data have measuring errors. In our numerical experiments, we assume that the source function, Dirichlet and Neumann data have certain level of noises. That is, intending to solve (1.1)–(1.3) we face the problem

$$Lu = \widehat{h}(x,y) \text{ in } \Omega, \tag{1.7}$$

$$u = \widehat{f}(x,y) \text{ on } \Gamma, \tag{1.8}$$

$$\frac{\partial u}{\partial n} = \widehat{g}(x,y) \text{ on } \Gamma, \tag{1.9}$$

where

$$\widehat{h}(x,y) = (1+\varepsilon_h(x,y))h(x,y), \tag{1.10}$$

$$\widehat{f}(x,y) = (1+\varepsilon_f(x,y))f(x,y), \tag{1.11}$$

$$\widehat{g}(x,y) = (1+\varepsilon_g(x,y))g(x,y), \tag{1.12}$$

with $\varepsilon_h(x,y)$, $\varepsilon_f(x,y)$, $\varepsilon_g(x,y)$ being the levels of random noises relative to the source function, Dirichlet data and Neumann data, respectively. Since the discretized linear system is highly ill-conditioned, regularization technique [13, 14, 17, 38, 39, 40] is applied for stabilizing its solution. In this paper we use the Delta-shaped basis together with the MFS and regularization to solve the problem (1.1)–(1.3) so that we can handle well not only the derivation of an accurate approximate particular solution but also a stable numerical solution against the disturbed data. That is, we achieve the numerical objectives through an effective basis in approximation, an efficient method of particular solutions for a general elliptic operator, and an efficient boundary method.

The outline of the paper is as follows: In Section 2, two sets of Delta-shaped basis functions are used for the approximation of the source function and the derivation of an approximate particular solution. In Section 3, the MFS and the Tikhonov regularization technique are described. In Section 4, numerical examples are given to demonstrate the effectiveness and stability of the method for inverse nonhomogeneous problems of elliptic operators. Conclusions are made in Section 5.

2. DERIVATION OF A PARTICULAR SOLUTION

To obtain a particular solution of a given nonhomogeneous partial differential equation, we use a new type of basis called the Delta-shaped basis [31, 32, 33, 35].

Due to their special characteristics, (i) they can handle effectively the approximation of $h(x, y)$ when $h(x, y)$ is given as scattered data in arbitrary domains; (ii) they allow easy derivation of a particular solution for a general elliptic differential operator.

The two dimensional Delta-shaped basis function with center (ξ, η) is represented as

$$I_{M,\chi}(x, y; \xi, \eta) = \sum_{n,m=1}^{M} c_{n,m}(\xi, \eta)\, \varphi_n(x)\, \varphi_m(y)$$

where

$$c_{n,m}(\xi, \eta) = r_n(M, \chi)\, r_m(M, \chi)\, \varphi_n(\xi)\, \varphi_m(\eta), \tag{2.1}$$

and

$$\varphi_n(x) = \sin\left(\frac{n\pi(x+1)}{2}\right) \text{ and } \mu_n = \left(\frac{n\pi}{2}\right)^2, \ n = 1, 2, 3, \ldots.$$

are the solutions of the Sturm-Liouville problem $-\varphi'' = \mu\varphi$, $\varphi(-1) = \varphi(1) = 0$, on the interval $[-1, 1]$. The coefficients $r_n(M, \chi)$ in (2.1) are the regularizing coefficients. When using the Riesz regularization technique,

$$\begin{aligned} r_n(M, \chi) &= \left(1 - \frac{\lambda_n^2}{\lambda_{M+1}^2}\right)^{\chi} \\ &= \left[1 - \left(\frac{n}{M+1}\right)^2\right]^{\chi}, \end{aligned}$$

where M and χ are positive integers with M playing the role of scaling and χ playing the role of regularizing. The parameters M and χ are taken in coupling. In this paper, we use $\chi = 4, 6, 9, 14, 22$ for $M = 10, 20, 30, 50, 100$. Since the eigenfunctions $\varphi_n(x)$ satisfy the homogeneous boundary condition, the 2-D Delta-shaped basis functions vanish on the boundary of the square $[-1, 1] \times [-1, 1]$. Therefore, when dealing with a 2-D problem, the data to be approximated should stay away from the boundary of the square $[-1, 1] \times [-1, 1]$. In order for the Delta-shaped basis to approximate a function with any boundary condition, we apply appropriate translation and scaling to the function such that the domain of the function is contained in $[-0.5, 0.5] \times [-0.5, 0.5]$.

We first approximate the source function $h(x, y)$ by the linear combination of the translates of two types of Delta-shaped basis functions [31, 35],

$$\widetilde{h} = \sum_{j=1}^{K_1} p_j I_{M_1,\chi_1}(x, y; \xi_j, \eta_j) + \sum_{j=K_1+1}^{K_1+K_2} p_j I_{M_2,\chi_2}(x, y; \xi_j, \eta_j),$$

where K_1 is the number of type one basis functions and K_2 is the number of type two basis functions. To determine the unknow coefficients p_j, we do collocation at the sampled data points $\{(x_i, y_i), h_i\}_{i=1}^{N}$ of the source function. The center points of the basis functions and the collocation points are randomly distributed inside the domain Ω. Without loss of generality, we assume $M_2 > M_1$. Then the role of I_{M_1,χ_1} is to capture the general shape and that of I_{M_2,χ_2} is to capture the oscillating details of the function $h(x, y)$. Compared with one-level approach, the two-level approach can significantly improve the fitting while reducing the number of basis functions needed.

Next we obtain a particular solution under the frame work of the DRM [28, 29] and the techniques of [33]. A particular solution $\Psi(x, y; \xi, \eta)$ associated with a Delta-shaped basis function $I_{M,\chi}(x, y; \xi, \eta)$ is $L\Psi(x, y; \xi, \eta) = I_{M,\chi}(x, y; \xi, \eta)$. By looking for $\Psi(x, y; \xi, \eta)$ in the form of $\Psi(x, y; \xi, \eta) = \sum_{n,m=1}^{M} d_{n,m}(\xi, \eta)\, \varphi_n(x)\, \varphi_m(y)$, the coefficients $d_{n,m}(\xi, \eta)$ can be determined for a general elliptic differential operator. For example, when $L = \Delta$,

$$d_{n,m}(\xi, \eta) = -\frac{c_{n,m}(\xi, \eta)}{\lambda_n^2 + \lambda_m^2},$$

and when $L = \Delta - k^2$,

$$d_{n,m}(\xi, \eta) = -\frac{c_{n,m}(\xi, \eta)}{\lambda_n^2 + \lambda_m^2 + k^2}.$$

Since the approximation to the source function is expressed as a linear combination of Delta-shaped basis functions, a particular solution can be obtained using the superposition principle.

3. REGULARIZATION OF THE DISCRETIZED ILL-POSED PROBLEM

The method of fundamental solutions [10, 22] has been widely used for solving homogeneous partial differential equations. A fundamental solution $G(X, Q)$ to a differential operator L is a function satisfying

$$L(G(X, Q)) = \delta(X - Q),$$

where δ is the Dirac delta function. The idea of the MFS is to approximate the solution $u_h(X)$ of a homogeneous equation by the fundamental solutions $G(X, Q_j)$, $j = 1, \ldots, M$, with the singularities Q_j placed outside the given domain. That is, letting $\widetilde{u}_h$ denote this approximation,

$$u_h(X) \approx \widetilde{u}_h(X) = \sum_{j=1}^{M} c_j G(X, Q_j).$$

To determine the coefficients c_j, we let $\widetilde{u}_h(X)$ satisfy the boundary condition (1.5)–(1.6) at a set of boundary points X_i, $i = 1, 2, \ldots, M$. The first M_1 collocation points $\{X_i\}_{i=1}^{M_1}$ are chosen to fit the Dirichlet boundary condition and the other M_2 collocation points $\{X_i\}_{i=M_1+1}^{M_1+M_2}$ are chosen to fit the Neumann boundary condition with $M_1 + M_2 = M$. This results in the following linear system,

$$\sum_{j=1}^{M} c_j G(X_i, Q_j) = f(X_i) - u_p(X_i),\ i = 1, \ldots, M_1, \tag{3.1}$$

$$\sum_{j=1}^{M} c_j \frac{\partial G}{\partial n}\left(X_i, Q_j\right) = g\left(X_i\right) - \frac{\partial u_p}{\partial n}\left(X_i\right),\ i = M_1+1,\ldots,M_1+M_2. \tag{3.2}$$

The numerical examples of the paper involve the Laplace operator $L = \Delta$ and the modified Helmholtz operator $L = \Delta - k^2$. It is known that in 2-D the fundamental solutions for these two operators are respectively

$$G\left(X,Q\right) = \frac{1}{2\pi}\ln\left(\left|X-Q\right|\right),$$

and

$$G\left(X,Q\right) = -\frac{1}{2\pi}K_0\left(k\left|X-Q\right|\right),$$

where $|\cdot|$ denotes the Euclidean distance in R^2 and $K_0\left(x\right)$ the modified Bessel function of the second kind of order zero.

The discretized problem (3.1)–(3.2) is ill-posed due to the ill-posed nature of the continuous model (1.1)–(1.3). Here we intend to solve the problem with perturbed data (1.7)–(1.9). Since the computed solution can sometimes be very sensitive to the perturbations in the input data, we need to be concerned about the accuracy of the output data. A measure of the instability of the system is the condition number κ. The matrices which have to be handled in this paper most often have huge condition numbers. With noises in the data, it is most likely that our computed solution is useless. To cure the ill-posedness, appropriate regularization is necessary.

Let

$$Ax = b \tag{3.3}$$

represents the matrix form of the system (3.1)–(3.2). In the following, we apply Tikhonov regularization and appropriate techniques of choosing a regularization parameter to find its regularized solution. More detailed discussion on the topic can be found in [3, 13, 14, 26, 27, 36, 37, 38, 39].

A method that is frequently used for ill-conditioned or rank-deficient systems is the singular value decomposition (the SVD) [3, 36, 37]. In the SVD, an m by n matrix A $(m \geq n)$ is factored into $A = U\Sigma V^T$ where $U = [u_1, u_2, \ldots, u_m] \in R^{m\times m}$ and $V = [v_1, v_2, \ldots v_n] \in R^{n\times n}$ are orthogonal matrices, and $\Sigma \in R^{m\times n}$ is a diagonal matrix with nonnegative diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ called singular values. It can be shown that every matrix has a singular value decomposition and the following properties hold for a matrix $A \in R^{m\times n}$ of rank r,

$$Av_i = \begin{cases} \sigma_i u_i,\ i = 1,\ldots,r, \\ 0,\ i = r+1,\ldots,n, \end{cases}$$

$$A^T u_i = \begin{cases} \sigma_i v_i,\ i = 1,\ldots,r, \\ 0,\ i = r+1,\ldots,m. \end{cases}$$

Another explanation for the SVD is that the matrix A of rank r can be represented as a sum of rank-one matrices

$$A=\sum_{j=1}^{r}\sigma_j u_j v_j^T,$$

and the solution to the system $Ax=b$ can be expressed as

$$x=\sum_{j=1}^{r}\frac{u_j^T b v_j}{\sigma_j}. \tag{3.4}$$

Although a generalized inverse solution can be obtained given the SVD of (3.3), the solution can become extremely unstable when one or more of the singular values σ_i are small. The norm of the numerical solution by (3.4) can be very large if the singular values of A tend to zero rapidly. Tikhonov regularization [3, 9, 19] controls simultaneously the norm of the residual $Ax-b$ and the norm of the approximate solution x. Considering the data containing noise as in (1.7)–(1.9), there is no point in fitting such data exactly. There can be many solutions that adequately fit the data in the sense that $\|Ax-b\|$ is small enough.

Let $\lambda>0$ be a given constant. The Tikhonov regularized solution x_λ is the minimizer of

$$F_\lambda(x)=\|Ax-b\|^2+\lambda\|x\|^2. \tag{3.5}$$

Here $\|\cdot\|$ denotes the 2-norm. The parameter $\lambda>0$ is called the regularization parameter. That is, the Tikhonov regularization minimizes the residual with a side constrain which is the norm of the solution. The minimization in (3.5) has a unique solution for every b and λ [21]. Clearly, the choice of λ plays an important role in this technique. While a small λ favors a solution with a small residual at the cost of a large norm of the solution, a large λ does the opposite. The value λ also controls the sensitivity of the solution to perturbations in A or b, and the perturbation bound is proportional to λ^{-1} [14]. Tikhonov regularization with the parameter λ gives a regularized solution

$$\widetilde{x}_\lambda=\sum_{j=1}^{r}f_j\frac{u_j^T\widetilde{b}v_j}{\sigma_j},$$

where $\widetilde{b}$ is the perturbed data and the filter factors f_j are

$$f_j=\frac{\sigma_j^2}{\sigma_j^2+\lambda^2}.$$

Since the filter factors control the contribution of singular values and their corresponding singular vectors to the solution, choosing an appropriate parameter value λ is an important part of the algorithm.

The L-curve [13] is a commen method for finding an appropriate parameter λ for the Tikhonov regularization. The L-curve is defined to be

$$L = \left\{ \left(\left\| A\widetilde{x}_\lambda - \widetilde{b} \right\|, \|\widetilde{x}_\lambda\| \right), \ \lambda \geq 0 \right\}.$$

It is called L-curve since it ususlly resembles an L. The vertical axis measures the norm of the solution while the horizontal axis measures the norm of the residual. From the curve, one can easily get an idea of the compromise between the minimization of these two quantities. The value λ that is associated with the point at the "edge" of the L is chosen for the Tikhonov regularization.

There are alternative approaches for finding a suitable value of the parameter such as the generalized cross validation (the GCV) [14]. Clearly, the parameters by different approaches do not necessarily equal each other. Based on the numerical results conducted by Wei et. al. [39] using different regularization techniques, the results by L-curve are consistently well behaved. Here we adopt Tikhonov and L-curve combination for our numerical experiments.

4. NUMERICAL EXAMPLES

In this section, we evaluate the sensitivity of the method towards noises in the source function and boundary data. The noisy data for the source function and boundary data in (1.10)–(1.12) are generated as follows:

$$\varepsilon_h = \beta_h(x, y)\varepsilon,$$
$$\varepsilon_f = \beta_f(x, y)\varepsilon,$$
$$\varepsilon_g = \beta_g(x, y)\varepsilon,$$

where $\beta_h(x, y)$, $\beta_f(x, y)$, and $\beta_g(x, y)$ are random numbers ranging between -1 and 1, and ε is a fixed noise level which lies between 10^{-5} and 1. The N_t test points $\{t_i\}_{i=1}^{N_t}$, randomly distributed inside Ω, are chosen for the calculation of the relative mean square root error,

$$E = \frac{\sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} \left(\widetilde{u}\left(t_i\right) - u\left(t_i\right)\right)^2}}{\sqrt{\frac{1}{N_t} \sum_{i=1}^{N_t} \left(u\left(t_i\right)\right)^2}},$$

where $\widetilde{u}$ and u are respectively the approximate and the exact solutions.

In the following examples, the number of type one and type two Delta-shaped basis functions are denoted by K_1 and K_2. Recall that they are used in the approximation of the source function. Type one basis is to get the general shape and the type two basis that has smaller support is to get the oscillating details of the source function. In general, we can distribute randomly in the domain the centers of the type one and type two basis functions. Since we know the feature of the source function, more of the type two centers can be added to the part of the domain where the source function is more oscillating. The purpose is to improve our approximation. Collocation points can be handled similarly. Examples and discussions on the approximation by Delta-shaped basis can be found in [31, 35]. For the MFS, we use 101 collocation points to fit the Dirichlet data and 100 points to fit the Neumann data. In the domain, the number of test points $N_t = 100$. Here we adopt a simple static approach that uses a fixed location for the source points. We place the source points on a circle of radius $r = 4$. The value of r we use here is satisfactory but not necessarily optimal with respect to the solution of the corresponding well-posed problem. It is known that the source location of the MFS is challenging. Although this is an important issue, it is not the emphasis of this paper. The readers can refer to the recent paper by Chen et al. [5] for a review of the choices of the location of source points. The difficulty of choosing the source location can be lessened substantially by using the method of approximate fundamental solutions [33]. For each example, we give error plots of the numerical solution against the noise in the source function, the noise in the boundary data, and the noise in both the source function and the boundary data. So there are three plots for each of the following examples.

Example 4.1. We consider the problem (1.1)–(1.3), where $L = \Delta - 9$, the domain Ω is bounded by a circle centered at origin with radius $r = 0.5$, Γ is the upper half of the circle, and the source and boundary data are given as,

$$\begin{aligned} h(x,y) &= 6x^4y + 12x^2y^3 - 9x^4y^3 - 9(x+y)^2 - 9xy^2 - 2x - 4, \\ f(x,y) &= x^4y^3 + (x+y)^2 - xy^2, \\ g(x,y) &= 4x^4y^4 + 2x(x+y) - xy^2 + 3x^4y^3 + 2y(x+y) - 2xy^2. \end{aligned}$$

The exact solution of the problem is $u(x,y) = x^4y^3 + (x+y)^2 - xy^2$. The basis functions used are $I_{10,4}$ and $I_{20,6}$ with $K_1 = 100$ and $K_2 = 225$. We place $N = 325$ collocation points inside the circular domain with higher density in $[-0.1, 0.1] \times [-0.1, 0.1]$. The generation of the center points of the basis functions follows the same pattern. The error plots of the numerical solution relative to the noise level are shown in Figures 1–3. The error plots show how the solution error varies with different levels of noise imposed on the data. The solution error increases reasonably with the noise level in the data, but never grows wild in spite of the ill-posedness of the problem.

Example 4.2. We consider a more oscillating source function $h(x,y)$ and a more complicated domain Ω. In the problem (1.1)–(1.3), $L = \Delta$, Ω is bounded by the oval of Cassini whose parametric equations are given by,

$$\begin{aligned} x(t) &= R(t)\cos(t), \; y(t) = R(t)\sin(t), \\ R(t) &= c^2\cos(2t) + \sqrt{b^4 - c^4\sin^2(2t)}, \; 0 \le t \le 2\pi, \end{aligned}$$

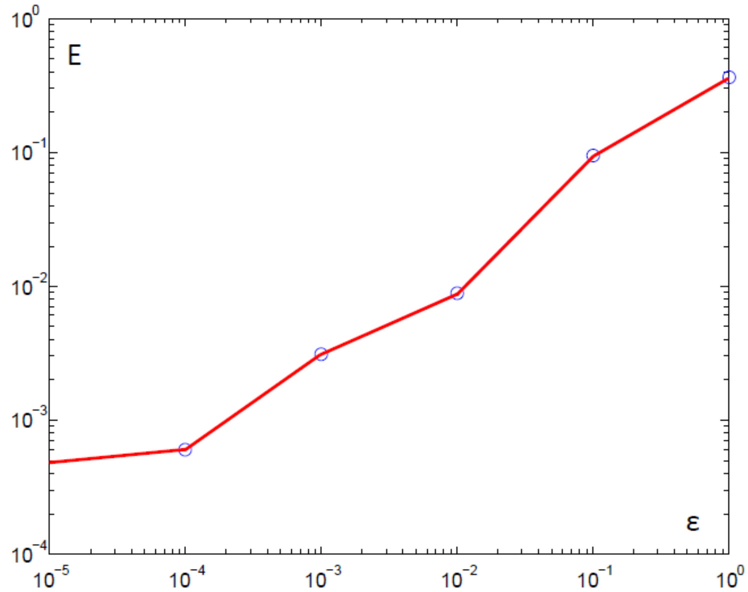

FIGURE 1. Example 1: solution error v.s. noise level ϵ in the source function

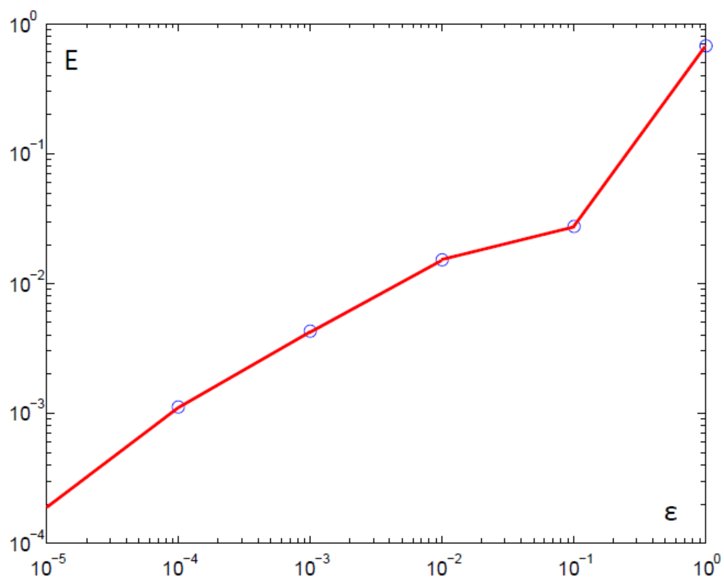

FIGURE 2. Example 1: solution error E v.s. noise level ϵ in the boundary data

with $c = 0.353$ and $b = \sqrt{0.25 - c^2}$. Here Γ denotes the upper curve of the oval of Cassini, and the data are given as

$$h(x,y) = 40\left(x^2+y^2\right)e^{x^2-y^2} - 10\sin\left(5\left(x+y\right)\right),$$

$$f(x,y) = 10e^{x^2-y^2} + \frac{1}{5}\cos\left(5\left(x+y\right)\right),$$

$$g(x,y) = \left(20xe^{x^2-y^2} + \cos\left(5\left(x+y\right)\right)\right)R'\left(t_{(x,y)}\right)\sin\left(t_{(x,y)}\right) + R\left(t_{(x,y)}\right)\cos\left(t_{(x,y)}\right)$$

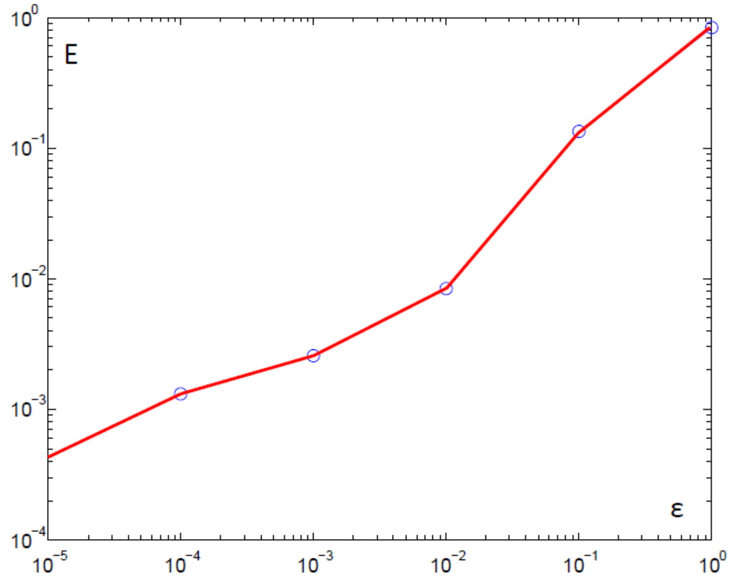

FIGURE 3. Example 1: solution error E v.s. noise level ϵ in both the source function and the boundary data

$$+\left(-20ye^{x^2-y^2}+\cos\left(5\left(x+y\right)\right)\right)R\left(t_{(x,y)}\right)\sin\left(t_{(x,y)}\right)-R'\left(t_{(x,y)}\right)\cos\left(t_{(x,y)}\right).$$

In the above expressions, $t_{(x,y)}$ denotes the parameter value corresponding to the point (x,y). The exact solution to the problem is $u\left(x,y\right)=10e^{x^2-y^2}+\frac{1}{5}\cos\left(5\left(x+y\right)\right)$. Basis functions of $I_{10,4}$ and $I_{20,6}$ with $K_1=150$, $K_2=100$, and $N=250$ collocation points are used in the two-level interpolation. The center points and the collocation points are chosen randomly in the oval of Cassini with a higher density of points near the part of Γ where the curvature is greater. We use more center and collocation points where the boundary quickly changes directions to effectively improve the approximation. The error plots of the numerical solution relative to the noise level are shown in Figures 4–6. Although the source function is highly oscillating, the numerical solution is stable against perturbations.

Example 4.3. In this example, a source function that has a pole close to the Cassini domain Ω is used. The operator $L=\Delta-1$. The partial boundary Γ is the upper curve of the oval of Cassini, and the data are given as follows:

$$\begin{aligned}
h\left(x,y\right)&=-\frac{\cosh\left(y\right)}{\cosh\left(y\right)-\cos\left(x+0.5\right)}+\frac{\sinh^2\left(y\right)}{\left(\cosh\left(y\right)-\cos\left(x+0.5\right)\right)^2}\\
&\quad-\frac{\cos\left(x+0.5\right)}{\cosh\left(y\right)-\cos\left(x+0.5\right)}+\frac{\sin^2\left(x+0.5\right)}{\left(\cosh\left(y\right)-\cos\left(x+0.5\right)\right)^2}\\
&\quad+\ln\left(\cosh\left(y\right)-\cos\left(x+0.5\right)\right),\\
f\left(x,y\right)&=-\ln\left(\cosh\left(y\right)-\cos\left(x+0.5\right)\right),\\
g\left(x,y\right)&=\left(-\frac{\sin\left(x+0.5\right)}{\cosh\left(y\right)-\cos\left(x+0.5\right)}\right)R'\left(t_{(x,y)}\right)\sin\left(t_{(x,y)}\right)+R\left(t_{(x,y)}\right)\cos\left(t_{(x,y)}\right)
\end{aligned}$$

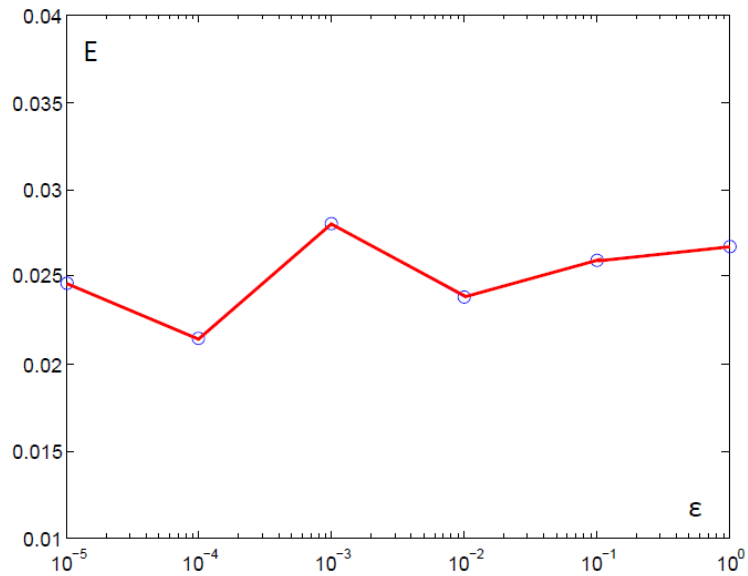

FIGURE 4. Example 2: solution error v.s. noise level ϵ in the source function

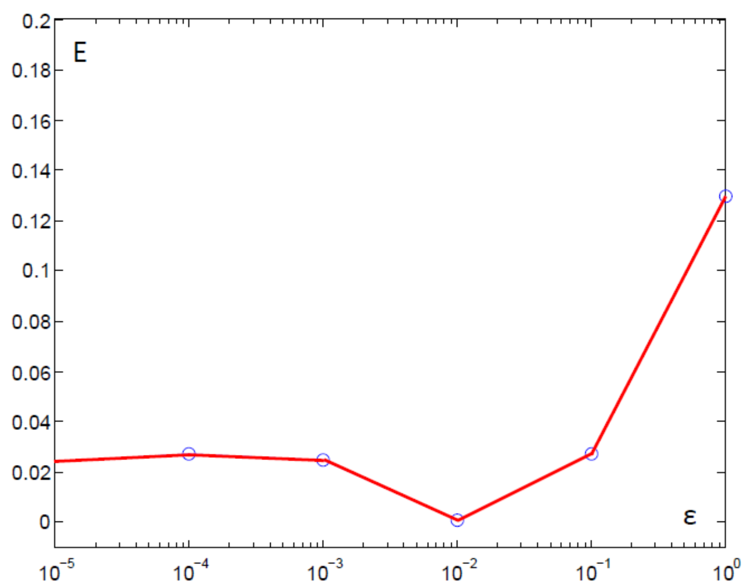

FIGURE 5. Example 2: solution error E v.s. noise level ϵ in the boundary data

$$+\left(-\frac{\sinh(y)}{\cosh(y)-\cos(x+0.5)}\right)R\left(t_{(x,y)}\right)\sin\left(t_{(x,y)}\right)-R'\left(t_{(x,y)}\right)\cos\left(t_{(x,y)}\right).$$

The exact solution to the problem is $u(x,y) = -\ln(\cosh(y)-\cos(x+0.5))$. The error plots of the numerical solution relative to the noise level are shown in Figures 7–9. Despite the fact that the source function has a pole at $(-0.5,0)$, a very accurate solution is obtained. Because of the dramatic changes in the values of the source function, we interpolate it with $I_{10,4}(x,y)$ and $I_{50,15}(x,y)$. We only need $K_1 = 50$, $K_2 = 100$, and $N = 150$ to get a good interpolation of $h(x,y)$. Again the numerical

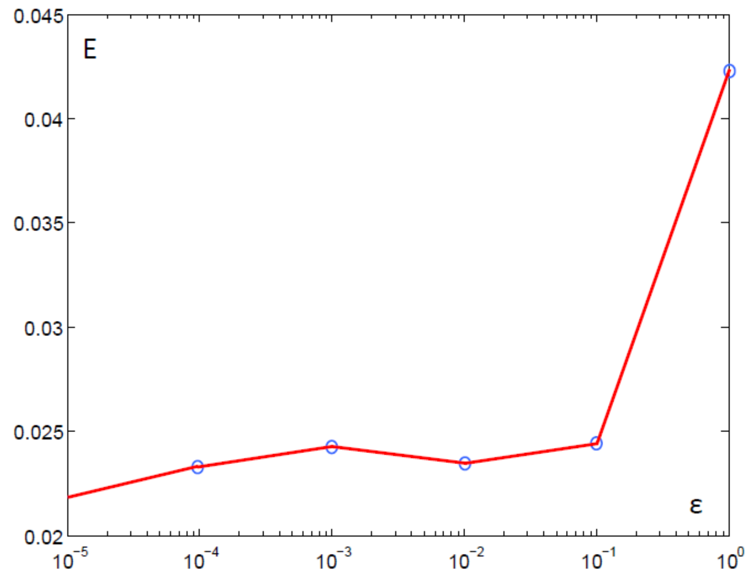

FIGURE 6. Example 2: solution error E v.s. noise level ϵ in both the source function and the boundary data

solution is reliable and stable against noises, and it shows similar behavior as in previous examples.

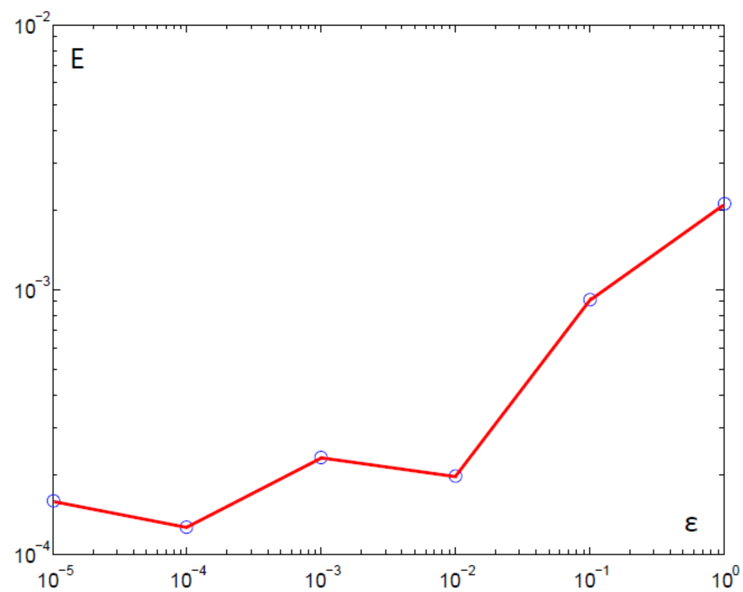

FIGURE 7. Example 3: solution error v.s. noise level ϵ in the source function

We remark that this paper indicates an approach for non-homogeneous ill-posed problems. The method used in this paper are equally applicable to homogeneous equations. The error behavior of the solutions are totally comparable to the results of the paper [39] where the homogeneous cases are solved. Here, the noise in the source term is transfered to the boundary through the approximate particular solution. The

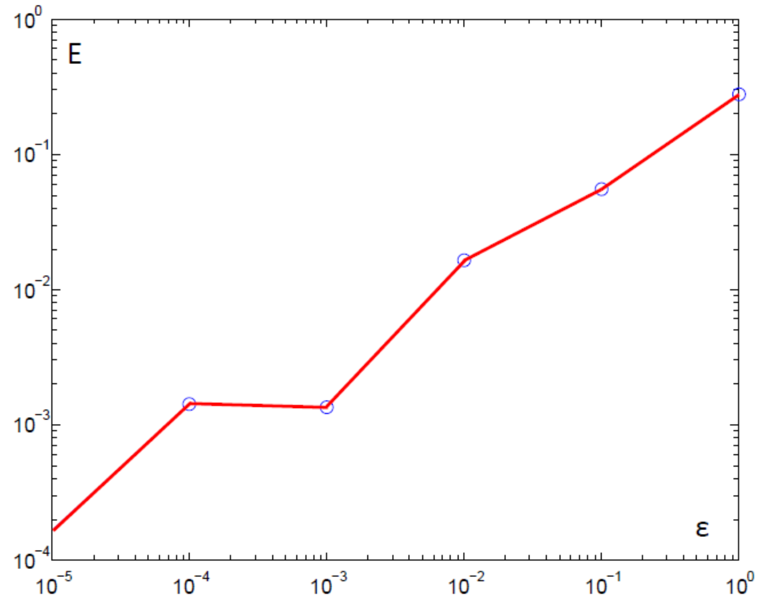

FIGURE 8. Example 3: solution error E v.s. noise level ϵ in the boundary data

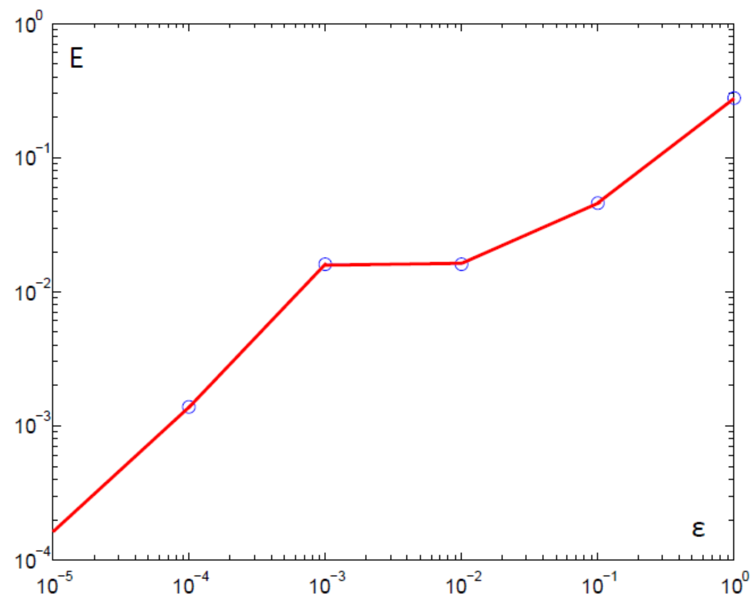

FIGURE 9. Example 3: solution error E v.s. noise level ϵ in both the source function and the boundary data

effective method of particular solutions by the Delta-shaped basis approximation helps keep the influence of the noise in the source function under control. In order for the Delta-shaped basis to be employed, the domain of the problem is embeded in a standard domain $[-0.5, 0.5] \times [-0.5, 0.5]$. If this is not the case originally, appropriate translation and scaling operations can be done to the domain. Interested readers can find the Delta-shaped basis applications on non-standard domains for time-dependent problems [33] and quenching problems [34].

5. CONCLUSIONS

An inverse nonhomogeneous elliptic boundary value problem is solved by a Delta-shaped basis method coupled with the MFS and regularization. The Delta-shaped basis is used for approximating the source function since it can handle effectively scattered data and give rapidly convergent approximation. The method is meshfree and it works well for domains of arbitrary shapes. Two-level approximation is used for the source function. An excellent fit can be achieved with reasonable number of center and collocation points. In addition to its nice properties in approximation, the use of the Delta-shaped basis results in an easy derivation of a particular solution. The corresponding homogeneous problem is solved by MFS and Tikhonov regularization. The L-curve technique is used to obtain the regularization parameter. The meshless method directly solves the given inverse problem, contrary to the traditional meshed and iterative methods. Numerical experiments show that the method is efficient and it is stable with respect to the increased noise level imposed to the source function and the boundary data. This approach can be extended to solving ill-posed multi-dimensional nonhomogeneous problems.

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